Poincare Journal of Analysis & Applications
Vol. 10, No. 1 (2023), 165-174

DOI: 10.46753/pjaa.2023.v010i01.012

SOLUTION OF FRACTIONAL INTEGRAL EQUATION VIA HYBRID CONTRACTIONS IN METRIC-LIKE SPACES

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Date of Receiving : 28. 12. 2022
Date of Revision : 15. 04. 2023
Date of Acceptance : 12. 06. 2023

Abstract. In this manuscript, we shall introduce hybrid contractions and prove fixed point theorems for such contractions in metric-like spaces. Some existing results of literature shall also be deduced from our main results. We shall provide an example to support the main result. In the end, a fractional integral equation shall also be solved using our main result.

1. Introduction

Since the last few decades solving fractional differential and integral equation in non linear functional analysis is the major topic of research. Fixed point technique is one of the technique to obtain the solution of such fractional equations. In 1922, Banach [14] was the first Mathematician, who gave the constructive method to get the fixed point. Later by seeing the usability of fixed point results Kannan ([11]-[12]) in 1968 tried to generalize the Banach contraction principle. In this sequence, in 2018, Karapinar [3] introduce the new notion of interpolation by revisiting Kannan contraction in metric space as follows:
Let (X, d) be a metric space and T be a self map on X. Then T is said to be interpolative Kannan type contraction, if there exists $\lambda \in (0, 1)$ such that

$$d(Tx, Ty) \leq \lambda[d(x, Tx)]^{\alpha}.[d(y, Ty)]^{1-\alpha}$$

for all $x, y \in X$.
In 2019, Karapinar and Fulga [4] defined hybrid contraction using interpolation in b-metric space. In the same year, Karapinar *et al.* [7] solved a Volterra fractional integral equation by defining hybrid contractions in metric spaces. For more results via hybrid and iterpolative contractions see([2], [6], [8], [9]). In this paper we shall also find the solution of Volterra fractional integral equation by defining hybrid contractions in metric

2010 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. Integral equation, hybrid contraction, metric-like spaces, fixed points.
Communicated by. Erdal Karapinar
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like spaces.

Definition 1.1. Let Φ be the set of all non-decreasing functions $\theta : [0,\infty) \to [0,\infty)$ such that there are $k_0 \in \mathbb{N}$ and $\delta \in (0,1)$ and a convergent series $\sum_{i=1}^{\infty} v_i$ such that $v_i \geq 0$ and

$$\theta^{(i+1)}(t) \leq \delta\theta^k(t) + v_i, \tag{1.1}$$

for $i \geq i_0$ and $t \geq 0$.

Each $\theta \in \Phi$ is called a (c) - comparison function. (see [10], [13]).
The following lemma demonstrate the usability and the power of such auxiliary functions:

Lemma 1.2. [13] *If $\theta \in \Phi$, then*

(1) The series $\sum_{i=1}^{\infty} \theta^k(\sigma)$ is convergent for $\sigma \geq 0$;
(2) $\{\theta^n(\sigma)\}_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$ for $\sigma \geq 0$;
(3) θ continuous;
(4) $\theta(\sigma) < \sigma$ for any $\sigma \in (0,\infty)$.

Definition 1.3. [1] Let X be a non-empty set. A function $d : X \times X \to [0,\infty)$ is said to be dislocated (metric-like) metric on X if for any $x, y, z \in X$, the following conditions hold:

(1) $d(x,y) = d(x,x) = 0 \Rightarrow x = y$;
(2) $d(x,y) = d(y,x)$;
(3) $d(x,y) \leq d(x,z) + d(z,y)$.

The pair (X,d) is then called dislocated (metric-like) metric space.

Definition 1.4. [1] Let (X,d) be a metric-like space.

(1) A sequence $\{x_n\}$ in X is a Cauchy sequence if $\lim_{n,m\to\infty} d(x_n,x_m)$ exists and is finite.
(2) (X,d) is complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$; that is,

$$\lim_{n\to\infty} d(x,x_n) = d(x,x) = \lim_{n,m\to\infty} d(x_n,x_m) = 0.$$

Definition 1.5. [1] Let (X,d) be a metric-like space. A mapping $T : (X,d) \to (X,d)$ is continuous if for any sequence $\{x_n\}$ in X such that $d(x_n,x) \to d(x,x)$ as $n \to \infty$, we have $d(Tx_n,Tx) \to d(Tx,Tx)$ as $n \to \infty$.

Definition 1.6. [5] Let (X,d) be a metric-like space. Let $\{x_n\}$ be a sequence in X such that $x_n \to x$ where $x \in X$ and $d(x,x) = 0$. Then, for all $y \in X$, we have $\lim_{n\to\infty} d(x_n,y) = d(x,x)$.

2. Main Results

In this section we shall define the new hybrid contractions and prove fixed point theorems for such contractions.

Definition 2.1. Let (X, d) be a metric-like space. A mapping $T : X \to X$ is called a hybrid contraction of type A, if there is θ in Φ such that

$$d(Tx, Ty) \leq \theta(G_T^s(x, y)) \tag{2.1}$$

where $s \geq 0$ and $\sigma_i \geq 0$, $i = 1, 2, 3, 4$ such that $\sum_{i=1}^{4} \sigma_i = 1$ and

$$G_T^s(x,y) = \begin{cases} [\sigma_1(d(x,y))^s + \sigma_2(d(x,Tx))^s + \sigma_3(d(y,Ty))^s + \sigma_4(\frac{d(x,Tx)+d(y,Ty)}{2})^s]^{\frac{1}{s}}, \\ \hfill \text{if } s > 0, x, y \in X \\ (d(x,y))^{\sigma_1}(d(x,Tx))^{\sigma_2}(d(y,Ty))^{\sigma_3}(\frac{d(x,Tx)+d(y,Ty)}{2})^{1-\sigma_1-\sigma_2-\sigma_3}, \\ \hfill \text{if } s = 0, x, y \in X - F_T(X). \end{cases}$$

where $F_T(X) = \{v \in X : Tv = v\}$.

Theorem 2.2. *Let (X, d) be a complete metric-like space and a self map T on X is a hybrid contraction of type A. Then, for any $v_0 \in X$, the sequence $\{T^n v_0\}$ converges to v if either*
(C_1) T is continuous at v;
or
(C_2) $(\sigma_3 + \frac{\sigma_4}{2})^{\frac{1}{s}} < 1$;
or
(C_3) $(\sigma_2 + \frac{\sigma_4}{2})^{\frac{1}{s}} < 1$.
Then T possesses a fixed point v.

Proof. Let $\{v_n\}$ be a sequence defined by the recursive relation $v_{n+1} = Tv_n, n \geq 0$, by taking an arbitrary point $v_0 \in X$. Hereafter, we shall assume that $v_n \neq v_{n+1} \Leftrightarrow d(v_n, v_{n+1}) > 0$ for all $n \in \mathbb{N}$.
Indeed, it is easy that the converse is trivial and terminate the proof. More precisely, if there is n_0 so that $v_{n_0} = v_{n_0+1} = Tv_{n_0}$, then v_{n_0} turns to be a fixed point of T.
Now, we shall examine the cases $s = 0$ and $s > 0$, separately.
We first consider the case $s > 0$.
In equation (2.1), taking $x = v_n$ and $y = v_{n-1}$, we obtain

$$d(v_{n+1}, v_n) \leq \theta(G_T^s(v_n, v_{n-1})) \tag{2.2}$$

where

$$\begin{aligned} G_T^s(v_n, v_{n-1}) &= [\sigma_1(d(v_n, v_{n-1}))^s + \sigma_2(d(v_n, Tv_n))^s + \sigma_3(d(v_{n-1}, Tv_{n-1}))^s \\ &+ \sigma_4(\frac{d(v_n, Tv_n) + d(v_{n-1}, Tv_{n-1})}{2})^s]^{\frac{1}{s}} \\ &= [\sigma_1(d(v_n, v_{n-1}))^s + \sigma_2(d(v_n, v_{n+1}))^s + \sigma_3(d(v_{n-1}, v_n))^s \\ &+ \sigma_4(\frac{d(v_n, v_{n+1}) + d(v_{n-1}, v_n)}{2})^s]^{\frac{1}{s}}. \end{aligned}$$

Suppose if possible that $d(v_n, v_{n+1}) \geq d(v_{n-1}, v_n)$, then by using $\sum_{i=1}^{4} \sigma_i = 1$, we get

$$G_T^s(v_n, v_{n-1}) \leq d(v_{n+1}, v_n)$$

Using this in equation (2.2), we have

$$d(v_{n+1}, v_n) \leq \theta(G_T^s(v_n, v_{n-1})) < d(v_{n+1}, v_n)$$

a contradiction.
So, $d(v_n, v_{n+1}) \leq d(v_{n-1}, v_n)$ and further

$$d(v_{n+1}, v_n) \leq \theta(d(v_n, v_{n-1})) < d(v_{n-1}, v_n).$$

Inductively, from the above inequalities, we deduce

$$d(v_{n+1}, v_n) \leq \theta^n(d(v_1, v_0)), \tag{2.3}$$

for all $n \in \mathbb{N}$.
From the equation (2.3) and the triangle inequality for all $k \geq 1$, we have

$$\begin{aligned} d(v_n, v_{n+k}) &\leq d(v_n, v_{n+1}) + ...d(v_{n+k-1}, v_{n+k}) \\ &\leq \sum_{r=n}^{n+k-1} \theta^r(d(v_1, v_0)) \\ &\leq \sum_{r=n}^{\infty} \theta^r(d(v_1, v_0)). \end{aligned}$$

Taking $n \to \infty$ in the above inequality, we have
$d(v_n, v_{n+k}) \to 0$.
Thus the constructive sequence $\{v_n\}$ is a Cauchy sequence in (X, d). Since, (X, d) is complete, we conclude that $\{v_n\}$ is a convergent sequence in (X, d), so there exist $v \in X$ such that

$$\lim_{n\to\infty} d(v_n, v) = d(v, v) = \lim_{n,m\to\infty} d(v_n, v_m) = 0. \tag{2.4}$$

Since T is continuous, from equation (2.4) we obtain that

$$\lim_{n\to\infty} d(v_{n+1}, Tv) = d(Tv, Tv) = \lim_{n\to\infty} d(Tv_n, Tv) = 0. \tag{2.5}$$

On the other hand, by equation (2.4) and Lemma 1.6

$$\lim_{n\to\infty} d(v_{n+1}, Tv) = d(v, Tv). \tag{2.6}$$

Comparing equations (2.5) and (2.6), we get

$$d(v, Tv) = d(Tv, Tv). \tag{2.7}$$

Now by using equations (2.1)and (2.7), we have

$$d(v, Tv) = d(Tv, Tv) \leq \theta(G_T^s(v, v)) \tag{2.8}$$

where

$$G_T^s(v, v) = [\sigma_1(d(v, v))^s + \sigma_2(d(v, Tv))^s + \sigma_3(d(v, Tv))^s + \sigma_4(d(v, Tv))^s]^{\frac{1}{s}}.$$

Now, either $\sigma_1 = 0$ or $\sigma_1 \neq 0$.
Case 1. If $\sigma_1 = 0$ then $\sigma_2 + \sigma_3 + \sigma_4 = 1$
So, $G_T^s(v, v) = d(v, Tv)$.
Now equation (2.8) implies that
$d(v, Tv) = d(Tv, Tv) \leq \theta(G_T^s(v, v)) = \theta(d(v, Tv)) < d(v, Tv)$, which holds when $d(v, Tv) = 0$.
So, v is the fixed point of T.
Case 2. If $\sigma_1 \neq 0$ then $\sigma_2 + \sigma_3 + \sigma_4 < 1$, so
$G_T^s(v, v) < d(v, Tv)$ and equation (2.8) implies that
$d(v, Tv) = d(Tv, Tv) \leq \theta(G_T^s(v, v)) = \theta(d(v, Tv)) < d(v, Tv)$, which holds when

$d(v, Tv) = 0$.
So, v is the fixed point of T.
Suppose that $d(v, Tv) > 0$ and condition (C_2) holds. Using equation (2.1), we get

$$\begin{aligned} d(v, Tv) &\leq d(v, v_{n+1}) + d(v_{n+1}, Tv) \\ &\leq d(v, v_{n+1}) + \theta(G_T^s(v_n, v)) \end{aligned}$$

where

$$G_T^s(v_n, v) =$$

$$[\sigma_1(d(v_n, v))^{\frac{1}{s}} + \sigma_2(d(v_n, v_{n+1}))^{\frac{1}{s}} + \sigma_3(d(v, Tv))^{\frac{1}{s}} + \sigma_4(\frac{d(v_n, v_{n+1}) + d(v, Tv)}{2})^s]^{\frac{1}{s}}$$

Taking limit as $n \to \infty$, we have
$d(v, Tv) \leq \mu d(v, Tv)$
where
$\mu = (\sigma_3 + \frac{\sigma_4}{2})^{\frac{1}{s}}$.
Since $\mu < 1$, a contradiction, that is
$v = Tv$.
The proof for condition (C_3) is same like (C_2).
As a last step we consider the case $s = 0$. Here the equation (2.1) becomes

$$d(Tx, Ty) \leq \theta((d(x, y))^{\sigma_1}(d(x, Tx))^{\sigma_2}(d(y, Ty))^{\sigma_3}(\frac{d(x, Tx) + d(y, Ty)}{2})^{1-\sigma_1-\sigma_2-\sigma_3}) \tag{2.9}$$

For all $x, y \in X - F_T(X)$ and $\sigma_1, \sigma_2, \sigma_3 \in (0, 1)$.
Set $x = \zeta_n, y = \zeta_{n-1}$ in the inequality (2.9), we find that

$$\begin{aligned} d(\zeta_{n+1}, \zeta_n) = d(T\zeta_n, T\zeta_{n-1}) &\leq \theta((d(\zeta_n, \zeta_{n-1}))^{\sigma_1}(d(\zeta_n, T\zeta_n))^{\sigma_2}(d(\zeta_{n-1}, T\zeta_{n-1}))^{\sigma_3} \\ &\quad . \ (\frac{d(\zeta_n, T\zeta_n) + d(\zeta_{n-1}, T\zeta_{n-1})}{2})^{1-\sigma_1-\sigma_2-\sigma_3}) \\ &\leq \theta((d(\zeta_n, \zeta_{n-1}))^{\sigma_1}(d(\zeta_n, \zeta_{n+1}))^{\sigma_2}(d(\zeta_{n-1}, \zeta_n))^{\sigma_3} \\ &\quad . \ (\frac{d(\zeta_n, \zeta_{n+1}) + d(\zeta_{n-1}, \zeta_n)}{2})^{1-\sigma_1-\sigma_2-\sigma_3}) \end{aligned} \tag{2.10}$$

Suppose that $d(\zeta_{n-1}, \zeta_n) < d(\zeta_n, \zeta_{n+1})$, then equation (2.10) implies that

$$[d(\zeta_n, \zeta_{n+1})]^{\sigma_1+\sigma_3} \leq \theta([d(\zeta_n, \zeta_{n-1})]^{\sigma_1+\sigma_3}) < [d(\zeta_n, \zeta_{n-1})]^{\sigma_1+\sigma_3}. \tag{2.11}$$

Thus we conclude that $d(\zeta_{n+1}, \zeta_n) \leq d(\zeta_n, \zeta_{n-1})$.
which is a contradiction. Thus, we have

$$d(\zeta_{n+1}, \zeta_n) \leq d(\zeta_n, \zeta_{n-1})$$

for all $n \in \mathbb{N}$.
Hence $\{d(\zeta_n, \zeta_{n-1})\}$ is a monotonically decreasing sequence of positive terms.
Together with simple elimination, using mathematical induction, the inequality (2.10) implies that

$$d(\zeta_{n+1}, \zeta_n) \leq \theta^n(d(\zeta_1, \zeta_0)). \tag{2.12}$$

Since the inequality (2.12) is equivalent to equation (2.3), by following corresponding lines, we derive that the iterated sequence $\{\zeta_n\}$ is Cauchy and hence converges to ζ^* in X, that is

$$\lim_{n\to\infty} d(\zeta_n, \zeta^*) = d(\zeta^*, \zeta^*) = \lim_{n,m\to\infty} d(\zeta_n, \zeta_m) = 0. \tag{2.13}$$

Since T is continuous, from equation (2.4), we obtain that

$$\lim_{n\to\infty} d(\zeta_{n+1}, T\zeta^*) = d(T\zeta^*, T\zeta^*) = \lim_{n\to\infty} d(T\zeta_n, T\zeta^*) = 0. \tag{2.14}$$

On the other hand, by equation (2.13) and Lemma 1.6

$$\lim_{n\to\infty} d(\zeta_{n+1}, T\zeta^*) = d(\zeta^*, T\zeta^*). \tag{2.15}$$

Comparing equations (2.14) and (2.15), we have

$$d(\zeta^*, T\zeta^*) = d(T\zeta^*, T\zeta^*). \tag{2.16}$$

From equations (2.9) and (2.16), we get

$$(\zeta^*, T\zeta^*) =$$
$$d(T\zeta^*, T\zeta^*) \leq \theta((d(\zeta^*, \zeta^*))^{\sigma_1}(d(\zeta^*, T\zeta^*))^{\sigma_2}(d(\zeta^*, T\zeta^*))^{\sigma_3}(d(\zeta^*, T\zeta^*))^{1-\sigma_1-\sigma_2-\sigma_3})$$

From equation (2.13), it is clear that
$d(\zeta^*, T\zeta^*) = 0$ that is $\zeta^* = T\zeta^*$.
Similarly one can prove for the conditions (C_2) and (C_3) easily. □

Definition 2.3. Let (X, d) be a metric-like space. A self mapping T on X is called a hybrid contraction of type B, if there is $\theta \in \Phi$ such that

$$d(Tx, Ty) \leq \theta(W_T^s(x, y)) \tag{2.17}$$

where $s \geq 0$ and $\sigma_i \geq 0$, $i = 1, 2, 3$ such that $\sum_{i=1}^{3} \sigma_i = 1$ and

$$W_T^s(x,y) = \begin{cases} [\sigma_1(d(x,y))^s + \sigma_2(d(x,Tx))^s + \sigma_3(d(y,Ty))^s]^{\frac{1}{s}}, & \text{if } s > 0, x, y \in X \\ (d(x,y))^{\sigma_1}(d(x,Tx))^{\sigma_2}(d(y,Ty))^{\sigma_3}, & \text{if } s = 0, x, y \in X - F_T(X). \end{cases}$$

where $F_T(X) = \{v \in X : Tv = v\}$.

Theorem 2.4. *Let (X, d) be a complete metric like space and a self map T on X is a hybrid contraction of type B or there is $k \in [0, 1)$ such that*

$$d(Tx, Ty) \leq kW_T^s(x, y)$$

Then T possesses a fixed point u if either
(C_1) T is continuous at v;
or
(C_2) $(\sigma_3 + \frac{\sigma_4}{2})^{\frac{1}{s}} < 1$;
or
(C_3) $(\sigma_2 + \frac{\sigma_4}{2})^{\frac{1}{s}} < 1$.
Then T possesses a fixed point v.

Proof. Taking $\theta = kt$ and $\sigma_4 = 0$ in Theorem 2.2, we obtain the proof. □

Example 2.5. Let $X = \mathbb{R}$ and define $d : X \times X \to [0, \infty)$ as $d(x, y) = \frac{|x-y|+|x|+|y|}{2}$, then (X, d) is a complete metric like space.
Taking $\sigma_1 = \frac{2}{3}, \sigma_2 = \sigma_3 = \frac{1}{6}$ and $s = 1$, and
$\theta = \frac{1}{2}t$, $Tx = \frac{x}{3}$ for all $x \in \mathbb{R}$.

$$\begin{aligned} d(Tx, Ty) &= \frac{|\frac{x}{3} - \frac{y}{3}| + |\frac{x}{3}| + |\frac{y}{3}|}{2} \\ &= \frac{|x - y| + |x| + |y|}{6} \end{aligned}$$

and

$$\begin{aligned}\theta(G_T^s(x,y)) &= \frac{1}{2}[\frac{2}{3}d(x,y)+\frac{1}{6}d(x,\frac{x}{3})+\frac{1}{6}d(y,\frac{y}{3})]\\ &= \frac{|x-y|+|x|+|y|}{6}+\frac{|x-\frac{x}{3}|+|x|+|\frac{x}{3}|}{24}+\frac{|y-\frac{y}{3}|+|y|+|\frac{y}{3}|}{24}\end{aligned}$$

This implies that
$d(Tx,Ty) \leq \theta(G_T^s(x,y))$ for all $x, y \in X$.
From the above, we conclude that equation (2.1) holds. Since T is continuous, so all the conditions of Theorem 2.2 holds.
Hence T has a fixed point.
Clearly, 0 is the fixed point of T.

3. Consequences

Some existing results from literature are deduced from our main results as follows:

Corollary 3.1. Let T be a self map on a complete metric-like space (X,d). Suppose that there is $k \in [0,1)$ such that $d(Tx,Ty) \leq k(G_T^s(x,y))$ for all $x, y \in X$.
Then T possesses a fixed point u if either
(C_1) T is continuous at v;
or
(C_2) $(\sigma_3 + \frac{\sigma_4}{2})^{\frac{1}{s}} < 1$;
or
(C_3) $(\sigma_2 + \frac{\sigma_4}{2})^{\frac{1}{s}} < 1$.
Then T possesses a fixed point u.

Proof. Taking $\theta = kt$ in Theorem 2.2, we obtain the proof. □

Corollary 3.2. Let T be a self map on a complete metric-like space (X,d). Suppose that

$$d(Tx,Ty) \leq kd^{\sigma_1}(x,y)d^{\sigma_2}(x,Tx)d^{\sigma_3}(y,Ty)$$

for all $x, y \in X - F_T(X)$ and $\sigma_i \geq 0$, $i = 1,2,3$ such that $\sum_{i=1}^3 \sigma_i = 1$. Then T possess a fixed point.

Proof. Putting $s = 0$ in Theorem 2.4. □

Corollary 3.3. Let T be a self map on a complete metric-like space (X,d). Suppose that

$$d(Tx,Ty) \leq k(d(x,y)d(x,Tx)d(y,Ty))^{\frac{1}{3}}$$

Then T has a fixed point.

Proof. Taking $s = 0$ and $\sigma_1 = \sigma_2 = \sigma_3 = \frac{1}{3}$ in Theorem 2.4, we get the proof. □

4. Solution for Volterra Fractional Integral Equations

The fundamental equation for fractional quantum mechanics is fractional Schrodinger equation. It differs from standard Schrodinger equation by fractional Laplacian operator. In this section, we shall find the solution of a nonlinear Volterra fractional equation using above main result.
Set $0 < \alpha < 1$ and $\mathfrak{E} = [\delta, \delta+\mathfrak{a}]$ in $\mathbb{R}(\mathfrak{a} > 0)$. Let $X = C(\mathfrak{E}, \mathbb{R})$, the set of continuous real-valued function on $\mathfrak{E}$. Now, particularly, we consider the following nonlinear Volterra fractional integral equation (VFIE).

$$\mu(t) = \omega(t) + \frac{1}{\gamma(\mathfrak{a})}\int_{\sigma_0}^{t}(t-s)^{\mathfrak{a}-1}\lambda(s,\mu(s))ds, \tag{4.1}$$

for all $t \in \mathfrak{E}$, where γ is the gamma function, $\omega : \mathfrak{E} \to \mathbb{R}$ and $\lambda : \mathfrak{E} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.
In the following results, under some assumptions, we ensure the existence of a solution for the VFIE (4.1).

Theorem 4.1. *Suppose that*
(Q1) There is constant $M > 0$ such that

$$\lambda(t,u) \leq Mu - \omega(t)\frac{\gamma(\alpha)}{(\delta-t)^{\alpha}} \tag{4.2}$$

for all $(t,u) \in \mathfrak{E} \times \mathbb{R}$;
(Q2) Such M verify that

$$\frac{M\mathfrak{a}}{\gamma(\tau+1)} \leq 1. \tag{4.3}$$

Then the VFIE (4.1) has a solution in X.

Proof. For $\mu, \sigma \in X$, consider the metric
$d(\mu,\sigma) = \max_{t\in\mathfrak{E}}\{\mu(t), \sigma(t)\}$.
Clearly (X,d) is a complete metric like space.
Define the map T as

$$T\mu(t) = \omega(t) + \frac{1}{\gamma(\mathfrak{a})}\int_{\delta}^{t}(t-s)^{\mathfrak{a}-1}\lambda(s,\mu(s))ds \tag{4.4}$$

$t \in \mathfrak{E}$.
Clearly, T is well defined. Let $\mu, \sigma \in X$ the for each $t \in \mathfrak{E}$,

$$d(T\mu(t), T\sigma(t)) = \max_{t\in\mathfrak{E}}\{T\mu(t), T\sigma(t)\} = T\mu(t)(say) \tag{4.5}$$

Then by using condition (Q1), we have

$$\begin{aligned} T\mu(t) &= \omega(t) + \frac{1}{\gamma(\mathfrak{a})}\int_{\delta}^{t}(t-s)^{\mathfrak{a}-1}\lambda(s,\mu(s))ds \\ &\leq \omega(t) + \frac{1}{\gamma(\mathfrak{a})}\int_{\delta}^{t}(t-s)^{\mathfrak{a}-1}(Mu(s) - \omega(s)\frac{\gamma(\alpha)}{(\delta-s)^{\alpha}})ds \\ &\leq \frac{\alpha}{\gamma(\alpha+1)}M\mu(t) \\ &\leq \frac{\alpha}{\tau+1)}Md(\mu,\sigma) = \theta(d(\mu,\sigma)). \end{aligned}$$

We deduce that

$$d(T\mu, T\sigma) \leq \theta(d(\mu, \sigma)). \tag{4.6}$$

where $\theta(t) = \frac{\alpha}{\gamma(\tau)+1)}$ for $t \geq 0$. By the hypothesis (Q2) $\theta \in \Phi$.
Now

$$d(T\mu, T\sigma) \leq \theta(G_T^s(\mu, \sigma)), \tag{4.7}$$

for $s > 0$ with $\sigma_1 = 1$ and $\sigma_2 = \sigma_3 = \sigma_4 = 0$. Applying Theorem 2.2, T has a fixed point in X, so the VFIE (4.1) has a solution in X. □

5. Conclusion

In this manuscript, we have defined hybrid contractions in metric like spaces and proved fixed point theorems for these contractions. We have also provided an example to illustrate the main result. Some result are proved as a corollaries of our main result. In the end, we have also solved an Volterra fractional integral equation using our main result.

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