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DECOMPOSITION OF $(\alpha\text{-}\mathcal{H}_\sigma, \lambda)$-CONTINUITY

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Abstract. In this new research paper we introduce and investigate the new kind of open sets α-$\mathcal{H}_\sigma$-open, σ-$\mathcal{H}_\sigma$-open, π-$\mathcal{H}_\sigma$-open, β-$\mathcal{H}_\sigma$-open sets in hereditary generalized topological spaces. Also, we obtained a decomposition of $(\alpha\text{-}\mathcal{H}_\sigma, \lambda)$-continuity and decompositions of (μ, λ)-continuity.

1. Introduction and Preliminaries

In the year 2002, Cs*á*sz*á*r [5] introduced very usefull notions of generalized topology and generalized continuity. Consider Z be a nonempty set and μ be a collection from the subsets of Z. Then μ is called a *generalized topology* (briefly GT) if $\emptyset \in \mu$ and an arbitrary union of elements from μ belongs to μ. Let μ be a generalized topology on Z, the elements of μ are called μ-open sets and the complement of μ-open sets are called μ-closed sets. The generalized-closure of a subset A of X, denoted by $c_\mu(A)$, is the intersection of all μ-closed sets containing A and the interior of A, denoted by $i_\mu(A)$, is the union of all μ-open sets contained in A. A subset L of a space (Z, μ) is called asμ-α-*open* [6] (resp. μ-σ-open [6], μ-π-open [6], μ-β-open [6]) if $L \subset i_\mu c_\mu i_\mu(L)$ (resp. $L \subset c_\mu i_\mu(L)$, $L \subset i_\mu c_\mu(L)$, $L \subset c_\mu i_\mu c_\mu(L)$). Let Z be a space. Then $\mu(x) = \{U : x \in U \in \mu\}$. A space Z is called a C_0-space [14], if $C_0 = Z$, where C_0 is the set of all representative elements of sets of μ and x is called a represent element of $u \in \mu$ if $u \subset v$ for each $v \in \mu(x)$. A nonempty family $\mathcal{H}$ of subsets of Z is called as a *hereditary class* [7], if $L \in \mathcal{H}$ and $B \subset L$, then $B \in \mathcal{H}$. For each $L \subseteq Z$, $L^*(\mathcal{H}, \mu)$=$\{z \in Z : L \cap V \notin \mathcal{H}$ for all $V \in \mu$ such that $z \in V\}$[7]. For $L \subset Z$, define $c^*_\mu(L) = L \cup L^*(\mathcal{H}, \mu)$ and $\mu^* = \{L \subset Z : Z - L = c^*_\mu(Z - L)\}$. If $\mathcal{H}$ is a hereditary class on Z, then $(Z, \mu, \mathcal{H})$ is

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called a hereditary generalized topological space $H.G.T.S.$ Also papers [1, 2, 3, 4] have introduced some property related to Decomposition of continuity and hereditary class $\mathcal{H}$.

Definition 1.1. [11] Consider L be a subset of H.G.T.S. $(Z, \mu, \mathcal{H})$. Then $L^*_\sigma(\mathcal{H}, \mu)$=$\{z \in Z : L \cap V \notin \mathcal{H}$ for all V $\in \mu$-σ-open such that z $\in V\}$.

Consider $(Z, \mu, \mathcal{H})$ be a hereditary generalized topological space. For $L \subset Z$, define $c^*_\sigma(L) = L \cup L^*_\sigma(\mathcal{H}, \mu)$ [11] and $c^*_\sigma(L)$ is enlarging, monotone and idempotent.

Definition 1.2. [12] A subset L of a H.G.T.S. $(Z, \mu, \mathcal{H})$ is said to be π^*-$\mathcal{H}_\sigma$-set, if $i_\mu c^*_\sigma(L) = i_\mu(L)$.

Definition 1.3. A function $f : (X, \tau) \to (Y, \sigma)$ is called α-continuous [13] (resp. Pre-continuos [9], $\mathcal{AB}$-continuous [8]), if $f^{-1}(U)$ is α-open (resp. pre open, $\mathcal{AB}$-set) in (X, τ) for each open set in (Y, σ).

2. α-$\mathcal{H}_\sigma$-open sets

Definition 2.1. A subset L of a H.G.T.S. $(Z, \mu, \mathcal{H})$ is called as

(1) α-$\mathcal{H}_\sigma$-open, if $L \subseteq i_\mu c^*_\sigma i_\mu(L)$.
(2) σ-$\mathcal{H}_\sigma$-open, if $L \subseteq c^*_\sigma i_\mu(L)$.
(3) π-$\mathcal{H}_\sigma$-open, if $L \subseteq i_\mu c^*_\sigma(L)$.
(4) β-$\mathcal{H}_\sigma$-open, if $L \subseteq c_\mu i_\mu c^*_\sigma(L)$.

Proposition 2.2. In H.G.T.S. $(Z, \mu, \mathcal{H})$ the following hold:

(1) All μ-open set is α-$\mathcal{H}_\sigma$-open.
(2) All μ-open set is σ-$\mathcal{H}_\sigma$-open.
(3) All μ-open set is π-$\mathcal{H}_\sigma$-open.
(4) All μ-open set is β-$\mathcal{H}_\sigma$-open.

Proof. Consider a subset L of H.G.T.S. $(Z, \mu, \mathcal{H})$ is μ-open. Then $L = i_\mu(L)$. Now $L \subseteq i_\mu(L) \subseteq i_\mu c^*_\sigma(L) \subseteq i_\mu c^*_\sigma i_\mu(L)$. Hence, L is α-$\mathcal{H}_\sigma$-open.
Proof of (2), (3) and (4) are similar to proof (1). □

Remark 2.3. The converse of Proposition 2.2 need not be true from the following example.

Example 2.4. Consider $Z = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$, $\mu = \{\emptyset, \{\zeta_1\}, \{\zeta_2\}, \{\zeta_1, \zeta_2\}, \{\zeta_2, \zeta_3, \zeta_4\}, Z\}$, $\mathcal{H} = \{\emptyset, \{\zeta_1\}, \{\zeta_3\}\}$. Then $L = \{\zeta_1, \zeta_2, \zeta_3\}$ is α-$\mathcal{H}_\sigma$-open (resp. σ-$\mathcal{H}_\sigma$-open, π-$\mathcal{H}_\sigma$-open, β-$\mathcal{H}_\sigma$-open) but not μ-open.

Proposition 2.5. For a subset of a H.G.T.S. $(Z, \mu, \mathcal{H})$ the following hold:

(1) All α-$\mathcal{H}_\sigma$-open is μ-α-open.
(2) All σ-$\mathcal{H}_\sigma$-open is μ-σ-open.
(3) All π-$\mathcal{H}_\sigma$-open is μ-π-open.
(4) All β-$\mathcal{H}_\sigma$-open is μ-β-open.

Proof. (1) Consider L be an α-$\mathcal{H}_\sigma$-open set. Then, we have $L \subseteq i_\mu c^*_\sigma i_\mu(L) \subseteq i_\mu c^*_\mu i_\mu(L) \subseteq i_\mu c_\mu i_\mu(L)$. Hence, L is μ-α-open.

Proof of (2), (3) and (4) are similar to proof (1). □

Remark 2.6. The converse of Proposition 2.5 need not be true from the following examples.

Example 2.7. Consider$Z = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5\}$, $\mu = \{\emptyset, \{\zeta_1\}, \{\zeta_2\}, \{\zeta_3\}, \{\zeta_1, \zeta_2\}, \{\zeta_1, \zeta_3\}, \{\zeta_2, \zeta_3\}, \{\zeta_1, \zeta_2, \zeta_3\}, \{\zeta_1, \zeta_2, \zeta_4\}, \{\zeta_2, \zeta_3, \zeta_4\}, \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}\}$, $\mathcal{H} = \{\emptyset, \{\zeta_1\}\}$. Then $L = \{\zeta_1, \zeta_2, \zeta_4\}$ is μ-α-open (resp. μ-π-open) but not α-$\mathcal{H}_\sigma$-open (resp.π-$\mathcal{H}_\sigma$-open) and $M = \{\zeta_3, \zeta_4\}$ is μ-σ-open but not σ-$\mathcal{H}_\sigma$-open.

Example 2.8. Consider$Z = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5\}$,$\mu = \{\emptyset, \{\zeta_1\}, \{\zeta_3\}, \{\zeta_1, \zeta_2\}, \{\zeta_1, \zeta_3\}, \{\zeta_1, \zeta_2, \zeta_3\}, \{\zeta_1, \zeta_3, \zeta_4, \zeta_5\}, \{\zeta_2, \zeta_3, \zeta_4, \zeta_5\}, Z\}$, $\mathcal{H} = \{\emptyset, \{\zeta_1\}, \{\zeta_4\}\}$. Then $L = \{\zeta_1, \zeta_4\}$ is μ-β-open but not β-$\mathcal{H}_\sigma$-open.

Proposition 2.9. For a subset of a H.G.T.S. $(Z, \mu, \mathcal{H})$ the following hold:

(1) All α-$\mathcal{H}_\sigma$-open is α-$\mathcal{H}$-open.
(2) All σ-$\mathcal{H}_\sigma$-open is σ-$\mathcal{H}$-open.
(3) All π-$\mathcal{H}_\sigma$-open is π-$\mathcal{H}$-open.
(4) All β-$\mathcal{H}_\sigma$-open is β-$\mathcal{H}$-open.

Proof. (1). Consider L be an α-$\mathcal{H}_\sigma$-open set. Then, we have $L \subseteq i_\mu c^*_\sigma i_\mu(L) \subseteq i_\mu c^*_\mu i_\mu(L)$. Hence, L is α-$\mathcal{H}$-open.

Proof of (2), (3) and (4) are similar to proof (1).

□

Remark 2.10. The converse of Proposition 2.9 need not be correct from the following examples.

Example 2.11. Consider $Z = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5\}$, $\mu = \{\emptyset, \{\zeta_1\}, \{\zeta_2\}, \{\zeta_3\}, \{\zeta_1, \zeta_2\}, \{\zeta_1, \zeta_3\}, \{\zeta_2, \zeta_3\}, \{\zeta_1, \zeta_2, \zeta_3\}, \{\zeta_1, \zeta_3, \zeta_4\}, \{\zeta_2, \zeta_3, \zeta_4\}, \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}\}$,$\mathcal{H}$=$\{\emptyset, \{\zeta_1\}\}$. Then, $L = \{\zeta_2, \zeta_3, \zeta_4, \zeta_5\}$ is σ-$\mathcal{H}$-open but not σ-$\mathcal{H}_\sigma$-open.

Example 2.12. Consider $Z = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5\}$, $\mu = \{\emptyset, \{\zeta_1\}, \{\zeta_3\}, \{\zeta_1, \zeta_3\}, \{\zeta_3, \zeta_4, \zeta_5\}, \{\zeta_1, \zeta_2, \zeta_3\}, \{\zeta_1, \zeta_3, \zeta_4, \zeta_5\}, Z\}$, $\mathcal{H} = \{\emptyset, \{\zeta_1\}, \{\zeta_2\}\}$. Then, $L = \{\zeta_3, \zeta_4\}$ is π-$\mathcal{H}$-open but not π-$\mathcal{H}_\sigma$-open.

Proposition 2.13. In H.G.T.S. $(Z, \mu, \mathcal{H})$, every α-$\mathcal{H}_\sigma$-open is σ-$\mathcal{H}_\sigma$-open.

Proof. Consider a subset L of H.G.T.S. $(Z, \mu, \mathcal{H})$ is α-$\mathcal{H}_\sigma$-open. Then, $L \subseteq i_\mu c^*_\sigma i_\mu(L)$. Now, $L \subseteq i_\mu c^*_\sigma i_\mu(L) \subseteq c^*_\sigma i_\mu(L)$. Hence, L is σ-$\mathcal{H}_\sigma$-open. □

Proposition 2.14. In H.G.T.S. $(Z, \mu, \mathcal{H})$, every α-$\mathcal{H}_\sigma$-open is π-$\mathcal{H}_\sigma$-open.

Proof. Consider a subset L of H.G.T.S. $(Z, \mu, \mathcal{H})$ is α-$\mathcal{H}_\sigma$-open. Then, $L \subseteq i_\mu c^*_\sigma i_\mu(L)$. Now, $L \subseteq i_\mu c^*_\sigma i_\mu(L) \subseteq i_\mu c^*_\sigma(L)$. Hence, L is π-$\mathcal{H}_\sigma$-open. □

Theorem 2.15. *A subset L of a H.G.T.S. $(Z, \mu, \mathcal{H})$, the following results are equivalent.*

(1) L is α-$\mathcal{H}_\sigma$-open.
(2) L is σ-$\mathcal{H}_\sigma$-open and π-$\mathcal{H}_\sigma$-open.

Proof. (1) $\Rightarrow$ (2). Consider L is α-$\mathcal{H}_\sigma$-open. Then by Proposition 2.13 and 2.14, L is σ-$\mathcal{H}_\sigma$-open and π-$\mathcal{H}_\sigma$-open.
(2) $\Rightarrow$ (1). Consider L is both σ-$\mathcal{H}_\sigma$-open and π-$\mathcal{H}_\sigma$-open. Then, $L \subseteq i_\mu c^*_\sigma(L) \subseteq i_\mu c^*_\sigma c^*_\sigma i_\mu(L) \subseteq i_\mu c^*_\sigma i_\mu(L)$. Hence, L is α-$\mathcal{H}_\sigma$-open. $\square$

Remark 2.16. Converse part of the Theorem 2.15 need not be true from the following counter example.

Example 2.17. Consider $Z = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5\}$, $\mu = \{\emptyset, \{\zeta_1\}, \{\zeta_2\}, \{\zeta_3\}, \{\zeta_1, \zeta_2\}, \{\zeta_1, \zeta_3\}, \{\zeta_2, \zeta_3\}, \{\zeta_1, \zeta_2, \zeta_3\}, \{\zeta_1, \zeta_3, \zeta_4\}, \{\zeta_1, \zeta_2, \zeta_3\}, \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}\}$, $\mathcal{H} = \{\emptyset, \{\zeta_1\}\}$. Then, $L = \{\zeta_1, \zeta_5\}$ is σ-$\mathcal{H}_\sigma$-open but not α-$\mathcal{H}_\sigma$-open.

Example 2.18. Consider $Z = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$, $\mu = \{\emptyset, \{\zeta_1, \zeta_3\}, \{\zeta_4\}, \{\zeta_1, \zeta_3, \zeta_4\}, Z\}$, $\mathcal{H} = \{\emptyset, \{\zeta_3\}\}$. Then, $L = \{\zeta_1, \zeta_2, \zeta_4\}$ is π-$\mathcal{H}_\sigma$-open but not α-$\mathcal{H}_\sigma$-open.

Remark 2.19. The notions of σ-$\mathcal{H}_\sigma$-open and π-$\mathcal{H}_\sigma$-open are independent.

Example 2.20. Consider $Z = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5\}$, $\mu = \{\emptyset, \{\zeta_1\}, \{\zeta_2\}, \{\zeta_3\}, \{\zeta_1, \zeta_2\}, \{\zeta_1, \zeta_3\}, \{\zeta_2, \zeta_3\}, \{\zeta_1, \zeta_2, \zeta_3\}, \{\zeta_1, \zeta_3, \zeta_4\}, \{\zeta_2, \zeta_3, \zeta_4\}, \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}\}$, $\mathcal{H}=\{\emptyset, \{\zeta_1\}\}$. Then, $L = \{\zeta_1, \zeta_3\}$ is σ-$\mathcal{H}_\sigma$-open but not π-$\mathcal{H}_\sigma$-open.

Example 2.21. Consider $Z = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$, $\mu = \{\emptyset, \{\zeta_1, \zeta_3\}, \{\zeta_4\}, \{\zeta_1, \zeta_3, \zeta_4\}, Z\}$, $\mathcal{H} = \{\emptyset, \{\zeta_3\}\}$. Then, $L = \{\zeta_1, \zeta_3, \zeta_4\}$ is π-$\mathcal{H}_\sigma$-open but not σ-$\mathcal{H}_\sigma$-open.

Proposition 2.22. Every π-$\mathcal{H}_\sigma$-open set is β-$\mathcal{H}_\sigma$-open but not conversely.

Proof. Consider a subset L of H.G.T.S. $(Z, \mu, \mathcal{H})$ is π-$\mathcal{H}_\sigma$-open. Then, $L \subseteq i_\mu c^*_\sigma(L)$. Now, $L \subseteq i_\mu c^*_\sigma(L) \subseteq c_\mu i_\mu c^*_\sigma(L)$. Hence, L is β-$\mathcal{H}_\sigma$-open set. $\square$

For several sets defined above, we have the following implications.

$$
\begin{tikzcd}
 & \alpha\text{-}\mathcal{H}\text{-open} \arrow[rr] \arrow[dr] & & \sigma\text{-}\mathcal{H}\text{-open} \arrow[dr] & \\
 & & \pi\text{-}\mathcal{H}\text{-open} \arrow[rr] & & \beta\text{-}\mathcal{H}\text{-open} \\
\mu\text{-open} \arrow[r] & \alpha\text{-}\mathcal{H}_\sigma\text{-open} \arrow[uu] \arrow[rr] \arrow[dr] \arrow[dd] & & \sigma\text{-}\mathcal{H}_\sigma\text{-open} \arrow[uu] \arrow[dr] \arrow[dd] & \\
 & & \pi\text{-}\mathcal{H}_\sigma\text{-open} \arrow[uu] \arrow[rr] \arrow[dd] & & \beta\text{-}\mathcal{H}_\sigma\text{-open} \arrow[uu] \arrow[dd] \\
 & \mu\text{-}\alpha\text{-open} \arrow[rr] \arrow[dr] & & \mu\text{-}\sigma\text{-open} \arrow[dr] & \\
 & & \mu\text{ -}\pi\text{-open} \arrow[rr] & & \mu\text{-}\beta\text{-open}
\end{tikzcd}
$$

Theorem 2.23. *A subset of H.G.T.S.* $(Z, \mu, \mathcal{H})$ *is* σ-$\mathcal{H}_\sigma$*-open if and only if* $c^*_\sigma(L) = c^*_\sigma i_\mu(L)$.

Proof. Consider a subset L of H.G.T.S. $(Z, \mu, \mathcal{H})$ is σ-$\mathcal{H}_\sigma$-open. Then, $L \subseteq c^*_\sigma i_\mu(L)$. Now, $c^*_\sigma(L) \subseteq c^*_\sigma c^*_\sigma i_\mu(L) \subseteq c^*_\sigma i_\mu(L)$. Always $c^*_\sigma i_\mu(L) \subseteq c^*_\sigma(L)$. Hence, $c^*_\sigma(L) = c^*_\sigma i_\mu(L)$. Conversely, consider $c^*_\sigma(L) = c^*_\sigma i_\mu(L)$. Then, $L \subseteq c^*_\sigma(L) = c^*_\sigma i_\mu(L)$. Hence, L is σ-$\mathcal{H}_\sigma$-open. □

Definition 2.24. A subset L of a H.G.T.S. $(Z, \mu, \mathcal{H})$ is said to be σ-$\mathcal{H}_\sigma$-closed, if $i_\mu c^*_\sigma(L) \subseteq L$.

Definition 2.25. A subset L of a H.G.T.S. $(Z, \mu, \mathcal{H})$ is said to be $R^{*\sigma}$-set, if $A = U \cap V$, where U is μ-open and V is σ-$\mathcal{H}_\sigma$-closed.

Proposition 2.26. Every μ-open is $R^{*\sigma}$-set but not conversely.

Proof. Obvious. □

Example 2.27. Let $Z = \{a, b, c, d\}$, $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}, Z\}$ and $\mathcal{H} = \{\emptyset, \{a\}, \{c\}\}$. Then, $L = \{c\}$ is $R^{*\sigma}$-set but not μ-open.

Theorem 2.28. *Let* $(Z, \mu, \mathcal{H})$ *be a strong H.G.T.S., where* Z *is* C_0*-space and* $L \subset Z$. *Then the following conditions are equivalent.*

(1) L is μ*-open.*
(2) L is α-$\mathcal{H}_\sigma$*-open and* $R^{*\sigma}$*-set.*
(3) L is π-$\mathcal{H}_\sigma$*-open and* $R^{*\sigma}$*-set.*

Proof. (1) ⇒ (2). Let a subset L of Z is μ-open. Then, it is α-$\mathcal{H}_\sigma$-open and $R^{*\sigma}$-set by Propositions 2.2 and 2.26.

(2) ⇒ (3). Let a subset L of Z is both α-$\mathcal{H}_\sigma$-open and $R^{*\sigma}$-set. Then, it is both π-$\mathcal{H}_\sigma$-open and $R^{*\sigma}$-set by Proposition 2.14.

(3) ⇒ (1). Let a subset L of Z is both π-$\mathcal{H}_\sigma$-open and $R^{*\sigma}$-set. Then, $L \subseteq i_\mu c^*_\sigma(L)$ and $L = U \cap V$, where U is μ-open and V is σ-$\mathcal{H}_\sigma$-closed.
Now, $L \subseteq i_\mu c^*_\sigma(L) = i_\mu c^*_\sigma(U \cap V) \subseteq i_\mu(c^*_\sigma(U) \cap c^*_\sigma(V)) \subseteq i_\mu c^*_\sigma(U) \cap i_\mu c^*_\sigma(V)$.
Since V is σ-$\mathcal{H}_\sigma$-closed, $i_\mu c^*_\sigma(V) \subseteq V \Rightarrow i_\mu c^*_\sigma(V) \subseteq i_\mu(V)$.
Hence, $L \subseteq i_\mu c^*_\sigma(U) \cap i_\mu(V)$.
Now as $L \subseteq U$, we have
$$\begin{aligned} L &= U \cap L \\ &\subseteq U \cap i_\mu c^*_\sigma(U) \cap i_\mu(V) \\ &= [U \cap i_\mu c^*_\sigma(U)] \cap i_\mu(V) \\ &= U \cap i_\mu(V) \\ &= i_\mu(U) \cap i_\mu(V) \\ &= i_\mu(U \cap V) \\ &= i_\mu(L). \end{aligned}$$
Therefore, $L \subseteq i_\mu(L)$. Hence, L is μ-open. □

Remark 2.29. The notions of α-$\mathcal{H}_\sigma$-open (resp. π-$\mathcal{H}_\sigma$-open) and $R^{*\sigma}$-set are independent.

Example 2.30. Let $Z = \{a,b,c,d\}$, $\mu = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{b,c,d\}, Z\}$ and $\mathcal{H} = \{\emptyset, \{a\}, \{c\}\}$. Then, $L = \{b,c\}$ is α-$\mathcal{H}_\sigma$-open (resp. π-$\mathcal{H}_\sigma$-open) but not $R^{*\sigma}$-set and $M = \{c\}$ is $R^{*\sigma}$-set but not α-$\mathcal{H}_\sigma$-open (resp. π-$\mathcal{H}_\sigma$-open).

Definition 2.31. A subset L of a H.G.T.S. $(Z, \mu, \mathcal{H})$ is said to be σ-$\mathcal{H}_\sigma$-regular, if L is both π^*-$\mathcal{H}_\sigma$-set and σ-$\mathcal{H}_\sigma$-open set.

Definition 2.32. A subset L of a H.G.T.S. $(Z, \mu, \mathcal{H})$ is said to be σ-$AR_{\mathcal{H}}$-set, if $A = U \cap V$, where U is μ-open and V is σ-$\mathcal{H}_\sigma$-regular set.

Proposition 2.33. Every μ-open is σ-$AR_{\mathcal{H}}$-set but not conversely.

Proof. Obvious. □

Example 2.34. Let $Z = \{a,b,c,d,e\}$, $\mu = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \{a,c,d\}, \{b,c,d\}, \{a,b,c,d\}\}$ and $\mathcal{H} = \{\emptyset, \{a\}\}$. Then, $L = \{c,d\}$ is σ-$AR_{\mathcal{H}}$-set but not μ-open.

Theorem 2.35. *Let $(Z, \mu, \mathcal{H})$ be a strong H.G.T.S., where Z is C_0-space and $L \subset Z$. Then the following conditions are equivalent.*

(1) L is μ-open.
(2) L is α-$\mathcal{H}_\sigma$-open and σ-$AR_{\mathcal{H}}$-set.
(3) L is π-$\mathcal{H}_\sigma$-open and σ-$AR_{\mathcal{H}}$-set.

Proof. (1) $\Rightarrow$ (2). Let a subset L of Z is μ-open. Then, it is α-$\mathcal{H}_\sigma$-open and σ-$AR_{\mathcal{H}}$-set by Propositions 2.2 and 2.33.

(2) $\Rightarrow$ (3). Let a subset L of Z is both α-$\mathcal{H}_\sigma$-open and σ-$AR_{\mathcal{H}}$-set. Then, it is both π-$\mathcal{H}_\sigma$-open and σ-$AR_{\mathcal{H}}$-set by Proposition 2.14.

(3) $\Rightarrow$ (1). Let a subset L of Z is both π-$\mathcal{H}_\sigma$-open and σ-$AR_{\mathcal{H}}$-set. Then, $L \subseteq i_\mu c^*_\sigma(L)$ and $L = U \cap V$, where U is μ-open and V is σ-$\mathcal{H}_\sigma$-regular set.
Now, $L \subseteq i_\mu c^*_\sigma(L) = i_\mu c^*_\sigma(U \cap V) \subseteq i_\mu(c^*_\sigma(U) \cap c^*_\sigma(V)) \subseteq i_\mu c^*_\sigma(U) \cap i_\mu c^*_\sigma(V)$.
Since V is σ-$\mathcal{H}_\sigma$-regular set, so $i_\mu c^*_\sigma(V) = i_\mu(V)$.
Hence, $L \subseteq i_\mu c^*_\sigma(U) \cap i_\mu(V)$.
Now, as $L \subseteq U$, we have
$$\begin{aligned} L &= U \cap L \\ &\subseteq U \cap i_\mu c^*_\sigma(U) \cap i_\mu(V) \\ &= [U \cap i_\mu c^*_\sigma(U)] \cap i_\mu(V) \\ &= U \cap i_\mu(V) \\ &= i_\mu(U) \cap i_\mu(V) \\ &= i_\mu(U \cap V) \\ &= i_\mu(L). \end{aligned}$$
Therefore, $L \subseteq i_\mu(L)$. Hence, L is μ-open. □

Remark 2.36. The notions of α-$\mathcal{H}_\sigma$-open (resp. π-$\mathcal{H}_\sigma$-open) and σ-$AR_{\mathcal{H}}$-set are independent.

Example 2.37. Let $Z = \{a, b, c, d\}$, $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}, Z\}$ and $\mathcal{H} = \{\emptyset, \{a\}, \{c\}\}$. Then $L = \{b, c\}$ is α-$\mathcal{H}_\sigma$-open (resp. π-$\mathcal{H}_\sigma$-open) but not σ-$AR_{\mathcal{H}}$-set.

Example 2.38. Let $Z = \{a, b, c, d, e\}$, $\mu = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\}$ and $\mathcal{H} = \{\emptyset, \{a\}\}$. Then $L = \{c, d\}$ is σ-$AR_{\mathcal{H}}$-set but not α-$\mathcal{H}_\sigma$-open (resp. π-$\mathcal{H}_\sigma$-open).

3. Decomposition of $(\alpha\text{-}\mathcal{H}_\sigma,\lambda)$-continuity

Definition 3.1. A map $j : (Z, \mu, \mathcal{H}) \to (W, \lambda)$ is $(\alpha\text{-}\mathcal{H}_\sigma, \lambda)$-continuity (resp.$(\sigma\text{-}\mathcal{H}_\sigma, \lambda)$-continuity, $(\pi\text{-}\mathcal{H}_\sigma, \lambda)$-continuity, $(R^{*\sigma}, \lambda)$-continuity, $(\sigma\text{-}AR_{\mathcal{H}}, \lambda)$-continuity), if $j^{-1}(V)$ is α-$\mathcal{H}_\sigma$-open (resp. σ-$\mathcal{H}_\sigma$-open, π-$\mathcal{H}_\sigma$-open, $R^{*\sigma}$-set, σ-$AR_{\mathcal{H}}$-set) for each λ-open set V in $(W, \lambda, \mathcal{H})$.

Proposition 3.2. Every $(\alpha\text{-}\mathcal{H}_\sigma, \lambda)$-continuity is $(\sigma\text{-}\mathcal{H}_\sigma, \lambda)$-continuity but not conversely.

Proof. Consider the map $j : (Z, \mu, \mathcal{H}) \to (W, \lambda)$ be a $(\alpha\text{-}\mathcal{H}_\sigma, \lambda)$-continuity. Then $j^{-1}(V)$ is α-$\mathcal{H}_\sigma$-open for each λ-open set V in $(W, \lambda, \mathcal{H})$. By Proposition 2.13, $j^{-1}(V)$ is σ-$\mathcal{H}_\sigma$-open. Hence, j is $(\sigma\text{-}\mathcal{H}_\sigma, \lambda)$-continuity. □

Example 3.3. Consider $Z = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5\}$, $\mu = \{\emptyset, \{\zeta_1\}, \{\zeta_2\}, \{\zeta_3\}, \{\zeta_1, \zeta_2\}, \{\zeta_1, \zeta_3\}, \{\zeta_2, \zeta_3\}, \{\zeta_1, \zeta_2, \zeta_3\}, \{\zeta_1, \zeta_3, \zeta_4\}, \{\zeta_2, \zeta_3, \zeta_4\}, \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}\}$, $\lambda = \{\emptyset, \{\zeta_1, \zeta_2\}\}$ and $\mathcal{H} = \{\emptyset, \{\zeta_1\}\}$. Then, the identy map $j : (Z, \mu, \mathcal{H}) \to (W, \lambda)$ is $(\sigma\text{-}\mathcal{H}_\sigma, \lambda)$-continuity and but not $(\alpha\text{-}\mathcal{H}_\sigma, \lambda)$-continuity.

Proposition 3.4. Every $(\alpha\text{-}\mathcal{H}_\sigma, \lambda)$-continuity is $(\pi\text{-}\mathcal{H}_\sigma, \lambda)$-continuity but not conversely.

Proof. Consider the map $j : (Z, \mu, \mathcal{H}) \to (W, \lambda, \mathcal{H})$ be a $(\alpha\text{-}\mathcal{H}_\sigma, \lambda)$-continuity. Then $j^{-1}(V)$ is α-$\mathcal{H}_\sigma$-open for each λ-open set V in $(W, \lambda, \mathcal{H})$. By Proposition 2.14, $j^{-1}(V)$ is π-$\mathcal{H}_\sigma$-open. Hence, j is $(\pi\text{-}\mathcal{H}_\sigma, \lambda)$-continuity. □

Example 3.5. Consider $Z = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$, $\mu = \{\emptyset, \{\zeta_1, \zeta_3\}, \{\zeta_4\}, \{\zeta_1, \zeta_3, \zeta_4\}, Z\}$, $\lambda = \{\emptyset, \{\zeta_1, \zeta_3, \zeta_4\}\}$,$\mathcal{H} = \{\emptyset, \{\zeta_3\}\}$. Then the identy map $j : (Z, \mu, \mathcal{H}) \to (W, \lambda)$ is $(\pi\text{-}\mathcal{H}_\sigma, \lambda)$-continuity but not $(\alpha\text{-}\mathcal{H}_\sigma, \lambda)$-continuity.

Remark 3.6. The notions of $(\sigma\text{-}\mathcal{H}_\sigma, \lambda)$-continuity and $(\pi\text{-}\mathcal{H}_\sigma, \lambda)$-continuity are independent.

Example 3.7. Consider $Z = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5\}$, $\mu = \{\emptyset, \{\zeta_1\}, \{\zeta_2\}, \{\zeta_3\}, \{\zeta_1, \zeta_2\}, \{\zeta_1, \zeta_3\}, \{\zeta_2, \zeta_3\}, \{\zeta_1, \zeta_2, \zeta_3\}, \{\zeta_1, \zeta_3, \zeta_4\}, \{\zeta_2, \zeta_3, \zeta_4\}, \{\zeta_2, \zeta_3, \zeta_4, \zeta_5\}\}$, $\lambda = \{\emptyset, \{\zeta_2, \zeta_3\}\}$ and $\mathcal{H} = \{\emptyset, \{\zeta_5\}\}$ Then, the identy map $j : (Z, \mu, \mathcal{H}) \to (W, \lambda)$ is $(\sigma\text{-}\mathcal{H}_\sigma, \lambda)$-continuity but not $(\pi\text{-}\mathcal{H}_\sigma, \lambda)$-continuity.

Example 3.8. Consider $Z = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$, μ=$\{\emptyset, \{\zeta_1, \zeta_3\}, \{\zeta_3\}, \{\zeta_1, \zeta_3, \zeta_4\}, Z\}$, $\lambda = \{\emptyset, \{\zeta_1, \zeta_3, \zeta_4\}\}$, $\mathcal{H} = \{\emptyset, \{\zeta_3\}\}$. Then, the identy map $j : (Z, \mu, \mathcal{H})\to (W, \lambda)$ is $(\pi\text{-}\mathcal{H}_\sigma, \lambda)$-continuity but not $(\sigma\text{-}\mathcal{H}_\sigma, \lambda)$-continuity.

Theorem 3.9. *For a map $j : (Z, \mu, \mathcal{H}) \to (W, \lambda, \mathcal{H})$, the following results are equivalent.*

(1) j is $(\alpha\text{-}\mathcal{H}_\sigma, \lambda)$-continuity.
(2) j is $(\sigma\text{-}\mathcal{H}_\sigma, \lambda)$-continuity and $(\pi\text{-}\mathcal{H}_\sigma, \lambda)$-continuity.

Proof. (1) $\Rightarrow$ (2) : Follows from the Proposition 3.2 and Proposition 3.4.

(2) $\Rightarrow$ (1) : Consider the map $j : (Z, \mu, \mathcal{H}) \to (W, \lambda, \mathcal{H})$ be $(\sigma\text{-}\mathcal{H}_\sigma, \lambda)$-continuity and $(\pi\text{-}\mathcal{H}_\sigma, \lambda)$-continuity. Then $j^{-1}(V)$ is σ-$\mathcal{H}_\sigma$-open and π-$\mathcal{H}_\sigma$-open for each λ-open. By Theorem 2.15, $j^{-1}(V)$ is α-$\mathcal{H}_\sigma$-open. Hence, j is $(\alpha\text{-}\mathcal{H}_\sigma, \lambda)$-continuity. □

Theorem 3.10. *For a map $j : (Z, \mu, \mathcal{H}) \to (W, \lambda, \mathcal{H})$ where Z is C_0-space, the following results are equivalent.*

(1) j is (μ, λ)-continuity.
(2) j is $(\alpha\text{-}\mathcal{H}_\sigma, \lambda)$-continuity and $(R^{*\sigma}, \lambda)$-continuity.
(3) j is $(\pi\text{-}\mathcal{H}_\sigma, \lambda)$-continuity and $(R^{*\sigma}, \lambda)$-continuity.

Proof. Proof is trivial from Theorem 2.28. □

Theorem 3.11. *For a map $j : (Z, \mu, \mathcal{H}) \to (W, \lambda, \mathcal{H})$ where Z is C_0-space, the following results are equivalent.*

(1) j is (μ, λ)-continuity.
(2) j is $(\alpha\text{-}\mathcal{H}_\sigma, \lambda)$-continuity and $(\sigma\text{-}AR_{\mathcal{H}}, \lambda)$-continuity.
(3) j is $(\pi\text{-}\mathcal{H}_\sigma, \lambda)$-continuity and $(\sigma\text{-}AR_{\mathcal{H}}, \lambda)$-continuity.

Proof. Proof is trivial from Theorem 2.35. □

If $\mathcal{H} = \emptyset$, and $\mu = \sigma = \tau$, then we obtain the following corollary.

Corollary 3.12. For a map $j : (Z, \mu, \mathcal{H}) \to (W, \lambda, \mathcal{H})$ the following results are equivalent.

(1) j is continuous.
(2) j is α-continuous and AB-continuous.
(3) j is pre-continuous and AB-continuous.

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