Poincare Journal of Analysis & Applications
Vol. 10, No. 1 (2023), 135-154

DOI: 10.46753/pjaa.2023.v010i01.010

MIXED INTEGRAL OPERATOR OF THE VOLTERRA CONVOLUTION TYPE IN WEIGHTED GENERALIZED HÖLDER SPACE

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Date of Receiving	:	26. 05. 2022
Date of Revision	:	19. 01. 2023
Date of Acceptance	:	01. 06. 2023

Abstract. Weighted Zygmund type estimates are obtained for the mixed continuity modulus of some convolution type integrals. In the case of mixed fractional integrals this strengthened to a result on isomorphism between certain weighted generalized Hölder type spaces

1. Introduction

In [3]-[10] the kernel and the weight function were power functions. Now, in the paper we deal here with arbitrary kernels and weights, i.e. not necessarily power ones. We study mixed integral of the Volterra convolution type of function of two variables in weighted generalized Hölder spaces of different orders in each variables.

We consider mixed integral operator of the Volterra convolution type

$$(\widetilde{K}\varphi)(x_1,\ x_2) = \int_0^{x_1}\int_0^{x_2} k(x_1 - t_1)k(x_2 - t_2)\varphi(t_1,\ t_2)dt_1dt_2 \tag{1.1}$$

in rectangle $Q = \{(x_1,\ x_2) :\ 0 \leq x_1 \leq b_1,\ \ 0 \leq x_2 \leq b_2\}$.

We obtain weighted Zygmund type estimates for mixed integrals of the Volterra convolution type in the weighted generalized Hölder space of the function of two variables defined by the mixed modulus of continuity. Estimates of Zygmund type make it possible to reveal the nature of the action of mixed integral operators of the Volterra convolution type in weighted generalized Hölder spaces. We consider weighted generalized Hölder spaces defined both by first order differences in each variable and also by the mixed second order difference, the main interest being in the evaluation of the latter for the

2010 *Mathematics Subject Classification.* 26A33.
Key words and phrases. function two variables, Zygmund type estimates, mixed finite differences, Hölder space, Volterra convolution type operator.
Communicated by. Shashank Goel
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mixed integral of Volterra convolution type in both the cases where the density of the integral belongs to the weighted generalized Hölder space.

2. Preliminaries

We follow the paper [11] in the definitions and theorems that we will need to present the issues under consideration below. Positive constants which can be different at different pales be denoted by $C_1,\ C_2$ and C_{12}.

Definition 2.1. [11] We say that $\psi(x) \in W_\mu = W_\mu([0,\ l]),\ \mu > 0$ if

(1) $\psi(x) \in \mathbf{C}([0,l]),\ \psi(0) = 0,\ \psi(x) > 0$ for $x > 0$;
(2) $\psi(x)$ is almost increasing;
(3) $\psi(x)/x^\mu$ is almost decreasing;
(4) there exists C_1 such that

$$\left|\frac{\psi(x) - \psi(y)}{x - y}\right| \leq C_1 \frac{\psi(x^\star)}{x^\star}, \quad x^\star = \max(x, y). \tag{2.1}$$

Definition 2.2. [11] We say that a non-negative function $k(x)$ on $[0,\ l]$ belongs to the class $V_\lambda,\ \lambda > 0$ if

(1) $k(x) \neq 0,\ x^\lambda k(x)$ is almost increasing and $x^\lambda k(x)\Big|_{x=0} = 0$;
(2) there exists $\varepsilon,\ 0 < \varepsilon < \lambda$, such that $x^{\lambda-\varepsilon} k(x)$ is almost decreasing;
(3) there exists C_1 such that

$$\left|\frac{k(x) - k(y)}{x - y}\right| \leq C_1 \frac{k(x^\star)}{x^\star}, \quad x^\star = \max(x, y). \tag{2.2}$$

In the sequel we shall use the following inequalities:
if $\omega(\varphi,\ h)$ is the continuity modulus, then

$$\frac{\omega(\varphi,\ x)}{x} \leq C_1 \frac{\omega(\varphi,\ y)}{y}, \ \ x \geq y; \tag{2.3}$$

if $0 < \alpha < 1$ then

$$\frac{\omega(\varphi,\ h)}{h^\alpha} \leq C_1 \int\limits_0^h \frac{\omega(\varphi,\ t)}{t^{1+\alpha}} dt; \tag{2.4}$$

if $\psi(x) \in W_\mu$ then

$$\psi(x) \leq C_1 \left(\frac{x}{y}\right)^\mu \psi(y), \ \ x \geq y; \tag{2.5}$$

if $\psi(x) \in W_\mu$ with $0 < \mu < 1$ then (2.1) holds with $x^\star$ replaced both by x and y:

$$\left|\frac{\psi(x) - \psi(y)}{x - y}\right| \leq C_1 \frac{\psi(x)}{x} \ \text{ and } \ \left|\frac{\psi(x) - \psi(y)}{x - y}\right| \leq C_1 \frac{\psi(y)}{y}; \tag{2.6}$$

if $k(x) \in V_\lambda$ then

$$k(x) \leq C_1 \left(\frac{y}{x}\right)^\lambda k(y), \ \ x \leq y, \tag{2.7}$$

and there exists $\varepsilon > 0$ such that

$$k(x) \leq C_1 \left(\frac{y}{x}\right)^{\lambda-\varepsilon} k(y), \quad x \geq y, \tag{2.8}$$

if $\lambda \leq 1$ then

$$\left|x^\lambda - y^\lambda\right| \leq C_1(x-y)y^{\lambda-1}, \quad x \geq y > 0, \tag{2.9}$$

and if $\lambda \geq 0$ then

$$\left|x^\lambda - y^\lambda\right| \leq C_1(x-y)x^{\lambda-1}, \quad x \geq y > 0, \tag{2.10}$$

Lemma 2.3. [11] *Let $k(x) \in V_\lambda, \quad \lambda > 0$ and let $\omega(x) \geq 0$ be an almost increasing function. Then*

$$\omega(x)k(x) \leq C_1 \int\limits_x^l \frac{k(t)}{t}\omega(t)dt$$

for $0 < x < l/2$.

Definition 2.4. [12] Let function $\varphi(x)$ bounded on $[a,\, b]$ The modulus of continuity is understood as expression

$$\omega(\varphi;\, \delta) = \sup_{|x_1-x_2|\leq\delta,\; a\leq x_1,x_2\leq b} \quad \sup |\varphi(x_1) - \varphi(x_2)|\,,$$

defined for all δ satisfying condition $0 < \delta \leq b-a$.

Definition 2.5. [12] We denote by Φ^1 the class of function $\omega(\delta) \in (0,\; b-a]$ satisfying the conditions:

(1) $\omega(\delta)$ is the continuity modulus;
(2) $\int\limits_0^\delta t^{-1}\omega(t)dt \leq C_1\omega(\delta)$;
(3) $\delta \int\limits_\delta^{b-a} t^{-2}\omega(t)dt \leq C_1\omega(\delta)$;
(4) $\omega'(\delta) \sim \delta^{-1}\omega(\delta)$.

For a continuous function $\varphi(x_1,\; x_2)$ on $\mathbf{R}$, we introduce the notation

$$\left(\overset{1,0}{\Delta}_{h_1}\varphi\right)(x_1,\; x_2) = \varphi(x_1+h_1,\; x_2)-\varphi(x_1,\; x_2), \quad \left(\overset{0,1}{\Delta}_{h_2}\varphi\right)(x_1,\; x_2) = \varphi(x_1,\; x_2+h_2)-\varphi(x_1,\; x_2),$$

$$\left(\overset{1,1}{\Delta}_{h_1,\,h_2}\varphi\right)(x_1,\; x_2) = \varphi(x_1+h_1,\; x_2+h_2)-\varphi(x_1+h_1,\; x_2)-\varphi(x_1,\; x_2+h_2)+\varphi(x_1,\; x_2),$$

so that

$$\varphi(x_1+h_1,\; x_2+h_2) = \left(\overset{1,1}{\Delta}_{h_1,\,h_2}\varphi\right)(x_1,\; x_2)+\left(\overset{0,1}{\Delta}_{h_2}\varphi\right)(x_1,\; x_2)+\left(\overset{1,0}{\Delta}_{h_1}\varphi\right)(x_1,\; x_2)+\varphi(x_1,\; x_2).$$

While studying the properties $\varphi(x_1, x_2)$ of continuous functions of several variables (that is, $\varphi(x_1, x_2) \in \mathbf{C}(Q)$) in particular the following class of functions arise [1]:

$$\widetilde{H}^{\varpi} = H^{\omega_1,\omega_2,\omega_{1,1}} = \Big\{\varphi(x_1,\; x_2) \in \mathbf{C}(Q) : \overset{1,0}{\omega}_\varphi(\delta_1,\; 0) = O(\omega_1(\delta_1)), \quad \overset{0,1}{\omega}_\varphi(0,\; \delta_2) = O(\omega_2(\delta_2)),$$

$$\overset{1,1}{\omega}_\varphi(\delta_1,\; \delta_2) = O(\omega_{1,1}(\delta_1,\; \delta_2))\Big\},$$

where
(a) private modules of continuity

$$\overset{1,0}{\omega}_{\varphi}(\delta_1,\ 0) = \sup_{x_2\in[a_2,\ b_2],\ 0\le h_1\le\delta_1}\left|\left(\overset{1,0}{\Delta}_{h_1}\varphi\right)(x_1,\ x_2)\right| \quad \text{and}$$

$$\overset{0,1}{\omega}_{\varphi}(0;\ \delta_2) = \sup_{x_1\in[a_1,\ b_1],\ 0\le h_2\le\delta_2}\left|\left(\overset{0,1}{\Delta}_{h_2}\varphi\right)(x_1,\ x_2)\right|;$$

(b) mixed modulus continuity of order 1, 1

$$\overset{1,1}{\omega}_{\varphi}(\delta_1,\ \delta_2) = \sup_{x_i\in[a_i,\ b_i],\ 0\le h_i\le\delta_i}\left|\left(\overset{1,1}{\Delta}_{h_1,\ h_2}\varphi\right)(x_1,\ x_2)\right|,$$

where $0<\delta_i\le b_i-a_i,\ \ i=1,\ 2.$

It follows from the definition that the function $\overset{1,1}{\omega}_{\varphi}(\delta_1,\ \delta_2)$ belongs to Φ^1 for each of the variables. In addition, we note the following inequality

$$\overset{1,1}{\omega}_{\varphi}(\delta_1,\ \delta_2) = \overset{1,1}{\omega}(\varphi;\ \delta_1,\ \delta_2) \le 2\min\left\{\overset{1,0}{\omega}(\varphi;\ \delta_1,\ 0),\ \overset{0,1}{\omega}(\varphi;\ 0,\ \delta_2)\right\}.$$

Definition 2.6. We denote by $\Phi^{1,1}$ the class of functions of two variables $\omega(\delta_1,\ \delta_2)$ satisfying the following conditions:

(1) $\omega(\delta_1,\ \delta_2)$ in δ_1, for any fixed δ_2;
(2) $\omega(\delta_1,\ \delta_2)$ in δ_2, for any fixed δ_1.

We call this as the class of mixed modulus of continuity of the first order of continuous functions of two variables. In [1] it is shown that the properties 1) and 2) are characteristic for continuity modulus in the sense that for every $\omega(x_1,\ x_2)\in\Phi^{1,1}$ there exists a function $\varphi\in\mathbf{C}(Q)$ such that

$$\overset{1,1}{\omega}(\varphi;\ \delta_1,\ \delta_2)\sim\omega_{1,1}(\delta_1,\ \delta_2),\quad \overset{1,0}{\omega}(\varphi;\ \delta_1,\ 0)\sim\omega_1(\delta_1),\quad \overset{0,1}{\omega}(\varphi;\ 0,\ \delta_2)\sim\omega_2(\delta_2).$$

Definition 2.7. We denote by Φ the set satisfying the conditions on $(\omega_1,\ \omega_2,\ \omega_{1,1})$ given by

(1) $\omega_1(\delta_1),\ \omega_2(\delta_2)\in\Phi^1$;
(2) $\omega_{1,1}(\delta_1,\ \delta_2)\in\Phi^{1,1}$;
(3) $\omega_{1,1}(\delta_1,\ \delta_2)\le C_{12}\min\{\omega_1(\delta_1),\ \omega_2(\delta_2)\}$.

Definition 2.8. Let $\omega(\delta_1,\delta_2)\in\Phi(Q)$. We will say that function $\varphi(x_1,x_2)$ belongs to class $\widetilde{H}^{\varpi}(Q)=\widetilde{H}^{\omega_1,\omega_2,\omega_{1,1}}(Q)$ if it is defined on Q and satisfies conditions

$$\sup_{0<\delta_1\le b_1-a_1}\frac{\overset{1,0}{\omega}(\varphi;\delta_1,0)}{\omega_1(\delta_1)}<\infty,\quad \sup_{0<\delta_2\le b_2-a_2}\frac{\overset{0,1}{\omega}(\varphi;0,\delta_2)}{\omega_2(\delta_2)}<\infty \text{ and } \sup_{0<\delta_i\le b_i-a_i}\frac{\overset{1,1}{\omega}(\varphi;\delta_1,\delta_2)}{\omega_{1,1}(\delta_1,\delta_2)}<\infty,$$

where $i=1,2$. Functions $\omega_1(\delta_1),\omega_2(\delta_2),\omega_{1,1}(\delta_1,\delta_2)$ will be called the characteristic functions of the generalized Hölder space $\widetilde{H}^{\omega_1,\omega_2,\omega_{1,1}}(Q)$.

Let $(\omega_1,\ \omega_2,\ \omega_{1,1})\in\Phi(Q)$. Then we introduce norm in $\widetilde{H}^{\varpi}=\widetilde{H}^{(\omega_1,\ \omega_2,\ \omega_{1,1})}$ by

$$\|\varphi\|_{\widetilde{H}^{\varpi}} = \left\{\|\varphi\|_{C(Q)},\ \sup_{\delta_1>0}\frac{\overset{1,0}{\omega}(\varphi;\delta_1,0)}{\omega_1(\delta_1)},\ \sup_{\delta_2>0}\frac{\overset{0,1}{\omega}(\varphi;0,\delta_2)}{\omega_2(\delta_2)},\ \sup_{\delta_i>0,i=1,2}\frac{\overset{1,1}{\omega}(\varphi;\delta_1,\delta_2)}{\omega_{1,1}(\delta_1,\delta_2)}\right\}.$$

We say that $\varphi \in \widetilde{H}_0^{\varpi}(Q)$ if $\varphi \in \widetilde{H}^{\varpi}(Q)$ and $\varphi(x_1,\ x_2)\Big|_{x_i=a_i} = \varphi(x_1,\ x_2)\Big|_{x_i=b_i} \equiv 0,\ i=1,2.$

Definition 2.9. Let $\widetilde{H}^{\varpi}(\rho) = \widetilde{H}^{\varpi}(Q;\rho)$ denote the set of all functions $\varphi(x_1,\ x_2)$ satisfying condition $\rho(x_1,\ x_2)\varphi(x_1,\ x_2) \in \widetilde{H}^{\varpi}(Q)$, where $\rho(x_1,\ x_2)$ is a non-negative weight function.

As usual the norm in $\widetilde{H}^{\varpi}(\rho)$ is given by

$$\|\varphi\|_{\widetilde{H}^{\varpi}(\rho)} = \|\rho\varphi\|_{\widetilde{H}^{\varpi}}.$$

Denote by $\widetilde{H}_0^{\varpi}(\rho) = \widetilde{H}_0^{\varpi}(Q;\rho)$ the spaces of functions $\varphi(x_1, x_2)$ such that $\rho(x_1,\ x_2)\varphi(x_1,\ x_2) \in \widetilde{H}_0^{\varpi}(Q)$.

Consider a one-dimensional integral operator of the Volterra convolution type

$$(K\varphi)(x) = \int_0^x k(x-t)\varphi(t)dt, \quad 0<x<b. \tag{2.11}$$

The following theorem is known [11].

Theorem 2.10. *Let $k(x) \in V_\lambda,\ \ 0<\lambda<1$ and $\varphi(x) \in \mathbf{C}([a,\ b]),\ \ \varphi(a)=0.$ Then*

$$\omega(K\varphi,\ h) \le C_1\left(hk(h)\omega(\varphi,\ h) + h\int_h^{b-a}\frac{k(t)}{t}\omega(\varphi,\ t)dt\right). \tag{2.12}$$

3. Zygmund type estimates

In [2] we give non-weighted Zygmund type estimates for mixed integrals of the Volterra convolution type.

Theorem 3.1. . *Let $k(x_i) \in V_{\lambda_i}, 0<\lambda_i<1, \varphi(x_1,\ x_2) \in C(Q)$ and $\varphi(x_1, x_2)\Big|_{x_i=a_i} = 0, i=1,2.$ Then the following Zygmund type estimates hold*

$$\overset{1,0}{\omega}(\widetilde{K}\varphi; h_1, 0) \le C_1\left[h_1k(h_1)\overset{1,1}{\omega}(\varphi; h_1, b_2-a_2) + h_1\int_{h_1}^{b_1-a_1}\frac{k(t_1)}{t_1}\overset{1,1}{\omega}(\varphi; t_1, b_2-a_2)dt_1\right],$$

$$\overset{0,1}{\omega}(\widetilde{K}\varphi; 0, h_2) \le C_2\left[h_2k(h_2)\overset{1,1}{\omega}(\varphi; b_1-a_1, h_2) + h_2\int_{h_2}^{b_2-a_2}\frac{k(t_2)}{t_2}\overset{1,1}{\omega}(\varphi; b_1-a_1, t_2)dt_2\right],$$

$$\overset{1,1}{\omega}(\widetilde{K}\varphi; h_1, h_2) \le C_3\left[h_1k(h_1)h_2k(h_2)\overset{1,1}{\omega}(\varphi; h_1, h_2) + h_1k(h_1)h_2\int_{h_2}^{b_2-a_2}\frac{k(t_2)}{t_2}\overset{1,1}{\omega}(\varphi; h_, t_2)dt_2 + \right.$$

$$h_1k(h_2)h_2\int\limits_{h_1}^{b_1-a_1}\frac{k(t_1)}{t_1}\overset{1,1}{\omega}(\varphi;t_1,h_2)dt_1+h_1h_2\int\limits_{h_1}^{b_1-a_1}\int\limits_{h_2}^{b_2-a_2}\frac{\overset{1,1}{\omega}(\varphi;t_1,t_2)}{t_1t_2}k(t_1)k(t_2)dt_1dt_2\Bigg].$$

Now we give weighted Zygmund type estimates for mixed integrals of the Volterra convolution type. To do this, we first give notation and auxiliary assertions. Let us introduce the notation:

$$M(x,\ t):=\frac{\psi(x-a)-\psi(t-a)}{\psi(t-a)}k(x-t),\ \text{and}\ M_1(x,\ h,\ t)=M(x+h,\ t)-M(x,\ t),$$

where $\psi(x)\in W_\mu,\ a<t<x,\ x+h<b$

Lemma 3.2. . *Let* $\rho=\psi(x-a),\psi(x)\in W_\mu,\ \mu\in R$ *and* $k(x)\in V_\lambda,\lambda>0.$ *Then*

$$|M(x,t)|\le C_1\left(\frac{x-a}{t-a}\right)^{\max(\mu-1,0)}\frac{x-t}{t-a}k(x-t),\tag{3.1}$$

$$|M_1(x,h,t)|\le C_1\left(\frac{x+h-a}{t-a}\right)^{\max(\mu-1,0)}\frac{h}{t-a}k(x+h-t).\tag{3.2}$$

Proof. Let $\mu<1$,then from inequalities (2.1) and (2.6) we have

$$|M(x,t)|=\left|\frac{\psi(x-a)-\psi(t-a)}{\psi(t-a)}k(x-t)\right|\le C_1\frac{x-t}{t-a}k(x-t).$$

If $1\le\mu$, then from (2.1) and (2.5), we obtain

$$|M(x,t)|\le C_1\frac{\psi(x-a)}{x-a}\frac{x-t}{\psi(t-a)}k(x-t)\le C_1\left(\frac{x-a}{t-a}\right)^{\mu-1}\frac{x-t}{t-a}k(x-t).$$

Now let's estimate $M_1(x,h,t)$. Let $\mu<1$, then from inequalities (2.1), (2.5), (2.2) and (2.7), we obtain

$$|M_1(x,h,t)|\le\left|\frac{\psi(x+h-a)-\psi(x-a)}{\psi(t-a)}\right|k(x+h-t)$$

$$+\left|\frac{\psi(x-a)-\psi(t-a)}{\psi(t-a)}\right||k(x+h-t)-k(x-t)|\le C_1\left[\frac{\psi(x-a)}{x-a}\frac{h}{\psi(t-a)}k(x+h-t)\right.$$

$$\left.+\frac{\psi(t-a)}{t-a}\frac{x-t}{\psi(t-a)}\frac{k(x+h-t)}{x+h-t}h\right]\le C_1\left[\frac{k(x+h-a)}{x-a}h\left(\frac{x-a}{t-a}\right)^\mu\right.$$

$$\left.+\frac{h}{t-a}k(x+h-t)\right]=C_1\frac{h}{t-a}\left[\left(\frac{x+h-a}{t-a}\right)^{\mu-1}+1\right]k(x+h-t).$$

Since $\mu<1$, then

$$|M_1(x,h,t)|\le C_1\frac{h}{t-a}k(x+h-t)\left[\left(\frac{t-a}{x-a}\right)^{1-\mu}+1\right]\le C_1\frac{h}{t-a}k(x+h-t).$$

If $1\le\mu$, then

$$|M_1(x,h,t)|\le\left|\frac{\psi(x+h-a)-\psi(x-a)}{\psi(t-a)}\right|k(x+h-t)$$

$$+\left|\frac{\psi(x-a)-\psi(t-a)}{\psi(t-a)}\right||k(x+h-t)-k(x-t)| \leq C_1\left[\frac{\psi(x+h-a)}{x+h-a}\frac{h}{\psi(t-a)}k(x+h-t)\right.$$

$$\left.+\frac{\psi(x-a)}{x-a}\frac{x-t}{\psi(t-a)}\frac{k(x+h-t)}{x+h-t}h\right] \leq C_1\left[\frac{k(x+h-t)}{x+h-a}\frac{\psi(x+h-a)}{\psi(t-a)}h\right.$$

$$\left.+\frac{\psi(x+h-a)}{\psi(t-a)}\frac{k}{x+h-a}k(x+h-t)\right] \leq 2C_1\frac{h}{t-a}\left(\frac{x+h-a}{t-a}\right)^{\mu-1}k(x+h-t).$$

□

The following theorem is already known [11], but since its proof is much more complicated, we use Lemma 3.2 to simplify its proof.

Theorem 3.3. . *Let $k(x) \in V_\lambda, 0 < \lambda < 1$ and $\rho(x) = \psi(x-a)$, where $\psi(x) \in W_\mu, 0 < \mu < 1+\lambda$. Assume that $\varphi_0(x) = \rho(x)\varphi(x) \in C([a,b]),\ \ \varphi_0(a) = 0$ and*

$$\int_0^{b-a}\frac{\omega(\varphi_0,t)}{t^\gamma}dt < \infty, \quad \text{where} \quad \gamma = \max(1,\mu).$$

Then the following Zygmund type estimate holds:

$$\omega(\rho K\varphi,h) \leq C_1\left[h^\gamma k(h)\int_0^h\frac{\omega(\varphi_0,t)}{t^\gamma}dt + h\int_h^{b-a}\frac{\omega(\varphi_0,t)}{t}k(t)dt\right]. \tag{3.3}$$

Proof. Let $\varphi_0(x) = \rho(x)\varphi(x) \in C([a,b)]$ and $\varphi_0(a) = 0$. We have

$$(\rho K\varphi)(x) = \int_a^x \varphi_0(t)k(x-t)dt + \int_a^x M(x,t)\varphi_0(t)dt = f_1(x) + f_2(x).$$

Here the question of estimating the modulus of continuity for the first term is known in Theorem 2.10.

Let us estimate the second term. Let be $h > 0; x, x+h \in [a,b]$. Consider the difference

$$f_2(x+h) - f_2(x) = \int_x^{x+h} M(x+h,t)\varphi_0(t)dt + \int_a^x M_1(x,h,t)\varphi_0(t)dt.$$

Further

$$|f_2(x+h)-f_2(x)| \leq \int_x^{x+h}|M(x+h,t)|\omega(\varphi_0,t-a)dt + \int_a^x|M_1(x,h,t)|\omega(\varphi_0,t-a)dt = f_2' + f"_2. \tag{3.4}$$

We estimate f_2'. I) Let $0 < \mu < 1$. Using (3.1), we have

$$f_2' \leq C_1\int_x^{x+h}\frac{x+h-t}{t-a}k(x+h-t)\omega(\varphi_0,t-a)dt \leq C_1hk(h)\int_x^{x+h}\frac{\omega(\varphi_0,t-a)}{t-a}dt.$$

Taking into account the almost decreasing $\frac{\omega(\varphi_0,t-a)}{t-a}$, we obtain

$$f_2' \le C_1hk(h)\int\limits_x^{x+h}\frac{\omega(\varphi_0,x-t)}{t-x}dt = C_1hk(h)\int\limits_0^h\frac{\omega(\varphi_0,t)}{t}dt. \tag{3.5}$$

II) If $1 \le \mu < 1+\lambda$, then from (3.1) we have

$$f_2' \le C_1hk(h)\int\limits_x^{x+h}\left(\frac{x+h-a}{x-a}\right)^{\mu-1}\frac{(x+h-t)}{t-a}k(x+h-t)\omega(\varphi_0,t-a)dt. \tag{3.6}$$

We distinguish two cases $x-a\le h$ and $x-a>h$. In the case

$$f_2' \le C_1h^\mu k(h)\int\limits_x^{x+h}\frac{\omega(\varphi_0,t-a)}{(t-a)^\mu}dt \le C_1h^\mu k(h)\int\limits_x^{x+h}\frac{\omega(\varphi_0,t-x)}{(t-x)^\mu}dt =$$

$$C_1h^\mu k(h)\int\limits_0^h\frac{\omega(\varphi_0,t)}{t^\mu}dt.$$

In the second case

$$f_2' \le C_1hk(h)(x+h-a)^{\mu-1}\int\limits_x^{x+h}\frac{\omega(\varphi_0,t-a)}{(t-a)^\mu}dt = \frac{C_1hk(h)}{(x+h-a)^{1-\mu}}\int\limits_0^h\frac{\omega(\varphi_0,x+t-a)}{(x+t-a)^\mu}dt \le$$

$$C_1hk(h)\int\limits_0^h\frac{\omega(\varphi_0,x+t-a)}{x+t-a}dt.$$

Taking into account the almost decreasing $\frac{\omega(\varphi_0,t)}{t}$, we obtain

$$f_2' \le C_1hk(h)\int\limits_0^h\frac{\omega(\varphi_0,t)}{t}dt.$$

Gathering all the estimates for f_2', we have

$$f_2' \le C_1h^\gamma k(h)\int\limits_0^h\frac{\omega(\varphi_0,t)}{t^\gamma}dt, \quad \gamma=\max(1,\mu). \tag{3.7}$$

Let us estimate $f"_2$. Here we also use the estimates from Lemma 3.2. In $0<\mu<1$, we have

$$f"_2 \le C_1h\int\limits_0^{x-a}\frac{\omega(\varphi_0,t)}{t}k(x+h-t-a)dt. \tag{3.8}$$

In the case $x-a\le h$ we obtain

$$f"_2 \le C_1hk(h)\int\limits_0^{x-a}\frac{\omega(\varphi_0,t)}{t}dt \le C_1hk(h)\int\limits_0^h\frac{\omega(\varphi_0,t)}{t}dt.$$

In the case $x-a>h$ we represent the right side of inequality (3.8) as the sum of three terms

$$f"_2 \le C_1 h \left(\int\limits_0^h + \int\limits_h^{\frac{x+h-a}{2}} + \int\limits_{\frac{x+h-a}{2}}^{x-a} \right) \frac{\omega(\varphi_0,t)}{t} k(x+h-t-a)dt = A_1 + A_2 + A_3.$$

Since $t \le x-a$, then it immediately follows

$$A_1 \le C_1 h k(h) \int\limits_0^h \frac{\omega(\varphi_0,t)}{t} dt.$$

The term A_2 satisfies the inequality $x+h-t-a \ge t$, so we have

$$A_2 \le C_1 h \int\limits_h^{\frac{x+h-a}{2}} \frac{\omega(\varphi_0,t)}{t} k(t) \left(\frac{t}{x+h-t-a}\right)^{\lambda-\varepsilon} dt \le C_1 h \int\limits_h^{b-a} \frac{\omega(\varphi_0,t)}{t} k(t)dt.$$

The term A_3 satisfies the inequality $x+h-t-a \le t$, so

$$A_3 \le C_1 \int\limits_{\frac{x+h-a}{2}}^{x-a} \frac{\omega(\varphi_0, x+h-t-a)}{x+h-t-a} k(x+h-t-a)dt.$$

We make the substitution $x+h-t-a=t$ and obtain

$$A_3 \le C_1 h \int\limits_h^{\frac{x+h-a}{2}} \frac{\omega(\varphi_0,t)}{t} k(t)dt \le C_1 h \int\limits_h^{b-a} \frac{\omega(\varphi_0,t)}{t} k(t)dt.$$

Thus, for $f"_2$, for $0<\mu<1$ we have

$$f"_2 \le C_1 \left(hk(h) \int\limits_0^h \frac{\omega(\varphi_0,t)}{t} dt + h \int\limits_h^{b-a} \frac{\omega(\varphi_0,t)}{t} k(t)dt \right). \tag{3.9}$$

Let $1 \le \mu < 1+\lambda$. Use estimates from Lemma 3.2 for $M_1(x,h,t)$ we obtain

$$f"_2 \le C_1 h \int\limits_0^{x-a} \left(\frac{x+h-a}{t}\right)^{\mu-1} \frac{\omega(\varphi_0,t)}{t} k(x+h-t-a)dt. \tag{3.10}$$

In the case $x-a \le h$ from (3.10) we obtain

$$f"_2 \le C_1 h^\mu \int\limits_0^{x-a} \frac{\omega(\varphi_0,t)}{t^\mu} k(x+h-t-a)dt \le C_1 h^\mu k(h) \int\limits_0^{x-a} \frac{\omega(\varphi_0,t)dt}{t^\mu (x+h-t-a)} \le$$

$$C_1 h^\mu k(h) \int\limits_0^h \frac{\omega(\varphi_0,t)}{t^\mu} dt.$$

If $x-a>h$, then we represent the right side of inequality (3.10) as the sum of three terms:

$$f"_2 \le C_1 h \left(\int\limits_0^h + \int\limits_h^{\frac{x+h-a}{2}} + \int\limits_{\frac{x+h-a}{2}}^{x-a} \right) \left(\frac{x+h-a}{t} \right)^{\mu-1} \frac{\omega(\varphi_0,t)}{t} k(x+h-t-a)dt = T_1+T_2+T_3.$$

Since $x+h-a \le 2(x+h-t-a)$ in T_1, we obtain

$$T_1 \le C_1 h \int\limits_0^h \left(\frac{x+h-t-a}{t} \right)^{\mu-1} \frac{\omega(\varphi_0,t)}{t} k(x+h-t-a)dt \le C_1 h^\mu k(h) \int\limits_0^h \frac{\omega(\varphi_0,t)}{t^\mu} dt.$$

Further $x+h-a \ge 2t$ in T_2, then

$$T_2 \le C_1 h \int\limits_h^{\frac{x+h-a}{2}} t^{\mu-1} k(t) \frac{\omega(\varphi_0,t)}{t^\mu} dt \le C_1 h \int\limits_h^{b-a} \frac{\omega(\varphi_0,t)}{t} k(t)dt.$$

Let us estimate the term T_3. Here $t \ge x+h-t-a$, therefore $\frac{\omega(\varphi_0,t)}{t} \le C_1 \frac{\omega(\varphi_0,x+h-t-a)}{x+h-t-a}$, hence it follows that

$$T_3 \le C_1 h \int\limits_{\frac{x+h-a}{2}}^{x-a} \left(\frac{x+h-a}{t} \right)^{\mu-1} \frac{\omega(\varphi_0,x+h-t-a)}{x+h-t-a} k(x+h-t-a)dt \le$$

$$C_1 h \int\limits_{\frac{x+h-a}{2}}^{x-a} \frac{\omega(\varphi_0,x+h-t-a)}{x+h-t-a} k(x+h-t-a)dt,$$

because $x+h-a \le 2t$. Making the substitution $\xi = x+h-t-a$ and passing again to the variable t, we obtain

$$T_3 \le C_1 h \int\limits_h^{b-a} \frac{\omega(\varphi_0,t)}{t} k(t)dt.$$

Gathering all estimates of $f"_2$ in the case $0 \le \mu < 1+\lambda$, we have

$$f"_2 \le C_1 \left(h^\gamma k(h) \int\limits_0^h \frac{\omega(\varphi_0,t)}{t^\gamma} dt + h \int\limits_h^{b-a} \frac{\omega(\varphi_0,t)}{t} k(t)dt \right), \quad \gamma = \max(1,\mu). \tag{3.11}$$

From inequalities (3.7) and (3.11), we obtain estimate for $0 \le \mu < 1+\lambda$

$$|f_2(x+h) - f_2(x)| \le C_1 \left(h^\gamma k(h) \int\limits_0^h \frac{\omega(\varphi_0,t)}{t^\gamma} dt + h \int\limits_h^{b-a} \frac{\omega(\varphi_0,t)}{t} k(t)dt \right), \tag{3.12}$$

where $\gamma = \max(1,\mu)$. □

We introduce the following notation

$$M(x_1,x_2;t_1,t_2)=\frac{\psi(x_1-a_1)\psi(x_2-a_2)-\psi(t_1-a_1)\psi(t_2-a_2)}{\psi(t_1-a_1)\psi(t_2-a_2)}k(x_1-t_1)k(x_2-t_1), \tag{3.13}$$

where $a_i<t_i<x_i<b_i, i=1,2.$

Lemma 3.4. *Let* $\rho(x_1,x_2)=\psi(x_1-a_1)\psi(x_2-a_2)$, *then*

$$M(x_1,x_2;t_1,t_2)=M(x_1,t_1)M(x_2,t_2)+M(x_1,t_1)k(x_2-t_2)+M(x_2,t_2)k(x_1-t_1), \tag{3.14}$$

here

$$M(x_i,t_i)=\frac{\psi(x_i-a_i)-\psi(t_i-a_i)}{\psi(t_i-a_i)}k(x_i-t_i), i=1,2.$$

Proof. Since $\rho(x_1,x_2)=\psi(x_1-a_1)\psi(x_2-a_2)$, then

$$\begin{aligned}&\psi(x_1-a_1)\psi(x_2-a_2)-\psi(t_1-a_1)\psi(t_2-a_2)\\&=\big[\psi(x_1-a_1)-\psi(t_1-a_1)\big]\big[\psi(x_2-a_2)-\psi(t_2-a_2)\big]\\&+\psi(t_2-a_2)\big[\psi(x_1-a_1)-\psi(t_1-a_1)\big]+\psi(t_1-a_1)\big[\psi(x_2-a_2)-\psi(t_2-a_2)\big].\end{aligned}$$

Taking into account this equality in (3.13), we have

$$M(x_1,x_2;t_1,t_2)=\frac{\big[\psi(x_1-a_1)-\psi(t_1-a_1)\big]\big[\psi(x_2-a_2)-\psi(t_2-a_2)\big]}{\psi(t_1-a_1)\psi(t_2-a_2)}k(x_1-t_1)k(x_2-t_2)$$

$$+\frac{\psi(x_1-a_1)-\psi(t_1-a_1)}{\psi(t_1-a_1)}k(x_1-t_1)k(x_2-t_2)+\frac{\psi(x_2-a_2)-\psi(t_2-a_2)}{\psi(t_2-a_2)}k(x_1-t_1)k(x_2-t_2)$$

$$=M(x_1,t_1)M(x_2,t_2)+M(x_1,t_1)k(x_2-t_2)+M(x_2,t_2)k(x_1-t_1).$$ □

Theorem 3.5. . *Let* $k(x_i)\in V_{\lambda_i}, 0<\lambda_i<1, \rho(x_1,x_2)=\psi(x_1-a_1)\psi(x_2-a_2)$, *where* $\psi(x_i)\in W_{\mu_i}, 0<\mu_i<1+\lambda_i, i=1,2.$ *Assume that* $\varphi_0(x_1,x_2)=\rho(x_1,x_2)\varphi(x_1,x_2)\in C(Q), \varphi_0(x_1,x_2)\big|_{x_i=a_i}=0$ *and*

$$\int_0^{b_1-a_1}\int_0^{b_2-a_2}\frac{\overset{1,1}{\omega}(\varphi_0;t_1,t_2)}{t_1^{\delta_1}t_2^{\delta_2}}dt_1dt_2<\infty, \quad \textit{where} \quad \delta_i=\max(1,\mu_i), i=1,2.$$

Then the following Zygmund type estimates hold

$$\begin{aligned}\overset{1,0}{\omega}(\rho\widetilde{K}\varphi;h_1,0)\le C_1&\left[h_1^{\delta_1}k(h_1)\int_0^{h_1}\frac{\overset{1,1}{\omega}(\varphi_0;t_1,b_2-a_2)}{t_1^{\delta_1}}dt_1\right.\\&\left.+h_1\int_{h_1}^{b_1-a_1}\frac{\overset{1,1}{\omega}(\varphi_0;t_1,b_2-a_2)}{t_1}k(t_1)dt_1\right],\end{aligned}\tag{3.15}$$

$$\overset{0,1}{\omega}(\rho\widetilde{K}\varphi;0,h_2)\le C_2\left[h_2^{\delta_2}k(h_2)\int_0^{h_2}\frac{\overset{1,1}{\omega}(\varphi_0;b_1-a_1,t_2)}{t_2^{\delta_2}}dt_2\right.$$

$$+h_2 \int\limits_{h_2}^{b_2-a_2} \frac{\overset{1,1}{\omega}(\varphi_0; b_1-a_1, t_2)}{t_2} k(t_2)dt_2 \Bigg], \tag{3.16}$$

$$\overset{1,1}{\omega}(\rho\widetilde{K}\varphi; h_1, h_2) \le C_{12} \Bigg[h_1^{\delta_1} h_2^{\delta_2} k(h_1)k(h_2) \int\limits_0^{h_1}\int\limits_0^{h_2} \frac{\overset{1,1}{\omega}(\varphi_0; t_1, t_2)}{t_1^{\delta_1} t_2^{\delta_2}} dt_1 dt_2$$

$$+h_2 h_1^{\delta_1} k(h_1) \int\limits_0^{h_1}\int\limits_{h_2}^{b_2-a_2} \frac{\overset{1,1}{\omega}(\varphi_0; t_1, t_2)}{t_1^{\delta_1} t_2} k(t_2)dt_1 dt_2 + h_1 h_2^{\delta_2} k(h_2) \int\limits_{h_1}^{b_1-a_1}\int\limits_0^{h_2} \frac{\overset{1,1}{\omega}(\varphi_0; t_1, t_2)}{t_2^{\delta_2} t_1} k(t_1)dt_1 dt_2$$

$$+h_2 h_1 \int\limits_{h_1}^{b_1-a_1}\int\limits_{h_2}^{b_2-a_2} \frac{\overset{1,1}{\omega}(\varphi_0; t_1, t_2)}{t_1 t_2} k(t_1)k(t_2)dt_1 dt_2 \Bigg]. \tag{3.17}$$

Proof. Let $\varphi_0(x_1, x_2) = \rho(x_1, x_2)\varphi(x_2, x_2)$. We note

$$D(x_1, x_2) := (\rho\widetilde{K}\varphi)(x_1, x_2) = \int\limits_{a_1}^{x_1}\int\limits_{a_2}^{x_2} \varphi_0(t_1, t_2)k(x_1-t_1)k(x_2-t_2)dt_1 dt_2$$

$$+\int\limits_{a_1}^{x_1}\int\limits_{a_2}^{x_2} M(x_1, x_2; t_1, t_2)\varphi_0(t_1, t_2)dt_1 dt_2 = D_1(x_1, x_2) + D_2(x_1, x_2). \tag{3.18}$$

Here the question of estimating the modulus of continuity for the first term is solved by us in Theorem 3.1. Let us estimate the second term. Taking into account (3.14), we represent $D_2(x_1, x_2)$ in the following form

$$D_2(x_1, x_2) = \int\limits_{a_1}^{x_1}\int\limits_{a_2}^{x_2} M(x_1, t_1)M(x_2, t_2)\varphi_0(t_1, t_2)dt_1 dt_2$$

$$+\int\limits_{a_1}^{x_1}\int\limits_{a_2}^{x_2} M(x_1, t_1)k(x_2-t_2)\varphi_0(t_1, t_2)dt_1 dt_2 + \int\limits_{a_1}^{x_1}\int\limits_{a_2}^{x_2} k(x_1-t_1)M(x_2, t_2)\varphi_0(t_1, t_2)dt_1 dt_2$$

$= m_1(x_1, x_2) + m_2(x_1, x_2) + m_3(x_1, x_2)$.

We shall estimate each term $m_j(x_1, x_2)$, $j = 1, 2, 3$ separately. Let $h_1 > 0; x_1, x_1+h_1 \in [a_1, b]$. Consider the difference

$$m_1(x_1+h_1, x_2) - m_1(x_1, x_2) = \int\limits_{x_1}^{x_1+h_1}\int\limits_{a_2}^{x_2} M(x_1+h_1, t_1)M(x_2, t_2)\varphi_0(t_1, t_2)dt_1 dt_2$$

$$+\int\limits_{a_1}^{x_1}\int\limits_{a_2}^{x_2} M_1(x_1, h_1, t_1)M(x_2, t_2)\varphi_0(t_1, t_2)dt_1 dt_2.$$

Since $\varphi_0(x_1, x_2)\Big|_{x_i=a_i} = 0, i = 1, 2$, then

$$|m_1(x_1+h_1, x_2) - m_1(x_1, x_2)|$$

$$\leq C_{12}\left[\int_{x_1}^{x_1+h_1}\int_{a_2}^{x_2}|M(x_1+h_1,t_1)|\,|M(x_2,t_2)|\overset{1,1}{\omega}(\varphi_0;t_1-a_1,t_2-a_2)dt_1dt_2\right.$$

$$\left.+\int_{a_1}^{x_1}\int_{a_2}^{x_2}|M_1(x_1,h_1,t_1)|\,|M(x_2,t_2)|\overset{1,1}{\omega}(\varphi_0;t_1-a_1,t_2-a_2)dt_1dt_2\right].$$

Let us estimate the integral

$$\int_{a_2}^{x_2}|M(x_2,t_2)|\overset{1,1}{\omega}(\varphi_0;t_1-a_1,t_2-a_2)dt_2.$$

Let $0<\mu_2<1$. From Lemma 3.2, we have

$$\int_{a_2}^{x_2}|M(x_2,t_2)|\overset{1,1}{\omega}(\varphi_0;t_1-a_1,t_2-a_2)dt_2\leq$$

$$C_2(x_2-a_2)k(x_2-a_2)\int_0^{x_2-a_2}\frac{\overset{1,1}{\omega}(\varphi_0;t_1-a_1,t_2)}{t_2}dt_2\leq C_2\overset{1,1}{\omega}(\varphi_0;t_1-a_1,b_2-a_2).$$

In $1\leq\mu_2<1+\lambda_2$

$$\int_{a_2}^{x_2}|M(x_2,t_2)|\overset{1,1}{\omega}(\varphi_0;t_1-a_1,t_2-a_2)dt_2\leq$$

$$C_2(x_2-a_2)k(x_2-a_2)\int_0^{x_2-a_2}\left(\frac{x_2-a_2}{t_2}\right)^{\mu_2-1}\frac{\overset{1,1}{\omega}(\varphi_0;t_1-a_1,t_2)}{t_2}dt_2\leq$$

$$C_2\overset{1,1}{\omega}(\varphi_0;t_1-a_1,b_2-a_2).$$

From here and from estimates $f_2', f"_2$ in Theorem 3.3, one can easily verify the validity of the inequality, for $0\leq\mu_1<1+\lambda_1$

$$|m_1(x_1+h_1,x_2)-m_1(x_1,x_2)|\leq C_{12}\left(h_1^{\delta_1}k(h_1)\int_0^{h_1}\frac{\overset{1,1}{\omega}(\varphi_0;t_1,b_2-a_2)}{t_1^{\delta_1}}dt_1\right.$$

$$\left.+h_1\int_{h_1}^{b_1-a_1}\frac{k(t_1)}{t_1}\overset{1,1}{\omega}(\varphi_0;t_1,b_2-a_2)dt_1\right),\tag{3.20}$$

where $\delta_1=\max(1,\mu_1)$.

Now we estimate $m_2(x_1,x_2)$. Let $h_1>0; x_1,x_1+h_1\in[a_1,b_1]$. Consider the difference

$$m_2(x_1+h_1,x_2)-m_2(x_1,x_2)=\int_{x_1}^{x_1+h_1}\int_{a_2}^{x_2}M(x_1+h_1,t_1)k(x_2-t_2)\varphi_0(t_1,t_2)dt_1dt_2$$

$$+\int_{a_1}^{x_1}\int_{a_2}^{x_2} M_1(x_1,h_1,t_1)k(x_2-t_2)\varphi_0(t_1,t_2)dt_1dt_2.$$

The following inequality holds

$$|m_2(x_1+h_1,x_2)-m_2(x_1,x_2)|$$

$$\leq C_1\left[\int_{x_1}^{x_1+h_1}\int_{a_2}^{x_2}|M(x_1+h_1,t_1)|\,k(x_2-t_2)\overset{1,1}{\omega}(\varphi_0;t_1-a_1,t_2-a_2)dt_1dt_2\right.$$

$$\left.+\int_{a_1}^{x_1}\int_{a_2}^{x_2}|M_1(x_1,h_1,t_1)|\,k(x_2-t_2)\overset{1,1}{\omega}(\varphi_0;t_1-a_1,t_2-a_2)dt_1dt_2\right]. \tag{3.21}$$

Since

$$\int_{a_2}^{x_2}\overset{1,1}{\omega}(\varphi_0;t_1-a_1,t_2-a_2)k(x_2-t_2)dt_2 \leq C_2\overset{1,1}{\omega}(\varphi_0;t_1-a_1,x_2-a_2)\int_{a_2}^{x_2}k(x_2-t_2)\leq$$

$$C_2\overset{1,1}{\omega}(\varphi_0;t_1-a_1,b_2-a_2). \tag{3.22}$$

Taking into account inequality (3.22), we obtain

$$|m_2(x_1+h_1,x_2)-m_2(x_1,x_2)|\leq C_1\left(\int_{x_1}^{x_1+h_1}|M(x_1+h_1,t_1)|\overset{1,1}{\omega}(\varphi_0;t_1-a_1,b_2-a_2)dt_1\right.$$

$$\left.+\int_{a_1}^{x_1}|M_1(x_1,h_1,t_1)|\overset{1,1}{\omega}(\varphi_0;t_1-a_1,b_2-a_2)dt_1\right).$$

It is easy to verify the validity of the estimate if estimates f_2' and $f"_2$ are taken into account in $0\leq\mu_1<1+\lambda_1$

$$|m_2(x_1+h_1,x_2)-m_2(x_1,x_2)|\leq C_1\left(h_1^{\delta_1}k(h_1)\int_0^{h_1}\frac{\overset{1,1}{\omega}(\varphi_0;t_1,b_2-a_2)}{t_1^{\delta_1}}dt_1\right.$$

$$\left.+h_1\int_{h_1}^{b_1-a_1}\frac{\overset{1,1}{\omega}(\varphi_0;t_1,b_2-a_2)}{t_1}k(t_1)dt_1\right),\ \text{ where }\ \delta_1=\max(\mu_1,1).$$

Now we estimate m_3. Imagine the difference

$$m_3(x_1+h_1,x_2)-m_3(x_1,x_2)=\int_{x_1}^{x_1+h_1}\int_{a_2}^{x_2}\varphi_0(t_1,t_2)k(x_1+h_1-t_1)M(x_2,t_2)dt_1dt_2$$

$$+\int_{a_1}^{x_1}\int_{a_2}^{x_2}\left[k(x_1+h_1-t_1)-k(x_1-t_1)\right]\varphi_0(t_1,t_2)M(x_2,t_2)dt_1dt_2.$$

Since $\varphi_0(x_1,a_2)=0$, we have

$$|m_3(x_1+h_1,x_2)-m_3(x_1,x_2)|\leq$$

$$C_{12}\left(\int\limits_{-h_1}^{0}\int\limits_{a_2}^{x_2}\overset{1,1}{\omega}(\varphi_0;t_1,t_2-a_2)|M(x_2,t_2)|k(h_1+t_1)dt_1dt_2\right.$$

$$\left.+\int\limits_{0}^{x_1-a_1}\int\limits_{a_2}^{x_2}|k(h_1+t_1)-k(t_1)|\overset{1,1}{\omega}(\varphi_0;t_1,t_2-a_2)|M(x_2,t_2)|dt_1dt_2\right).$$

Using the integral estimate (3.20), we obtain

$$|m_3(x_1+h_1,x_2)-m_3(x_1,x_2)|\leq C_1\left(\int\limits_{-h_1}^{0}\overset{1,1}{\omega}(\varphi_0;t_1,b_2-a_2)k(h_1+t_1)dt_1\right.$$

$$\left.+\int\limits_{0}^{x_1-a_1}|k(h_1+t_1)-k(t_1)|\overset{1,1}{\omega}(\varphi_0;t_1,b_2-a_2)dt_1\right).$$

Using the estimates for A_1, A_2 from Theorem 2.10, we obtain the estimate

$$|m_3(x_1+h_1,x_2)-m_3(x_1,x_2)|\leq C_1\left(h_1k(h_1)\overset{1,1}{\omega}(\varphi_0;h_1,b_2-a_2)\right.$$

$$\left.+h_1\int\limits_{h_1}^{b_1-a_1}\frac{\overset{1,1}{\omega}(\varphi_0;t_1,b_2-a_2)}{t_1}k(t_1)dt_1\right).$$

Gathering estimates of $m_j(x_1,x_2), j=1,2,3$ we obtain

$$|D_2(x_1+h_1,x_2)-D_2(x_1,x_2)|\leq C_1\left(h_1^{\delta_1}k(h_1)\int\limits_{0}^{h_1}\frac{\overset{1,1}{\omega}(\varphi_0;t_1,b_2-a_2)}{t_1^{\delta_1}}dt_1\right.$$

$$\left.+h_1\int\limits_{h_1}^{b_1-a_1}\frac{\overset{1,1}{\omega}(\varphi_0;t_1,b_2-a_2)}{t_1}k(t_1)dt_1\right),\quad \text{where}\quad \delta_1=\max(\mu_1,1). \tag{3.22}$$

Symmetrically, we can obtain the estimate

$$|D_2(x_1,x_2+h_2)-D_2(x_1,x_2)|\leq C_2\left(h_2^{\delta_2}k(h_2)\int\limits_{0}^{h_2}\frac{\overset{1,1}{\omega}(\varphi_0;b_1-a_1,t_2)}{t_2^{\delta_2}}dt_2\right.$$

$$\left.+h_2\int\limits_{h_2}^{b_2-a_2}\frac{\overset{1,1}{\omega}(\varphi_0;b_1-a_1,t_2)}{t_2}k(t_2)dt_2\right),\quad \text{where}\quad \delta_2=\max(\mu_2,1). \tag{3.23}$$

Now, let us prove the validity of estimate (3.17). Let $h_i>0; x_i, x_i+h_i\in[a_i,b_i], i=1,2$. We represent the difference

$$\left(\overset{1,1}{\Delta}_{h_1,h_2}D_2\right)(x_1,x_2):=\sum_{k=1}^{12}Y_k,$$

where

$$Y_1 = \int\limits_{x_1}^{x_1+h_1} \int\limits_{x_2}^{x_2+h_2} M(x_1+h_1,t_1)M(x_2+h_2,t_2)\varphi_0(t_1,t_2)dt_1dt_2,$$

$$Y_2 = \int\limits_{a_1}^{x_1} \int\limits_{a_2}^{x_2} M_1(x_1,h_1,t_1)M_2(x_2,h_2,t_2)\varphi_0(t_1,t_2)dt_1dt_2,$$

$$Y_3 = \int\limits_{x_1}^{x_1+h_1} \int\limits_{a_2}^{x_2} M(x_1+h_1,t_1)M_1(x_2,h_2,t_2)\varphi_0(t_1,t_2)dt_1dt_2,$$

$$Y_4 = \int\limits_{a_1}^{x_1} \int\limits_{x_2}^{x_2+h_2} M_1(x_1,h_1,t_1)M(x_2+h_2,t_2)\varphi_0(t_1,t_2)dt_1dt_2,$$

$$Y_5 = \int\limits_{x_1}^{x_1+h_1} \int\limits_{x_2}^{x_2+h_2} M(x_1+h_1,t_1)k(x_2+h_2-t_2)\varphi_0(t_1,t_2)dt_1dt_2,$$

$$Y_6 = \int\limits_{x_1}^{x_1+h_1} \int\limits_{a_2}^{x_2} M(x_1+h_1,t_1)\left[k(x_2+h_2-t_2)-k(x_2-t_2)\right]\varphi_0(t_1,t_2)dt_1dt_2,$$

$$Y_7 = \int\limits_{a_1}^{x_1} \int\limits_{x_2}^{x_2+h_2} M_1(x_1,h_1,t_1)k(x_2+h_2-t_2)\varphi_0(t_1,t_2)dt_1dt_2,$$

$$Y_8 = \int\limits_{a_1}^{x_1} \int\limits_{a_2}^{x_2} M_1(x_1+h_1,t_1)\left[k(x_2+h_2-t_2)-k(x_2-t_2)\right]\varphi_0(t_1,t_2)dt_1dt_2,$$

$$Y_9 = \int\limits_{x_1}^{x_1+h_1} \int\limits_{x_2}^{x_2+h_2} k(x_1+h_1-t_1)M(x_2+h_2,t_2)\varphi_0(t_1,t_2)dt_1dt_2,$$

$$Y_{10} = \int\limits_{a_1}^{x_1} \int\limits_{x_2}^{x_2+h_2} \left[k(x_1+h_1-t_1)-k(x_1-t_1)\right]M(x_2+h_2,t_2)\varphi_0(t_1,t_2)dt_1dt_2,$$

$$Y_{11} = \int\limits_{x_1}^{x_1+h_1} \int\limits_{a_2}^{x_2} k(x_1+h_1-t_1)M_1(x_2,h_2,t_2)\varphi_0(t_1,t_2)dt_1dt_2,$$

$$Y_{12} = \int\limits_{a_1}^{x_1} \int\limits_{a_2}^{x_2} \left[k(x_1+h_1-t_1)-k(x_1-t_1)\right]M_1(x_2,h_2,t_2)\varphi_0(t_1,t_2)dt_1dt_2.$$

Since $\varphi_0(x_1,x_2)\Big|_{x_i=a_i}=0$, then

$$|Y_1|\le C_{12}\int\limits_{x_1}^{x_1+h_1}\int\limits_{x_2}^{x_2+h_2} M(x_1+h_1,t_1)M(x_2+h_2,t_2)\overset{1,1}{\omega}(\varphi_0;t_1-a_1,t_2-a_2)dt_1dt_2,$$

$$|Y_2|\le C_{12}\int\limits_{a_1}^{x_1}\int\limits_{a_2}^{x_2} M_1(x_1,h_1,t_1)M_2(x_2,h_2,t_2))\overset{1,1}{\omega}(\varphi_0;t_1-a_1,t_2-a_2)dt_1dt_2,$$

$$|Y_3|\le C_{12}\int\limits_{x_1}^{x_1+h_1}\int\limits_{a_2}^{x_2} M(x_1+h_1,t_1)M_1(x_2,h_2,t_2))\overset{1,1}{\omega}(\varphi_0;t_1-a_1,t_2-a_2)dt_1dt_2,$$

$$|Y_4|\le C_{12}\int\limits_{a_1}^{x_1}\int\limits_{x_2}^{x_2+h_2} M_1(x_1,h_1,t_1)M(x_2+h_2,t_2))\overset{1,1}{\omega}(\varphi_0;t_1-a_1,t_2-a_2)dt_1dt_2,$$

$$|Y_5|\le C_{12}\int\limits_{x_1}^{x_1+h_1}\int\limits_{x_2}^{x_2+h_2} M(x_1+h_1,t_1)k(x_2+h_2-t_2))\overset{1,1}{\omega}(\varphi_0;t_1-a_1,t_2-a_2)dt_1dt_2,$$

$$|Y_6|\le C_{12}\int\limits_{x_1}^{x_1+h_1}\int\limits_{a_2}^{x_2} M(x_1+h_1,t_1)\left|k(x_2+h_2-t_2)-k(x_2-t_2)\right|)\overset{1,1}{\omega}(\varphi_0;t_1-a_1,t_2-a_2)dt_1dt_2,$$

$$|Y_7|\le C_{12}\int\limits_{a_1}^{x_1}\int\limits_{x_2}^{x_2+h_2} M_1(x_1,h_1,t_1)k(x_2+h_2-t_2))\overset{1,1}{\omega}(\varphi_0;t_1-a_1,t_2-a_2)dt_1dt_2,$$

$$|Y_8|\le C_{12}\int\limits_{a_1}^{x_1}\int\limits_{a_2}^{x_2} M_1(x_1+h_1,t_1)\left|k(x_2+h_2-t_2)-k(x_2-t_2)\right|)\overset{1,1}{\omega}(\varphi_0;t_1-a_1,t_2-a_2)dt_1dt_2,$$

$$|Y_9|\le C_{12}\int\limits_{x_1}^{x_1+h_1}\int\limits_{x_2}^{x_2+h_2} k(x_1+h_1-t_1)M(x_2+h_2,t_2))\overset{1,1}{\omega}(\varphi_0;t_1-a_1,t_2-a_2)dt_1dt_2,$$

$$|Y_{10}|\le C_{12}\int\limits_{a_1}^{x_1}\int\limits_{x_2}^{x_2+h_2} \left|k(x_1+h_1-t_1)-k(x_1-t_1)\right|M(x_2+h_2,t_2))\overset{1,1}{\omega}(\varphi_0;t_1-a_1,t_2-a_2)dt_1dt_2,$$

$$|Y_{11}|\le C_{12}\int\limits_{x_1}^{x_1+h_1}\int\limits_{a_2}^{x_2} k(x_1+h_1-t_1)M_1(x_2,h_2,t_2))\overset{1,1}{\omega}(\varphi_0;t_1-a_1,t_2-a_2)dt_1dt_2,$$

$$|Y_{12}|\le C_{12}\int\limits_{a_1}^{x_1}\int\limits_{a_2}^{x_2} \left|k(x_1+h_1-t_1)-k(x_1-t_1)\right|M_1(x_2,h_2,t_2))\overset{1,1}{\omega}(\varphi_0;t_1-a_1,t_2-a_2)dt_1dt_2.$$

We skip the details of calculating each $Y_k, k = 1, 2, ?, 12$. Their calculation is carried out in the standard way using Lemma 3.2, inequalities (3.1)-(3.2) and using the estimates for $f_2', f"_2$ from Theorem 3.3, it is easy to verify the validity of the inequality

$$\left|\left(\overset{1,1}{\Delta}_{h_1,h_2} D_2\right)\right| \leq C_{12}\left[h_1^{\delta_1} h_2^{\delta_2} k(h_1)k(h_2)\int_0^{h_1}\int_0^{h_2} \frac{\omega(\varphi_0; t_1, t_2)}{t_1^{\delta_1} t_2^{\delta_2}} dt_1 dt_2 + \right.$$

$$h_1 h_2^{\delta_2} k(h_2)\int_{h_1}^{b_1-a_1}\int_0^{h_2} \frac{\omega(\varphi_0; t_1, t_2)}{t_1 t_2^{\delta_2}} k(t_1) dt_1 dt_2 +$$

$$h_1^{\delta_1} h_2 k(h_1)\int_0^{h_1}\int_{h_2}^{b_2-a_2} \frac{\omega(\varphi_0; t_1, t_2)}{t_1^{\delta_1} t_2} k(t_2) dt_1 dt_2 +$$

$$\left. h_1 h_2 \int_0^{h_1}\int_0^{h_2} \frac{\omega(\varphi_0; t_1, t_2)}{t_1 t_2} k(t_1)k(t_2) dt_1 dt_2\right], \text{ if } 0 \leq \mu_i < 1 + \lambda_i, i = 1, 2. \tag{3.24}$$

From inequalities (3.22), (3.23) and (3.24), we obtain the corresponding estimates (3.15), (3.16) and (3.17). □

4. Main Result

Theorem 4.1. *Let $\rho(x_1, x_2) = \psi(x_1 - a_1)\psi(x_2 - a_2), \psi(x_i) \in W_{\mu_i}, k(x_i) \in V_{\lambda_i}, 0 < \lambda_i < 1, 0 < \mu_i < \lambda_i + 1, i = 1, 2$. Assume that*

$$\int_0^{h_1}\int_0^{h_2} \left(\frac{h_1}{t_1}\right)^{\max(\mu_1,0)} \left(\frac{h_2}{t_2}\right)^{\max(\mu_2,0)} \frac{\omega(t_1, t_2)}{t_1 t_2} dt_1 dt_2 \leq C_{12}\omega(h_1, h_2)$$

and

$$\int_{h_1}^{b_1-a_1}\int_{h_2}^{b_2-a_2} \frac{\omega(t_1, t_2)}{t_1 t_2 k(t_1)k(t_2)} dt_1 dt_2 \leq C_{12}\omega(h_1, h_2)k(h_1)k(h_2).$$

Then the operator $\widetilde{K}$ is bounded from $\widetilde{H}_0^{\varpi}(\rho)$ into $\widetilde{H}_0^{\varpi_k}(\rho)$ with characteristic $\varpi_k(h_1, h_2) = h_1 h_2 k(h_1)k(h_2)\omega(h_1, h_2)$.

Proof. Let $f(x_1, x_2) = (\widetilde{K}\varphi)(x_1, x_2), \varphi \in \widetilde{H}_0^{\varpi}(\rho)$. We must show that $f \in \widetilde{H}_0^{\varpi_k}(\rho)$. For this, it suffices to show that

$$\sup_{h_1>0} \frac{\overset{1,0}{\omega}(\rho f; h_1, 0)}{h_1 k(h_1)\omega_1(h_1)} = C_1 < \infty, \quad \sup_{h_2>0} \frac{\overset{0,1}{\omega}(\rho f; 0, h_2)}{h_2 k(h_2)\omega_2(h_2)} = C_2 < \infty,$$

$$\sup_{h_1>0, i=1,2} \frac{\overset{1,1}{\omega}(\rho f; h_1, h_2)}{h_1 k(h_1) h_2 k(h_2)\omega_{1,1}(h_1, h_2)} = C_{12} < \infty.$$

Note that in the case $0 < \mu_i < 1 + \lambda_i$

$$\int\limits_0^{b_1-a_1}\int\limits_0^{b_2-a_2}\frac{\overset{1,1}{\omega}(\rho\varphi;t_1,t_2)}{t_1^{\delta_1}t_2^{\delta_2}}dt_1dt_2 < \infty, \quad \text{where} \quad \delta_i = \max(1,\mu_i),$$

for each argument. Therefore, the Zygmund type estimates (3.15), (3.16), and (3.17) hold. Where should

$$\frac{\overset{1,0}{\omega}(\rho f;h_1,0)}{h_1k(h_1)\omega_1(h_1)} \le C_1\left(\frac{h_1^{\delta_1-1}}{\omega_1(h_1)}\int\limits_0^{h_1}\frac{\overset{1,1}{\omega}(\rho\varphi;t_1,b_2-a_2)}{t_1^{\delta_1}}dt_1+\right.$$

$$\left.\frac{1}{k(h_1)\omega_1(h_1)}\int\limits_{h_1}^{b_1-a_1}\frac{\overset{1,1}{\omega}(\rho\varphi;t_1,b_2-a_2)}{t_1}k(t_1)dt_1\right) \le C_1\|\rho\varphi\|_{\widetilde{H}_0^{\varpi}}\left(\frac{h_1^{\delta_1-1}}{\omega_1(h_1)}\int\limits_0^{h_1}\frac{\omega_1(t_1)}{t_1^{\delta_1}}dt_1+\right.$$

$$\left.\frac{1}{k(h_1)\omega_1(h_1)}\int\limits_{h_1}^{b_1-a_1}\frac{\omega_1(t_1)}{t_1}k(t_1)dt_1\right) \le C_1\|\rho\varphi\|_{\widetilde{H}_0^{\varpi}}, \tag{4.1}$$

$$\frac{\overset{0,1}{\omega}(\rho f;0,h_2)}{h_2k(h_2)\omega_2(h_2)} \le C_2\left(\frac{h_2^{\delta_2-1}}{\omega_2(h_2)}\int\limits_0^{h_2}\frac{\overset{1,1}{\omega}(\rho\varphi;b_1-a_1,t_2)}{t_2^{\delta_2}}dt_2+\right.$$

$$\left.\frac{1}{k(h_2)\omega_2(h_2)}\int\limits_{h_2}^{b_2-a_2}\frac{\overset{1,1}{\omega}(\rho\varphi;b_1-a_1,t_2)}{t_2}k(t_2)dt_2\right) \le C_2\|\rho\varphi\|_{\widetilde{H}_0^{\varpi}}\left(\frac{h_2^{\delta_2-2}}{\omega_2(h_2)}\int\limits_0^{h_2}\frac{\omega_2(t_2)}{t_2^{\delta_2}}dt_2+\right.$$

$$\left.\frac{1}{k(h_2)\omega_2(h_2)}\int\limits_{h_2}^{b_2-a_2}\frac{\omega_2(t_2)}{t_2}k(t_2)dt_2\right) \le C_2\|\rho\varphi\|_{\widetilde{H}_0^{\varpi}}, \tag{4.2}$$

$$\frac{\overset{1,1}{\omega}(\rho f;h_1,h_2)}{h_1k(h_1)h_2k(h_2)\omega_{1,1}(h_1,h_2)} \le C_{12}\|\rho\varphi\|_{\widetilde{H}_0^{\varpi}}\left(\frac{h_1^{\delta_1-1}h_2^{\delta_2-1}}{\omega_{1,1}(h_1,h_2)}\int\limits_0^{h_1}\int\limits_0^{h_2}\frac{\overset{1,1}{\omega}(\rho\varphi;t_1,t_2)}{t_1^{\delta_1}t_2^{\delta_2}}dt_1dt_2+\right.$$

$$\frac{h_1^{\delta_1-1}}{k(h_2)\omega_{1,1}(h_1,h_2)}\int\limits_0^{h_1}\int\limits_{h_2}^{b_2-a_2}\frac{\overset{1,1}{\omega}(\rho\varphi;t_1,t_2)}{t_1^{\delta_1}t_2}k(t_2)dt_1dt_2+$$

$$\frac{h_2^{\delta_2-1}}{k(h_1)\omega_{1,1}(h_1,h_2)}\int\limits_{h_1}^{b_1-a_1}\int\limits_0^{h_2}\frac{\overset{1,1}{\omega}(\rho\varphi;t_1,t_2)}{t_1t_2^{\delta_2}}k(t_1)dt_1dt_2+$$

$$\left.\frac{1}{k(h_1)k(h_2)\omega_{1,1}(h_1,h_2)}\int\limits_{h_1}^{b_1-a_1}\int\limits_{h_2}^{b_2-a_2}\frac{\overset{1,1}{\omega}(\rho\varphi;t_1,b_2-a_2)}{t_1t_2}k(t_1)k(t_2)dt_1dt_2\right) \le C_{12}\|\rho\varphi\|_{\widetilde{H}_0^{\varpi}}. \tag{4.3}$$

It remains to show that $\varphi_0(x_1,x_2)\Big|_{x_i=a_i} = \rho(x_1,x_2)\varphi(x_1,x_2)\Big|_{x_i=a_i} = 0, i = 1,2.$ After substitutions $t_i = x_i - s_i(x_i - a_i), i = 1,2$, we obtain

$$\rho(x_1,x_2)f(x_1,x_2) = \int\limits_{a_1}^{x_1}\int\limits_{a_2}^{x_2} \frac{\psi(x_1-a_1)\psi(x_2-a_2)}{\psi(t_1-a_1)\psi(t_2-a_2)} k(x_1-t_1)k(x_2-t_2)\varphi_0(t_1,t_2)dt_1dt_2$$

$$= \psi(x_1-a_1)\psi(x_2-a_2)(x_1-a_1)(x_2-a_2)\times$$

$$\int\limits_0^1\int\limits_0^1 \frac{\varphi_0\left[x_1-(x_1-a_1)s_1, x_1-(x_2-a_2)s_2\right]}{\psi\left[(1-s_1)(x_1-a_1)\right]\psi\left[(1-s_2)(x_2-a_2)\right]} k\left[s_1(x_1-a_1)\right]k\left[s_2(x_2-a_2)\right]ds_1ds_2.$$

Since $\varphi_0(x_1,x_2)\Big|_{x_i=a_i} = 0, i = 1,2$, then

$$\Big|\varphi_0\left[x_1-(x_1-a_1)s_1, x_2-(x_2-a_2)s_2\right] - \varphi_0\left[a_1, x_2-(x_2-a_2)s_2\right]\Big| \le C_1 \overset{1,0}{\omega}(\varphi_0; 1-s_1, 0),$$

$$\Big|\varphi_0\left[x_1-(x_1-a_1)s_1, x_2-(x_2-a_2)s_2\right] - \varphi_0\left[x_1-(x_1-a_1)s_1, a_2\right]\Big| \le C_2 \overset{0,1}{\omega}(\varphi_0; 0, 1-s_2),$$

$$\left|\left(\overset{1,1}{\Delta}_{(x_1-a_1)(1-s_1),(x_2-a_2)(1-s_2)}\varphi\right)(a_1,a_2)\right| \le C_{12} \overset{1,1}{\omega}(\varphi_0; 1-s_1, 1-s_2).$$

These estimates and inequalities (2.5), (2.7) , we obtain

$$|\rho(x_1,x_2)f(x_1,x_2)| \le C_{12}\frac{k(x_1-a_1)k(x_2-a_2)}{(x_1-a_1)^{-1}(x_2-a_2)^{-1}}\int\limits_0^1\int\limits_0^1 \frac{\overset{1,1}{\omega}(\varphi_0; 1-s_1, 1-s_2)ds_1ds_2}{s_1^{\lambda_1}s_2^{\lambda_2}(1-s_1)^{\mu_1}(1-s_2)^{\mu_2}} =$$

$$C_{12}(x_1-a_1)k(x_1-a_1)(x_2-a_=2)k(x_2-a_2).$$

If $x_i \to a_i, i = 1,2$, then $\rho(x_1,x_2)f(x_1,x_2) \to 0$. From this we obtain $\rho(x_1,x_2)\varphi(x_1,x_2)\Big|_{x_i=a_i} = 0, i = 1,2$. So

$$\|\rho f\|_{\mathbf{C}(Q)} \le C_{12}\|\varphi\|_{\widetilde{H}_0^{\overline{\omega}}(\rho)}. \tag{4.4}$$

Inequalities (4.1), (4.2), (4.3), and (4.4) imply the assertion of the theorem. □

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