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EXISTENCE AND HYERS-ULAM STABILITY OF THE SOLUTIONS TO THE IMPLICIT SECOND-ORDER DIFFERENTIAL EQUATION

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Abstract. The main objective of this research is to investigate the existence and stability of solutions for implicit second order differential equations with fractional-orders integral boundary conditions. Our studies have relied on Krasnoselskii's fixed point theorem and Banach contraction principle. We provide examples to clarify the achieved conclusions.

1. Introduction

Mathematicians effectively used the tools of fractional calculus to describe real-life problems requiring highly accurate mathematical modeling. It is observed that the proposed models are more appropriate to be represented by fractional-order differential equations that yield more precise outcomes. Recently, fractional-order derivatives have been increasingly used in a diversity of scientific fields. Such as electromagnetics, electrochemistry, fluid mechanics, quantum mechanics, viscoelasticity, ecological systems, optics, and signal processing. Therefore, it is a mathematical challenge to develop and implement new methods for solving fractional-order differential equations, see [8, 10, 12, 13, 14, 15, 16, 23].
On the other hand, the stability analysis of the fractional order differential equations is critical in many applications and essential in the Ulam-Hyers-Rassias stability types and their generalizations, see [7, 18, 20].

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The nonlinear fractional differential equation

$$\begin{aligned} &D^{\zeta}\mu(\mathfrak{r}) = g\big(\mathfrak{r}, \mu(\mathfrak{r}), D^{\delta}\mu(\mathfrak{r})\big), \quad \mathfrak{r} \in (0,1), 1<\zeta\leq 2,\ 0<\delta\leq 1,\\ &\mu(0) = \mu_{\circ},\ \mu(1) = \int_0^1 h(\varsigma)\mu(\varsigma))\ d\varsigma, \end{aligned}$$

considered in [17], Murad and Hadid [21] studied the fractional differential equation's boundary value problem:

$$\begin{aligned} &D^{\zeta}\mu(\mathfrak{r}) = g\big(\mathfrak{r}, \mu(\mathfrak{r}), D^{\delta}\mu(\mathfrak{r})\big), \quad \mathfrak{r} \in (0,1), 1<\zeta\leq 2,\ 0<\delta\leq 1,\\ &\mu(0) = 0,\ \mu(1) = I_0^{\gamma}\mu(\mathfrak{r}). \end{aligned}$$

Chasreechai and Tariboon in [11] demonstrated existence theorems for the positive solutions of the following types of problems

$$\begin{aligned} &\mu''(\mathfrak{r}) + \chi g(\mathfrak{r})\ h(\mu(\mathfrak{r})) = 0, \quad \mathfrak{r} \in (0,1),\\ &\mu(0) = \alpha_1 \int_0^{\eta} \mu(\varsigma)\ d\varsigma, \quad \mu(1) = \alpha_2 \int_0^{\eta} \mu(\varsigma)\ d\varsigma. \end{aligned}$$

Motivated by the mentioned results in the above articles, here we develop existence and stability theorem for the implicit second-order fractional order differential problem, abbreviated by ISDP:

$$\frac{d^2}{d\mathfrak{r}^2}y(\mathfrak{r}) = \theta\big(\mathfrak{r}, y(\mathfrak{r}), {}^{c}D^{\zeta}y(\mathfrak{r}), \int_0^{\mathfrak{r}} \phi(\mathfrak{r},\varsigma)\, {}^{c}D^{\delta}y(\varsigma)d\varsigma\big),\ \mathfrak{r} \in [0,1] \tag{1.1}$$

under two sets of boundary conditions

$$y(\tau) = \frac{1}{\Gamma(\sigma)} \int_0^{\tau} (\tau-\varsigma)^{\sigma-1}\ F_1(\varsigma, y(\varsigma))\ d\varsigma,\ \tau \in [0,1) \tag{1.2}$$

$$y(\eta) = \frac{1}{\Gamma(\sigma)} \int_0^{\eta} (\eta-\varsigma)^{\sigma-1}\ F_2(\varsigma, y(\varsigma))\ d\varsigma,\ \eta \in (0,1], \tag{1.3}$$

where $\mathfrak{r} \in \mathbb{I} = [0,1],\ \tau \neq \eta,\ 1<\zeta<\delta\leq 2,\ 0<\sigma<1,\ \theta : \mathbb{I}\times\mathbb{R}^3 \to \mathbb{R}$, and $\phi : \mathbb{I}\times\mathbb{I} \to \mathbb{R}$ are continuous functions, and ${}^{c}D^{\gamma}$ is the Caputo fractional derivative of order $\gamma \in \{\zeta, \delta\}$. Under some appropriate assumptions, we use the Banach and Krasnoselskiis fixed point Theorems to establish our main results.

The article is structured as follows. In Sect.2, we collect the essential preliminary definitions and facts. In Sect.3. contains the main findings, and it is divided into two parts. In the first part, we focused on the equivalence of the ISDP (1.1)-(1.3) and the integral equation. In the second part, we study the existence results for issue ISDP (1.1)-(1.3). Sect.4 is devoted to Ulam-Hyers-Rassias stability for our problem. We present some particular cases and examples to illustrate our findings in Sect.5. Finally, in Sect.6, conclusions are presented.

2. Preliminaries

Here, we present certain definitions and basic auxiliary results essential to achieving our aim.

Definition 2.1. [26] The Riemann-Liouville of fractional integral of the function $h \in L^1([a,b],\mathbb{R}^+)$ of order $\alpha \in \mathbb{R}^+$ is defined by

$$I_a^\alpha\ h(\mathfrak{r}) = \int_a^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\alpha-1}}{\Gamma(\alpha)}\ h(\varsigma)\ d\varsigma.$$

and when $a = 0$, we have $I^\alpha\ h(\mathfrak{r}) = I_0^\alpha\ h(\mathfrak{r})$, where $\Gamma(.)$ is Euler's Gamma function.

Definition 2.2. [30] The Caputo fractional derivative of order $\alpha > 0$ of function $h \in L^1([a,b],\mathbb{R}^+)$, with $\mathfrak{r} \in [a,b]$ is given by

$$({}^cD_{a+}^\alpha h)(\mathfrak{r}) = I_a^{n-\alpha}\frac{d}{d\mathfrak{r}}h(\mathfrak{r}) = \int_a^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{n-\alpha-1}}{\Gamma(1-\alpha)}h^{(n)}(\varsigma)d\varsigma,$$

where $n = [\alpha]+1$. If $\alpha \in (0,1]$, then

$$({}^cD_{a+}^\alpha h)(\mathfrak{r}) = I_{a^+}^{1-\alpha}\ \frac{d}{d\mathfrak{r}}\ h(\mathfrak{r}) = \int_a^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{-\alpha}}{\Gamma(1\ -\ \alpha)}\ \frac{d}{d\varsigma}h(\varsigma)d\varsigma.$$

For further properties of fractional calculus operator see [24, 25, 26].

Lemma 2.3. [30]. *Let $\alpha > 0$, then the differential equation*

$$({}^cD_{a+}^\alpha h)(\mathfrak{r}) = 0$$

has solution

$$h(\mathfrak{r}) = a_0 + a_1\mathfrak{r} + a_2\mathfrak{r}^2 + ... + a_{n-1}\mathfrak{r}^{n-1},\ a_i \in \mathbb{R},$$
$$i = 0,1,2,...,n-1,\ n = [\alpha]+1.$$

Lemma 2.4. [30] *Let $\alpha > 0$, then*

$$I^\alpha\ ({}^cD_{a_+}^\alpha h(\mathfrak{r})) = h(\mathfrak{r}) + a_0 + a_1\mathfrak{r} + a_2\mathfrak{r}^2 + ... + a_{n-1}\mathfrak{r}^{n-1},$$

for arbitrary $a_i \in \mathbb{R},\ i = 0,1,2,...,n-1,\ n = [\alpha]+1.$

3. Existence and Uniqueness

We show the existence of a solution for ISDP (1.1)-(1.3), by assuming the following conditions:

(H_1) $F_i : \mathbb{I} \times \mathbb{R} \to \mathbb{R}$, are continuous and there exist constants $k_i \in [0,1)$ with

$$|F_i(\mathfrak{r},u) - F_i(\mathfrak{r},v)| \le k_i|u-v|,\ i = 1,2.$$

(H_2) $\theta : \mathbb{I} \times \mathbb{R}^3 \to \mathbb{R}$ is continuous and there exists $\psi \in C(\mathbb{I},\mathbb{R}_+)$, with norm $\|\psi\|$, such that:

$$|\theta(\mathfrak{r},u_1,u_2,u_3) - \theta(\mathfrak{r},v_1,v_2,v_3)| \le \psi(\mathfrak{r})\big(|u_1-v_1| + |u_2-v_2| + |u_3-v_3|\big),$$

$\forall\ \mathfrak{r} \in \mathbb{I},\ u_i, v_i \in \mathbb{R},\ (i = 1,2,3).$

(H_3) $\phi(\mathfrak{r},\varsigma)$ is continuous for all $(\mathfrak{r},\varsigma) \in \mathbb{I} \times \mathbb{I}$.

Observations:

From assumptions (H_1) and (H_2)

$$|F_i(\mathfrak{r},u)| - |F_i(\mathfrak{r},0)| \le |F_i(\mathfrak{r},u) - F_i(\mathfrak{r},0)| \le k_i|u-0|,$$

$$|\theta(\mathfrak{r},u_1,u_2,u_3)| - |\theta(\mathfrak{r},0,0,0)| \le |\theta(\mathfrak{r},u_1,u_2,u_3) - \theta(\mathfrak{r},0,0,0)| \le \psi(\mathfrak{r})(|u_1|+|u_2|+|u_3|),$$

then

$$|F_i(\mathfrak{r},u)| \leq F_i + k_i|u|, \text{ with } F_i = \sup_{\mathfrak{r}\in\mathbb{I}} |F_i(\mathfrak{r},0)|,$$

and

$$|\theta(\mathfrak{r},u_1,u_2,u_3)| \leq \|\psi\|(|u_1|+|u_2|+|u_3|)+\Theta, \text{ with } \Theta = \sup_{\mathfrak{r}\in\mathbb{I}} |\theta(\mathfrak{r},0,0,0)|.$$

Also, from From assumptions (H_3), let Φ be a positive constant such that

$$\max_{\mathfrak{r},\varsigma\in\mathbb{I}} |\phi(\mathfrak{r},\varsigma)| = \Phi.$$

For the solution existence of an ordinary second-order functional differential equation, the following lemma is required.

Lemma 3.1. *The ISDP (1.1)-(1.3) is identical to the integral equation*

$$y(\mathfrak{r}) = F(\mathfrak{r},y(\mathfrak{r})) + \int_0^{\mathfrak{r}} (\varsigma-\mathfrak{r})u(\varsigma)d\varsigma, \tag{3.1}$$

with

$$u(\mathfrak{r}) = \theta\big(\mathfrak{r}, F(\mathfrak{r},y(\mathfrak{r})) + \int_0^{\mathfrak{r}} (\varsigma-\mathfrak{r})u(\varsigma)d\varsigma, I^{2-\zeta}u(\mathfrak{r}), \int_0^{\mathfrak{r}} \phi(\mathfrak{r},\varsigma)I^{2-\delta}u(\varsigma)d\varsigma\big)$$

and

$$\begin{aligned} F(\mathfrak{r},y(\mathfrak{r})) &= \frac{\mathfrak{r}-\eta}{(\tau-\eta)\Gamma(\sigma)}\int_0^{\tau} (\tau-\varsigma)^{\sigma-1}F_1(\varsigma,y(\varsigma))d\varsigma \\ &\quad + \frac{\tau-\mathfrak{r}}{(\tau-\eta)\Gamma(\sigma)}\int_0^{\eta} (\eta-\varsigma)^{\sigma-1}F_2(\varsigma,y(\varsigma))d\varsigma. \end{aligned} \tag{3.2}$$

Proof. We know that $D^{\zeta}y(\mathfrak{r}) = I^{2-\zeta}\frac{d^2}{d\mathfrak{r}^2}\, y(\mathfrak{r})$ and $D^{\delta}y(\mathfrak{r}) = I^{2-\delta}\frac{d^2}{d\mathfrak{r}^2}\, y(\mathfrak{r})$ for $\mathfrak{r} \in I$. Consequently, if y is a solution of ISDP (1.1), then the equality

$$\frac{d^2}{dt^2}y(\mathfrak{r}) = \theta\big(\mathfrak{r}, y(\mathfrak{r}), I^{2-\zeta}\,\frac{d^2}{d\mathfrak{r}^2}\,y(\mathfrak{r}), \int_0^{\mathfrak{r}} \phi(\mathfrak{r},\varsigma)I^{2-\delta}\,\frac{d^2}{d\mathfrak{r}^2}\,y(\varsigma)d\varsigma\big)$$

holds.

Now, let $\frac{d^2}{d\mathfrak{r}^2}y(\mathfrak{r}) = u(\mathfrak{r})$, then

$$u(\mathfrak{r}) = \theta(\mathfrak{r}, y(\mathfrak{r}), I^{2-\zeta}u(\mathfrak{r}), \int_0^{\mathfrak{r}} \phi(\mathfrak{r},\varsigma)I^{2-\delta}u(\varsigma)d\varsigma),$$

with

$$y(\mathfrak{r}) = a_\circ + a_1\mathfrak{r} + \int_0^{\mathfrak{r}} (t-\varsigma)u(\varsigma)d\varsigma. \tag{3.3}$$

From (1.2) and (1.3), a straightforward calculation yields

$$a_\circ + a_1\tau + \int_0^{\tau} (\tau-\varsigma)u(\varsigma)ds = \frac{1}{\Gamma(\sigma)}\int_0^{\tau} (\tau-\varsigma)^{\sigma-1}\, F_1(\varsigma,y(\varsigma))\, d\varsigma, \tag{3.4}$$

$$a_\circ + a_1\eta + \int_0^{\eta} (\eta-\varsigma)u(\varsigma)ds = \frac{1}{\Gamma(\sigma)}\int_0^{\eta} (\eta-\varsigma)^{\sigma-1}\, F_2(\varsigma,y(\varsigma))\, d\varsigma. \tag{3.5}$$

By solving (3.4)-(3.5), we obtain

$$
\begin{aligned}
a_1 &= \frac{-1}{(\tau-\eta)} \int_0^\tau (\tau-\varsigma)u(\varsigma)ds + \frac{1}{(\tau-\eta)} \int_0^\eta (\eta-\varsigma)u(\varsigma)d\varsigma \\
&+ \frac{1}{(\tau-\eta)\Gamma(\sigma)} \int_0^\tau (\tau-\varsigma)^{\sigma-1} F_1(\varsigma,y(\varsigma))d\varsigma - \frac{1}{(\tau-\eta)\Gamma(\sigma)} \int_0^\eta (\eta-\varsigma)^{\sigma-1} F_2(\varsigma,y(\varsigma))d\varsigma,
\end{aligned}
$$

and

$$
a_\circ = \frac{\eta}{(\tau-\eta)} \int_0^\tau (\tau-\varsigma)u(\varsigma)d\varsigma - \frac{\tau}{(\tau-\eta)} \int_0^\eta (\eta-\varsigma)u(\varsigma)d\varsigma \tag{3.6}
$$

$$
+ \frac{-\eta}{(\tau-\eta)\Gamma(\sigma)} \int_0^\tau (\tau-\varsigma)^{\sigma-1} F_1(\varsigma,y(\varsigma))\, d\varsigma + \frac{\tau}{(\tau-\eta)\Gamma(\sigma)} \int_0^\eta (\eta-\varsigma)^{\sigma-1} F_2(\varsigma,y(\varsigma))\, d\varsigma. \tag{3.7}
$$

Replacing a_0 and a_1 by their equivalents in (3.3), the solution of (1.1)-(1.3) is provided by

$$
\begin{aligned}
y(\mathfrak{r}) &= \frac{\eta-\mathfrak{r}}{(\tau-\eta)} \int_0^\tau (\tau-\varsigma)u(\varsigma)d\varsigma + \frac{\mathfrak{r}-\tau}{(\tau-\eta)} \int_0^\eta (\eta-\varsigma)u(\varsigma)d\varsigma \\
&+ \frac{\mathfrak{r}-\eta}{(\tau-\eta)\Gamma(\sigma)} \int_0^\tau (\tau-\varsigma)^{\sigma-1} F_1(\varsigma,y(\varsigma))\, d\varsigma + \frac{\tau-\mathfrak{r}}{(\tau-\eta)\Gamma(\sigma)} \int_0^\eta (\eta-\varsigma)^{\sigma-1} F_2(\varsigma,y(\varsigma))\, d\varsigma \\
&+ \int_0^{\mathfrak{r}} (\mathfrak{r}-\varsigma)\, u(\varsigma)d\varsigma.
\end{aligned}
$$

Consequently, ISDP (1.1)-(1.3) is equivalent to the integral equation

$$
y(\mathfrak{r}) = F(\mathfrak{r}, y(\mathfrak{r})) + \int_0^1 G(\mathfrak{r},\varsigma)u(\varsigma)d\varsigma, \tag{3.8}
$$

where u is the solution of the functional integral equation

$$
u(\mathfrak{r}) = \theta\big(\mathfrak{r}, F(\mathfrak{r}, y(\mathfrak{r})) + \int_0^1 G(\mathfrak{r},\varsigma)u(\varsigma)d\varsigma, I^{2-\zeta}u(\mathfrak{r}), \int_0^{\mathfrak{r}} \phi(\mathfrak{r},\varsigma) I^{2-\delta}u(\varsigma)d\varsigma\big) \tag{3.9}
$$

with $G(\mathfrak{r},\varsigma)$ is the Green's function defined by

$$
G(\mathfrak{r},\varsigma) = \begin{cases} (\mathfrak{r}-\varsigma) + \frac{\eta-\mathfrak{r}}{(\tau-\eta)}(\tau-\varsigma) + \frac{\mathfrak{r}-\tau}{(\tau-\eta)}(\eta-\varsigma) = 0, & 0 \le \varsigma \le \mathfrak{r} \le \eta < \tau \le 1, \\[2ex] \frac{\eta-\mathfrak{r}}{(\tau-\eta)}(\tau-\varsigma) + \frac{\mathfrak{r}-\tau}{(\tau-\eta)}(\eta-\varsigma) = (\varsigma-\mathfrak{r}), & 0 \le \mathfrak{r} \le \varsigma \le \eta < \tau \le 1, \end{cases}
$$

Replace the value of $G(\mathfrak{r},\varsigma)$ by its value in equation (3.8) and (3.9). This prove the ISDP (1.1)-(1.3) corresponding to Eq.(3.1). □

Lemma 3.2. $F : \mathbb{I} \times \mathbb{R} \to \mathbb{R}$, *is Lipschitzian function with a Lipschitz constant* c

$$
\|F(\mathfrak{r}, x) - F(\mathfrak{r}, y)\| \le c\|x - y\|.
$$

Proof. For all $u, v \in X$ and $\mathfrak{r} \in \mathbb{I}$, we obtain

$$
\begin{aligned}
&|\digamma(\mathfrak{r}, x(\mathfrak{r})) - \digamma(\mathfrak{r}, y(\mathfrak{r}))| \\
&\leq \Big| \frac{\mathfrak{r}-\eta}{(\tau-\eta)\Gamma(\sigma)} \int_0^\tau (\tau-\varsigma)^{\sigma-1} \digamma_1(\varsigma, x(\varsigma)) d\varsigma + \frac{\tau-\mathfrak{r}}{(\tau-\eta)\Gamma(\sigma)} \int_0^\eta (\eta-\varsigma)^{\sigma-1} \digamma_2(\varsigma, x(\varsigma)) d\varsigma \\
&- \frac{\mathfrak{r}-\eta}{(\tau-\eta)\Gamma(\sigma)} \int_0^\tau (\tau-\varsigma)^{\sigma-1} \digamma_1(\varsigma, y(\varsigma)) d\varsigma - \frac{\tau-\mathfrak{r}}{(\tau-\eta)\Gamma(\sigma)} \int_0^\eta (\eta-\varsigma)^{\sigma-1} \digamma_2(\varsigma, y(\varsigma)) d\varsigma \Big| \\
&\leq \frac{\mathfrak{r}-\eta}{(\tau-\eta)\Gamma(\sigma)} \int_0^\tau (\tau-\varsigma)^{\sigma-1} \big|\digamma_1(\varsigma, x(\varsigma)) - \digamma_1(\varsigma, y(\varsigma))\big| d\varsigma \\
&+ \frac{\tau-\mathfrak{r}}{(\tau-\eta)\Gamma(\sigma)} \int_0^\eta (\eta-\varsigma)^{\sigma-1} \big|\digamma_2(\varsigma, x(\varsigma)) - \digamma_2(\varsigma, y(\varsigma))\big| d\varsigma \\
&\leq \frac{k_1(\mathfrak{r}-\eta)}{(\tau-\eta)\Gamma(\sigma+1)} \|x-y\| + \frac{k_2(\tau-\mathfrak{r})}{(\tau-\eta)\Gamma(\sigma+1)} \|x-y\|,
\end{aligned}
$$

then

$$\|\digamma(\mathfrak{r}, x) - \digamma(\mathfrak{r}, y)\| \leq \frac{k_1(\mathfrak{r}-\eta) + k_2(\tau-\mathfrak{r})}{(\tau-\eta)\Gamma(\sigma+1)} \|x-y\|,$$

take $c = \frac{k_1(\mathfrak{r}-\eta)+k_2(\tau-\mathfrak{r})}{(\tau-\eta)\Gamma(\sigma+1)}$. Thus $\digamma$ is Lipschitz function. $\square$

3.1. Main results. Our first existence result for ISDP (1.1)-(1.3) is based on the fixed point Theorem of Krasnoselskii [9].

Theorem 3.3. *Let assumptions* (H_1)-(H_3) *be satisfied. If*

$$\left(\frac{\|\psi\|\tau}{\mathfrak{M}} + \frac{(\mathfrak{r}-\eta)k_1 + (\tau-\mathfrak{r})k_2}{(\tau-\eta)\Gamma(\sigma+1)} \right) < 1, \tag{3.10}$$

then the ISDP (1.1)-(1.3) has at least one solution.

Proof. Define the operator $A : C(\mathbb{I}, \mathbb{R}) \to C(\mathbb{I}, \mathbb{R})$ by:

$$Ay(\mathfrak{r}) = \digamma(\mathfrak{r}, y(\mathfrak{r})) + \int_0^{\mathfrak{r}} (\varsigma - \mathfrak{r})\ v(\varsigma)\ d\varsigma, \tag{3.11}$$

where $v \in C(\mathbb{I}, \mathbb{R})$ satisfies the implicit functional equation

$$v(\mathfrak{r}) = \theta\big(\mathfrak{r}, y(\mathfrak{r}), I^{2-\zeta} v(\mathfrak{r}), \int_0^{\mathfrak{r}} \phi(\mathfrak{r}, \varsigma) I^{2-\delta} v(\varsigma) d\varsigma\big),$$

with $\digamma$ is the functions defined by (3.2).
We choose

$$\varrho \geq \left(\frac{(\mathfrak{r}-\eta)F_1 + (\tau-\mathfrak{r})F_2}{(\tau-\eta)\Gamma(\sigma+1)} + \frac{\Theta\tau}{\mathfrak{M}} \right) \left(1 - \left[\frac{\|\psi\|\tau}{\mathfrak{M}} + \frac{(\mathfrak{r}-\eta)k_1 + (\tau-\mathfrak{r})k_2}{(\tau-\eta)\Gamma(\sigma+1)} \right] \right)^{-1}$$

where $\mathfrak{M} = 1 - \|\psi\| \ \big(\frac{1}{\Gamma(3-\zeta)} + \frac{\Phi}{\Gamma(3-\delta)}\big)$.
Define the ball

$$\mathfrak{B}_\varrho = \{y \in C(\mathbb{I}, \mathbb{R}) : \|y\| \leq \varrho\}.$$

Moreover, we characterize the operators A_1 and A_2 on $\mathfrak{B}_\varrho$ by

$$A_1 y(\mathfrak{r}) = F(\mathfrak{r}, y(\mathfrak{r}))$$
$$A_2 y(\mathfrak{r}) = \int_0^{\mathfrak{r}} (\varsigma - t) v(\varsigma) d\varsigma.$$

Consider that A_1 and A_2 are defined on B_ϱ, and for any $y \in C(\mathbb{I}, \mathbb{R})$,

$$Ay(\mathfrak{r}) = A_1 y(\mathfrak{r}) + A_2 y(\mathfrak{r}), \ \mathfrak{r} \in \mathbb{I}.$$

The proof will be decomposed into steps:

Step 1: Let $y_1, y_2 \in B_\varrho$ and $\mathfrak{r} \in \mathbb{I}$, we have

$$\begin{aligned} |A_1 y_1(\mathfrak{r}) + A_2 y_2(\mathfrak{r})| &\le |A_1 y_1(\mathfrak{r})| + |A_2 y_2(\mathfrak{r})| \\ &\le |F(\mathfrak{r}, y_1(\mathfrak{r}))| + \int_0^{\mathfrak{r}} |(\varsigma - \mathfrak{r})||v(\varsigma)| d\varsigma, \end{aligned} \tag{3.12}$$

where $v(\mathfrak{r}) = \theta\big(\mathfrak{r}, y_2(\mathfrak{r}), I^{2-\zeta} v(\mathfrak{r}), \int_0^{\mathfrak{r}} \phi(\mathfrak{r}, \varsigma) I^{2-\delta} v(\varsigma) d\varsigma\big)$, and

$$\begin{aligned} |v(\mathfrak{r})| &= \Big|\theta\big(\mathfrak{r}, y_2(\mathfrak{r}), I^{2-\zeta} v(\mathfrak{r}), \int_0^{\mathfrak{r}} \phi(\mathfrak{r}, \varsigma) I^{2-\delta} v(\varsigma) d\varsigma\big)\Big| \\ &\le \ \Theta \ + |\psi(\mathfrak{r})| \ \Big(|y_2(\mathfrak{r})| \\ &\quad + \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \varsigma)^{1-\zeta}}{\Gamma(2-\zeta)} |v(\varsigma)| \ d\varsigma + \int_0^{\mathfrak{r}} |\phi(\mathfrak{r}, \varsigma)| \int_0^{\varsigma} \frac{(\varsigma - \tau)^{1-\delta}}{\Gamma(2-\delta)} |v(\tau)| \ ds \ d\tau\Big) \\ &\le \ \Theta + \|\psi\| \ \Big(\|y_2\| + \frac{\varsigma^{2-\zeta}}{\Gamma(3-\zeta)} \|v\| + \Phi \frac{\varsigma^{2-\delta}}{\Gamma(3-\delta)} \|v\|\Big) \\ \|v\| &\le \ \Theta + \|\psi\| \ \Big(\|y_2\| + \frac{1}{\Gamma(3-\zeta)} \|v\| + \Phi \frac{1}{\Gamma(3-\delta)} \|v\|\Big). \end{aligned}$$

Then

$$\|v\| \le \ \frac{\varrho \ \|\psi\| + \Theta}{\mathfrak{M}}$$

and

$$\begin{aligned} |F(\mathfrak{r}, y(\mathfrak{r}))| &\le \frac{\mathfrak{r} - \eta}{(\tau - \eta)\Gamma(\sigma)} \int_0^{\tau} \ (\tau - \varsigma)^{\sigma - 1} \ |F_1(\varsigma, y(\varsigma))| d\varsigma \\ &\qquad + \frac{\tau - \mathfrak{r}}{(\tau - \eta)\Gamma(\sigma)} \int_0^{\eta} \ (\eta - \varsigma)^{\sigma - 1} \ |F_2(\varsigma, y(\varsigma))| d\varsigma \\ &\le \frac{\mathfrak{r} - \eta}{(\tau - \eta)\Gamma(\sigma)} \int_0^{\tau} \ (\tau - \varsigma)^{\sigma - 1} [F_1 + k_1 |y(\varsigma)|] d\varsigma \\ &\qquad + \frac{\tau - \mathfrak{r}}{(\tau - \eta)\Gamma(\sigma)} \int_0^{\eta} \ (\eta - \varsigma)^{\sigma - 1} [F_2 + k_2 |y(\varsigma)|] d\varsigma \\ &\le \frac{(\mathfrak{r} - \eta)[F_1 + k_1 \|y\|]}{(\tau - \eta)\Gamma(\sigma + 1)} + \frac{(\tau - \mathfrak{r})[F_2 + k_2 \|y\|]}{(\tau - \eta)\Gamma(\sigma + 1)} \\ &\le \frac{(\mathfrak{r} - \eta) F_1 + (\tau - \mathfrak{r}) F_2}{(\tau - \eta)\Gamma(\sigma + 1)} + \frac{\varrho[(\mathfrak{r} - \eta) k_1 + (\tau - \mathfrak{r}) k_2]}{(\tau - \eta)\Gamma(\sigma + 1)}. \end{aligned}$$

Hence (3.12) implies that, for each $\mathfrak{r} \in \mathbb{I}$,

$$|A_1 y_1(\mathfrak{r}) + A_2 y_2(\mathfrak{r})| \leq \frac{(\mathfrak{r}-\eta)F_1 + (\tau-\mathfrak{r})F_2}{(\tau-\eta)\Gamma(\sigma+1)} + \frac{\varrho[(\mathfrak{r}-\eta)k_1 + (\tau-\mathfrak{r})k_2]}{(\tau-\eta)\Gamma(\sigma+1)} + \frac{(\varrho\ \|\psi\| + F)\tau}{\mathfrak{M}}$$
$$\leq\ \varrho.$$

Then

$$\|A_1 y_1 + A_2 y_2\| \leq\ \varrho.$$

This proves that $A_1 y_1 + A_2 y_2 \in B_\varrho$ for every $y_1, y_2 \in B_\varrho$.

Step.2: The operator A_1 is contraction mapping on B_ϱ.
It is clear that from lemma 3.2, A_1 is a contraction mapping for $c < 1$.

Step.3: The operator A_2 is completely continuous on B_ϱ.
First, prove operator A_2 is continuous.
For $\{y_n\}_{n\in N}$ is a sequence with $y_n \to y$ as $n \to \infty$ in $C(\mathbb{I}, \mathbb{R})$.
For all $\mathfrak{r} \in I$, we obtain

$$|A_2 y_n - A_2 y| \leq \int_0^{\mathfrak{r}} |(\varsigma - \mathfrak{r})||v_n(\varsigma) - v(\varsigma)| d\varsigma, \tag{3.13}$$

where $v_n, v \in C(\mathbb{I}, \mathbb{R})$, such that

$$v_n(\mathfrak{r}) = \theta\big(\mathfrak{r}, y_n(\mathfrak{r}), I^{2-\zeta} v_n(\mathfrak{r}), \int_0^{\mathfrak{r}} \phi(\mathfrak{r}, \varsigma) I^{2-\delta}\ v_n(\varsigma) d\varsigma\big),$$
$$v(\mathfrak{r}) = \theta\big(\mathfrak{r}, y(\mathfrak{r}), I^{2-\zeta} v(\mathfrak{r}), \int_0^{\mathfrak{r}} \phi(\mathfrak{r}, \varsigma) I^{2-\delta}\ v(\varsigma) d\varsigma\big).$$

but by condition (H_2), we have

$$\begin{aligned}
|v_n(\mathfrak{r}) - v(\mathfrak{r})| &= \Big|\theta\big(\mathfrak{r}, y_n(\mathfrak{r}), I^{2-\zeta} v_n(\mathfrak{r}), \int_0^{\mathfrak{r}} \phi(\mathfrak{r}, \varsigma) I^{2-\delta} v_n(\mathfrak{r}) ds\big) \\
&\qquad - \theta\big(\mathfrak{r}, y(\mathfrak{r}), I^{2-\zeta} v(\mathfrak{r}), \int_0^{\mathfrak{r}} \phi(\mathfrak{r}, \varsigma) I^{2-\delta} v(\varsigma) d\varsigma\big)\Big| \\
&\leq \psi(\mathfrak{r}) \Bigg(|y_n(\mathfrak{r}) - y(\mathfrak{r})| + \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{1-\zeta}}{\Gamma(2-\zeta)} |v_n(\varsigma) - v(\varsigma)| d\varsigma \\
&\qquad + \int_0^{\mathfrak{r}} \phi(\mathfrak{r}, \varsigma) \int_0^{\tau} \frac{(\varsigma-\tau)^{1-\delta}}{\Gamma(2-\delta)} |v_n(\tau) - v(\tau)| d\varsigma d\tau \Bigg) \\
&\leq\ \|\psi\|\ \big(\|y_n - y\| + \frac{\varsigma^{2-\zeta}}{\Gamma(3-\zeta)} \|v_n - v\| + \Phi \frac{\varsigma^{3-\delta}}{\Gamma(3-\delta)} \|v_n - v\|\ \big) \\
&\leq\ \|\psi\|\ \big(\|y_n - y\| + \frac{1}{\Gamma(3-\zeta)} \|v_n - v\| + \Phi \frac{1}{\Gamma(3-\delta)} \|v_n - v\|\ \big).
\end{aligned}$$

Thus

$$\|v_n - v\| \leq \frac{\|\psi\|}{\mathfrak{M}} \|y_n - y\|.$$

Since $y_n \to y$, then we get $v_n(\mathfrak{r}) \to v(\mathfrak{r})$ as $n \to \infty$ for each $\mathfrak{r} \in \mathbb{I}$. And let $\varepsilon > 0$ be such that, for each $\mathfrak{r} \in \mathbb{I}$, we have $|v_n(\mathfrak{r})| \leq \varepsilon$, and $|v(\mathfrak{r})| \leq \varepsilon$. Then, we have

$$\begin{aligned}
|(\varsigma - \mathfrak{r})||v_n(\varsigma) - v(\varsigma)| &\leq |(\varsigma - \mathfrak{r})|[|v_n(\varsigma)| + |v(\varsigma)|] \\
&\leq 2\varepsilon|(\varsigma - \mathfrak{r})|
\end{aligned}$$

The function $s \to 2\varepsilon|(\varsigma - \mathfrak{r})|$ is integrable on $\mathbb{I}$, for each $t \in \mathbb{I}$. By Lebesgue Dominated Convergence Theorem, and (3.13) we obtain

$$\|A_2 y_n - A_2 y\| \to 0 \text{ as } n \to \infty.$$

Consequently, A_2 is continuous.
Next, it is easy to confirm

$$\|A_2 y\| \leq \frac{(\varrho\ \|\psi\| + \Theta)\ \tau}{\mathfrak{M}} \leq \varrho.$$

This proves that A_2 is uniformly bounded on B_ϱ.

Finally, we prove that bounded sets are transformed into equicontinuous sets of $C(\mathbb{I}, \mathbb{R})$ by A_2, i.e., B_ϱ is equicontinuous.

Now, $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ and $\mathfrak{r}_1, \mathfrak{r}_2 \in \mathbb{I},\ \mathfrak{r}_1 < \mathfrak{r}_2,\ |\mathfrak{r}_2 - \mathfrak{r}_1| < \delta$. Then we obtain

$$\begin{aligned}
|A_2 y(\mathfrak{r}_2) - A_2 y(\mathfrak{r}_1)| &\leq \int_0^{\mathfrak{r}} |(\varsigma - \mathfrak{r}_2) - (\varsigma - \mathfrak{r}_1)|\ |v(\varsigma)| d\varsigma \\
&\leq \|v\| \int_0^{\mathfrak{r}} |\mathfrak{r}_2 - \mathfrak{r}_1|\ d\varsigma \\
&\leq (\frac{\varrho\ \|\psi\| + \Theta}{\mathfrak{M}}) |\mathfrak{r}_2 - \mathfrak{r}_1|.
\end{aligned}$$

Consequently,

$$|A_2 y(\mathfrak{r}_2) - A_2 y(\mathfrak{r}_1)| \to 0,\ \forall\ |\mathfrak{r}_2 - \mathfrak{r}_1| \to 0.$$

Thus, $\{Ay\}$ is equi-continuous on B_ϱ and A is a compact operator by the Arzela-Ascoli Theorem [27], we establish that $A : C(\mathbb{I}, \mathbb{R}) \to C(\mathbb{I}, \mathbb{R})$ is completely continuous.

Hence, all the hypotheses of Krasnoselskii's fixed point Theorem have been validated and show that $A_1 + A_2$ has a fixed point on B_ϱ. Therefore, the ISDP (1.1)-(1.3) has a solution, and the proof is complete. □

Our second finding is dependent on Banach's fixed point Theorem, which establishes the existence of a unique solution of the ISDP (1.1)-(1.3).

Theorem 3.4. *Let the assumptions of Theorem 3.3 hold. If*

$$c + \frac{\tau\ \|\psi\|}{\mathfrak{M}} < 1, \tag{3.14}$$

then the the ISDP (1.1)-(1.3) has a unique solution.

Proof. As a result of Theorem 3.3, that ISDP (1.1)-(1.3) has at least one solution. Therefore, we only need to show that the operator A described in (3.11) is a contraction.

Now take $x, y \in C(\mathbb{I}, \mathbb{R})$. Then for $\mathfrak{r} \in \mathbb{I}$, we obtain

$$Ax(\mathfrak{r}) - Ay(\mathfrak{r}) = F(\mathfrak{r}, x(\mathfrak{r})) + \int_0^{\mathfrak{r}} (\varsigma - \mathfrak{r}) u(\varsigma) ds - F(\mathfrak{r}, y(\mathfrak{r})) - \int_0^{\mathfrak{r}} (\varsigma - \mathfrak{r}) v(\varsigma) d\varsigma, \tag{3.15}$$

with

$$\begin{aligned}
u(\mathfrak{r}) &= \theta\big(\mathfrak{r}, x(\mathfrak{r}), I^{2-\beta} u(\mathfrak{r}), \int_0^{\mathfrak{r}} \phi(\mathfrak{r}, \varsigma) I^{2-\alpha}\ u(\varsigma) d\varsigma\big), \\
v(\mathfrak{r}) &= \theta\big(\mathfrak{r}, y(\mathfrak{r}), I^{2-\beta} v(\mathfrak{r}), \int_0^{\mathfrak{r}} \phi(\mathfrak{r}, \varsigma) I^{2-\alpha}\ v(\varsigma) ds\big).
\end{aligned}$$

Then, for $\mathfrak{r} \in \mathbb{I}$

$$|Ax(\mathfrak{r}) - Ay(\mathfrak{r})| \leq \ |F(\mathfrak{r}, x(\mathfrak{r})) - F(\mathfrak{r}, y(\mathfrak{r}))| + \int_0^{\mathfrak{r}} (\varsigma - \mathfrak{r}) \ |u(\varsigma) - v(\varsigma)| d\varsigma, \tag{3.16}$$

by condition (H_2), we obtain

$$\begin{aligned}
|v_n(\mathfrak{r}) - v(\mathfrak{r})| &= \big|\theta\big(\mathfrak{r}, x(\mathfrak{r}), I^{2-\zeta}u(\mathfrak{r}), \int_0^{\mathfrak{r}} \phi(\mathfrak{r},\varsigma) I^{2-\delta} u(\mathfrak{r}) d\varsigma\big) \\
&\qquad - \theta\big(\mathfrak{r}, y(\mathfrak{r}), I^{2-\zeta}v(\mathfrak{r}), \int_0^{\mathfrak{r}} \phi(\mathfrak{r},\varsigma) I^{2-\delta} v(\varsigma) d\varsigma\big)\big| \\
&\leq \psi(\mathfrak{r}) \Big(|x(\mathfrak{r}) - y(\mathfrak{r})| + \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \varsigma)^{1-\zeta}}{\Gamma(2-\zeta)} |u(\varsigma) - v(\varsigma)| d\varsigma \\
&\qquad + \int_0^{\mathfrak{r}} \phi(\mathfrak{r},\varsigma) \int_0^{\tau} \frac{(\varsigma - \tau)^{1-\delta}}{\Gamma(2-\delta)} |u(\tau) - v(\tau)| d\varsigma d\tau \Big) \\
&\leq \ |\psi(t)| \ \big(\|x - y\| + \frac{\varsigma^{2-\zeta}}{\Gamma(3-\zeta)} \|u - v\| + \Phi \frac{\varsigma^{3-\delta}}{\Gamma(3-\delta)} \|u - v\| \ \big) \\
&\leq \ \|\psi\| \ \big(\|x - y\| + \frac{1}{\Gamma(3-\zeta)} \|u - v\| + \Phi \frac{1}{\Gamma(3-\delta)} \|u - v\| \ \big).
\end{aligned}$$

Thus

$$\|u - v\| \leq \frac{\|\psi\|}{\mathfrak{M}} \|x - y\|.$$

Return to (3.16) and by lemma 3.2, we have

$$\begin{aligned}
|Ax(\mathfrak{r}) - Ay(\mathfrak{r})| &\leq \ c\|x - y\| + \frac{\tau \ |\psi\|}{\mathfrak{M}} \|x - y\| \\
&\leq \big(c + \frac{\tau \ \|\psi\|}{\mathfrak{M}}\big) \|x - y\|.
\end{aligned}$$

$$\|Ax - Ay\| \leq \big(c + \frac{\tau \ |\psi\|}{\mathfrak{M}}\big) \|x - y\|.$$

By $\big(c + \frac{\tau \ |\psi\|}{\mathfrak{M}}\big) < 1$, A is a contraction operator. As a result of Banach's contraction principle, A has a unique fixed point which is the ISDP (1.1)-(1.3) solution on $\mathbb{I}$. $\square$

4. Stability of solutions

Now, we examine the Ulam stability for ISDP (1.1)-(1.3). Let $\epsilon > 0$ and $\Phi : \mathbb{I} \to \mathbb{R}_+$ be a continuous function. Inequalities such as these are taken into account:

$$\Big| \frac{d^2}{dt^2} y(\mathfrak{r}) - \theta\big(\mathfrak{r}, y(\mathfrak{r}), {}^c D^{\zeta} y(\mathfrak{r}), \int_0^{\mathfrak{r}} \phi(\mathfrak{r},\varsigma) {}^c D^{\delta} y(\varsigma) d\varsigma\big) \Big| \leq \epsilon, \ \ \mathfrak{r} \in \mathbb{I} \tag{4.1}$$

$$\Big| \frac{d^2}{dt^2} y(\mathfrak{r}) - \theta\big(\mathfrak{r}, y(\mathfrak{r}), {}^c D^{\zeta} y(\mathfrak{r}), \int_0^{\mathfrak{r}} \phi(\mathfrak{r},\varsigma) {}^c D^{\delta} y(\varsigma) d\varsigma\big) \Big| \leq \Phi(\mathfrak{r}), \ \ \mathfrak{r} \in \mathbb{I} \tag{4.2}$$

$$\Big| \frac{d^2}{d\mathfrak{r}^2} y(\mathfrak{r}) - \theta\big(\mathfrak{r}, y(\mathfrak{r}), {}^c D^{\zeta} y(\mathfrak{r}), \int_0^{\mathfrak{r}} \phi(\mathfrak{r},\varsigma) {}^c D^{\delta} y(\varsigma) d\varsigma\big) \Big| \leq \epsilon \ \Phi(\mathfrak{r}), \ \ \mathfrak{r} \in \mathbb{I}. \tag{4.3}$$

Definition 4.1. [28] The ICFDP (1.1)-(1.3) is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $y \in C(\mathbb{I}, \mathbb{R})$ of the inequality (4.1) there exists a solution $x \in C(\mathbb{I}, \mathbb{R})$ of (1.1)-(1.3) with

$$|y(\mathfrak{r}) - x(\mathfrak{r})| \leq \epsilon\ c_f,\ \mathfrak{r} \in \mathbb{I}.$$

Definition 4.2. [28] The problem ICFDP (1.1)-(1.3) is generalized Ulam-Hyers stable if there exists $c_f \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $c_f(0) = 0$ such that for each $\epsilon > 0$ and for each solution $y \in C(\mathbb{I}, \mathbb{R})$ of the inequality (4.1) there exists a solution $x \in C(\mathbb{I}, \mathbb{R})$ of (1.1)-(1.3) with

$$|y(\mathfrak{r}) - x(\mathfrak{r})| \leq c_f(\epsilon),\ \mathfrak{r} \in \mathbb{I}.$$

Definition 4.3. [28] The problem ICFDP (1.1)-(1.3) is Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{f,\Phi} > 0$ such that for each $\epsilon > 0$ and for each solution $y \in C(\mathbb{I}, \mathbb{R})$ of the inequality (4.2) there exists a solution $x \in C(\mathbb{I}, \mathbb{R})$ of (1.1)-(1.3) with

$$|y(\mathfrak{r}) - x(\mathfrak{r})| \leq \epsilon\ c_{f,\Phi}\Phi(\mathfrak{r}),\ \mathfrak{r} \in \mathbb{I}.$$

Definition 4.4. [28] The problem ICFDP (1.1)-(1.3) is generalized Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{f,\Phi} > 0$ such that for each solution $y \in C(\mathbb{I}, \mathbb{R})$ of the inequality (4.3) there exists a solution $x \in C(\mathbb{I}, \mathbb{R})$ of (1.1)-(1.3) with

$$|y(\mathfrak{r}) - x(\mathfrak{r}) > | \leq c_{f,\Phi}\Phi(\mathfrak{r}),\ \mathfrak{r} \in \mathbb{I}.$$

4.1. Ulam-Hyers Stability.

Next, we present the following Ulam−Hyers stable result.

Theorem 4.5. *Assume that assumptions of Theorem 3.4 hold. Then ISDP (1.1)-(1.3) is Ulam−Hyers stable.*

Proof. Let $\epsilon > 0$ and let $z \in C(\mathbb{I}, \mathbb{R})$ meets the inequality (4.1), i.e.,

$$\left| \frac{d^2}{d\mathfrak{r}^2} z(\mathfrak{r}) - \theta\big(\mathfrak{r}, z(\mathfrak{r}), D^{\zeta} z(\mathfrak{r}), \int_0^{\mathfrak{r}} \phi(\mathfrak{r}, \varsigma)^c D^{\delta} z(\varsigma) d\varsigma\big) \right| \leq \epsilon,\ \mathfrak{r} \in I$$

and let $y \in C(\mathbb{I}, \mathbb{R})$ be the unique solution of ISDP (1.1)-(1.3) which is by lemma 3.1 the ISDP (1.1)-(1.3) equivalence to fractional order integral equation

$$y(\mathfrak{r}) = F(\mathfrak{r}, y(\mathfrak{r})) + \int_0^{\mathfrak{r}} (\varsigma - \mathfrak{r}) u(\varsigma) d\varsigma,$$

with

$$u(\mathfrak{r}) = \theta\big(\mathfrak{r}, y(\mathfrak{r}), I^{2-\zeta} u(\mathfrak{r}), \int_0^{\mathfrak{r}} \phi(\mathfrak{r}, \varsigma) I^{2-\delta} u(\varsigma) d\varsigma\big)$$

Operating by I^2 to (4.1), we obtain

$$\left| z(\mathfrak{r}) - F(\mathfrak{r}, z(\mathfrak{r})) - \int_0^{\mathfrak{r}} (\varsigma - \mathfrak{r}) v(\varsigma) d\varsigma \right| \leq \frac{\epsilon}{2}. \tag{4.4}$$

For each $\mathfrak{r} \in \mathbb{I}$, we obtain

$$\begin{aligned}
|z(\mathfrak{r}) - y(\mathfrak{r})| &= \Big|z(\mathfrak{r}) - F(\mathfrak{r}, y(\mathfrak{r})) - \int_0^{\mathfrak{r}} (\varsigma - \mathfrak{r}) u(\varsigma) d\varsigma\Big| \\
&\le \Big|z(\mathfrak{r}) - F(\mathfrak{r}, z(\mathfrak{r})) + \int_0^{\mathfrak{r}} (\varsigma - \mathfrak{r}) v(\varsigma) d\varsigma\Big| \\
&+ \Big|F(\mathfrak{r}, z(\mathfrak{r})) + \int_0^{\mathfrak{r}} (\varsigma - \mathfrak{r}) v(\varsigma) d\varsigma - F(\mathfrak{r}, y(\mathfrak{r}) - \int_0^{\mathfrak{r}} (\varsigma - \mathfrak{r}) u(\varsigma) d\varsigma\Big| \\
&\le \frac{\epsilon}{2} + \big|F(\mathfrak{r}, z(\mathfrak{r})) - F(\mathfrak{r}, y(\mathfrak{r}))\big| + \int_0^{\mathfrak{r}} (\varsigma - \mathfrak{r}) |v(\varsigma) - u(\varsigma)| d\varsigma \\
&\le \frac{\epsilon}{2} + c\, |z(\mathfrak{r}) - y(\mathfrak{r})| + \int_0^{\mathfrak{r}} (\varsigma - \mathfrak{r}) |v(\varsigma) - u(\varsigma)| d\varsigma \\
&\le \frac{\epsilon}{2} + c\, \|z - y\| + \tau \|u - v\|.
\end{aligned}$$

However, from proof of Theorem 3.4, we obtain

$$\|u - v\| \le \frac{\|\psi\|}{1 - \|\psi\|\big(\Phi\ \frac{1}{\Gamma(3-\delta)} + \frac{1}{\Gamma(3-\zeta)}\big)} \|z - y\|.$$

and

$$\|z - y\| \le \frac{\epsilon}{2} + c\, \|z - y\| + \frac{\tau\, \|\psi\|}{1 - \|\psi\|\big(\Phi\ \frac{1}{\Gamma(3-\delta)} + \frac{1}{\Gamma(3-\zeta)}\big)} \|z - y\|.$$

Thus

$$\|z - y\| \quad \le \quad \frac{\epsilon}{2}\Bigg[1 - \Big(c\ + \frac{\tau\, \|\psi\|}{1 - \|\psi\|\big(\Phi\ \frac{1}{\Gamma(3-\delta)} + \frac{1}{\Gamma(3-\zeta)}\big)}\Big)\Bigg]^{-1} = \varsigma\ \epsilon,$$

for let $\varsigma = \frac{1}{2}\Bigg[1 - \Big(c\ + \frac{\tau\ \|\psi\|}{1 - \|\psi\|\big(\Phi + \frac{1}{\Gamma(\delta - \zeta + 1)}\big)}\Big)\Bigg]^{-1}$. So, the ISDP (1.1)-(1.3) is Ulam-Hyers stable. □

By setting $\Delta(\epsilon) = \varsigma\ \epsilon,\ \Delta(0) = 0$ gives that the ISDP (1.1)-(1.3) is generalized Ulam-Hyers stable.

4.2. Ulam-Hyers-Rassias Stability.

Theorem 4.6. *Assume that assumptions $(H_1) - (H_3)$ and*

(H_4) *The function $\Theta \in C(\mathbb{I}, \mathbb{R}_+)$ is increasing and there exists $\lambda_\Theta > 0$ such that, for each $\mathfrak{r} \in J$, we have*

$$I^2\ \Theta(\mathfrak{r}) \le \lambda_\Theta\ \Theta(\mathfrak{r}).$$

are satisfied. Then ISDP (1.1)-(1.3) is Ulam−Hyers−Rassias stable with respect to Θ.

Proof. Let $z \in C(\mathbb{I}, \mathbb{R})$ be a solution of the inequality (4.3), i.e.,

$$\left|\frac{d^2}{d\mathfrak{r}^2} z(\mathfrak{r}) - \theta\big(\mathfrak{r}, z(\mathfrak{r}),^c D^\zeta z(\mathfrak{r}), \int_0^{\mathfrak{r}} \phi(\mathfrak{r}, \varsigma)^c D^\delta z(\varsigma) d\varsigma\big)\right| \le \epsilon\ \Theta,\ \ \mathfrak{r} \in \mathbb{I}$$

assume that y is a solution to (1.1)−(1.3), as a result, we obtain

$$y(\mathfrak{r}) = F(\mathfrak{r}, y(\mathfrak{r})) + \int_0^{\mathfrak{r}} (\varsigma - \mathfrak{r}) u(\varsigma) d\varsigma,$$

where $u \in C(\mathbb{I}, \mathbb{R})$ such that

$$u(\mathfrak{r}) = \theta\big(\mathfrak{r}, y(\mathfrak{r}), I^{2-\zeta}u(\mathfrak{r}), \int_0^{\mathfrak{r}} \phi(\mathfrak{r}, \varsigma) I^{2-\delta}u(\varsigma)d\varsigma\big).$$

Operating by I^2 on both sides of the inequality (4.3) and then integrating, we get

$$\begin{aligned}\left| z(\mathfrak{r}) - F(\mathfrak{r}, z(\mathfrak{r})) - \int_0^{\mathfrak{r}} (\varsigma - \mathfrak{r}) v(\varsigma) d\varsigma \right| &\leq \epsilon \int_0^{\mathfrak{r}} (\mathfrak{r} - \varsigma)\ \Theta(\varsigma) d\varsigma \\ &\leq \epsilon \lambda_\Theta \Theta(\mathfrak{r}),\end{aligned}$$

where $v \in C(\mathbb{I}, \mathbb{R})$ such that

$$v(\mathfrak{r}) = \theta\big(\mathfrak{r}, z(\mathfrak{r}), I^{2-\zeta}v(\mathfrak{r}), \int_0^{\mathfrak{r}} \phi(\mathfrak{r}, \varsigma) I^{2-\delta}v(\varsigma)d\varsigma\big).$$

For each $t \in \mathbb{I}$, we obtain

$$\begin{aligned}|z(\mathfrak{r}) - y(\mathfrak{r})| &= \big|z(\mathfrak{r}) - \big|F(\mathfrak{r}, y(\mathfrak{r})) + \int_0^{\mathfrak{r}} (\varsigma - \mathfrak{r}) u(\varsigma) d\varsigma\big| \\ &\leq \big|z(\mathfrak{r}) - F(\mathfrak{r}, z(\mathfrak{r})) + \int_0^{\mathfrak{r}} (\varsigma - \mathfrak{r}) v(\varsigma) d\varsigma\big| \\ &+ \big|F(\mathfrak{r}, z(\mathfrak{r})) + \int_0^{\mathfrak{r}} (\varsigma - \mathfrak{r}) v(\varsigma) ds - F(\mathfrak{r}, y(\mathfrak{r})) - \int_0^{\mathfrak{r}} (\varsigma - \mathfrak{r}) u(\varsigma) d\varsigma\big| \\ &\leq \epsilon\ \lambda_\Theta\ \Theta(\mathfrak{r}) + \big|F(\mathfrak{r}, z(\mathfrak{r})) - F(\mathfrak{r}, y(\mathfrak{r}))\big| + \int_0^{\mathfrak{r}} (\varsigma - \mathfrak{r}) |v(\varsigma) - u(\varsigma)| d\varsigma \\ &\leq \epsilon\ \lambda_\Theta\ \Theta(\mathfrak{r}) + c\ |z(\mathfrak{r})) - y(\mathfrak{r})| + \int_0^{\mathfrak{r}} (\varsigma - \mathfrak{r}) |v(\varsigma) - u(\varsigma)| d\varsigma \\ &\leq \epsilon\ \lambda_\Theta\ \Theta(\mathfrak{r}) + c\ \|z - y\| + \tau \|v - u\|.\end{aligned}$$

Indeed, from proof of Theorem 3.4, we obtain

$$\|u - v\| \leq \frac{\|\psi\|}{1 - \|\psi\|\big(\Phi\ \frac{1}{\Gamma(3-\delta)} + \frac{1}{\Gamma(3-\zeta)}\big)} \|z - y\|.$$

Then, for each $t \in \mathbb{I}$

$$\|z - y\| \leq \epsilon\ \lambda_\Theta\ \Theta(t) + c\ \|z - y\| + \frac{\tau\ \|\psi\|}{1 - \|\psi\|\big(\Phi\ \frac{1}{\Gamma(3-\delta)} + \frac{1}{\Gamma(3-\zeta)}\big)} \|z - y\|.$$

Thus

$$\|z - y\| \leq \left[1 - \big(c + \frac{\tau\ \|\psi\|}{1 - \|\psi\|\big(\Phi\ \frac{1}{\Gamma(3-\delta)} + \frac{1}{\Gamma(3-\zeta)}\big)}\big)\right] \epsilon\ \lambda_\Theta\ \Theta(t) = c_\Theta \epsilon\ \Theta(t),$$

where

$$c_\Theta = \left[1 - \big(c + \frac{\tau\ \|\psi\|}{1 - \|\psi\|\big(\Phi\ \frac{1}{\Gamma(3-\delta)} + \frac{1}{\Gamma(3-\zeta)}\big)}\big)\right] \lambda_\Theta.$$

So, the problem ISDP (1.1)-(1.3) is Ulam-Hyers-Rassias stable with respect to Θ. □

5. Discussion and examples

As special cases of our main result, in this section, we present certain existence results for some boundary value problems.

- Setting $\Upsilon_1(\varsigma) = \frac{(\tau-\varsigma)^\sigma}{\Gamma(\sigma+1)}$, $\Upsilon_2(\varsigma) = \frac{(\eta-\varsigma)^\sigma}{\Gamma(\sigma+1)}$, then using Riemann-Stieltjes integrals, we construct second order nonlocal boundary value problems with nonlocal boundary conditions.
$$\frac{d^2}{d\mathfrak{r}^2}y(\mathfrak{r}) = \theta\big(\mathfrak{r}, y(\mathfrak{r}), {}^c D^\zeta y(\mathfrak{r}), \int_0^{\mathfrak{r}} \phi(\mathfrak{r},\varsigma){}^c D^\delta y(\varsigma)d\varsigma\big),\ \mathfrak{r} \in (0,1)$$
Given the boundary condition
$$\begin{aligned} y(\tau) &= \tfrac{1}{\Gamma(\gamma)} \textstyle\int_0^\tau F_1(\varsigma, y(\varsigma))d\Upsilon_1(\varsigma), \\ y(\eta) &= \tfrac{1}{\Gamma(\gamma)} \textstyle\int_0^\eta F_2(\varsigma, y(\varsigma))\ d\Upsilon_2(\varsigma). \end{aligned}$$
Many studies have investigated this form of such boundary conditions, for example [29].
- Letting $\gamma \to 1$, $\theta(\mathfrak{r}, y(\mathfrak{r}), u(\mathfrak{r}), v(\mathfrak{r})) = -\delta\ m(\mathfrak{r})\theta(y(\mathfrak{r}))$, $F_1(\varsigma, y(\varsigma)) = F_2(\varsigma, y(\varsigma)) = y(\varsigma)$, we get a nonlocal value problem including integral condition
$$\frac{d^2}{d\mathfrak{r}^2}y(\mathfrak{r}) + \delta\ m(\mathfrak{r})\theta(y(\mathfrak{r})) = 0,\ \mathfrak{r} \in (0,1)$$
$$y(\tau) = \int_0^\tau\ y(\varsigma)d\varsigma \text{ and } y(\eta) = \int_0^\eta\ y(\varsigma)d\varsigma,$$
which is investigated in [11].
- Letting $\gamma \to 1$, $\theta(\mathfrak{r}, y(\mathfrak{r}), u(\mathfrak{r}), v(\mathfrak{r})) = -\theta(\mathfrak{r}, y) - \varpi^2 y(\mathfrak{r})$, $F_1(\varsigma, y(\varsigma)) = F_2(\varsigma, y(\varsigma)) = \varsigma y(\varsigma)$, then we have the following boundary value problem [22]
$$\frac{d^2}{d\mathfrak{r}^2}y(\mathfrak{r}) + \varpi^2 y(\mathfrak{r}) = -\theta(\mathfrak{r}, y(\mathfrak{r})),\ \mathfrak{r} \in (0,1)$$
$$y(\tau) = \int_0^\tau \varsigma\ y(\varsigma)d\varsigma \text{ and } y(\eta) = \int_0^\eta\ \varsigma\ y(\varsigma)d\varsigma,$$

Furthermore, some examples are provided to support the main findings.

Example 5.1. Consider the following boundary value problem with non-local conditions
$$\frac{d^2}{d\mathfrak{r}^2}y(\mathfrak{r}) = \frac{3 + y(\mathfrak{r}) + {}^c D^{\frac{4}{3}} y(\mathfrak{r}) + \int_0^1 e^{\mathfrak{r}-\varsigma}\ {}^C D^{\frac{3}{2}} y(\varsigma)d\varsigma}{3e^{\mathfrak{r}+1}\big(1 + y(\mathfrak{r}) + {}^c D^{\frac{4}{3}} y(\mathfrak{r}) + \int_0^1 e^{\mathfrak{r}-\varsigma}\ {}^C D^{\frac{3}{2}} y(\varsigma)ds\big)},\ \mathfrak{r} \in \mathbb{I} \tag{5.1}$$
$$y(\frac{1}{3}) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^{\frac{1}{3}} (\frac{1}{3} - \varsigma)^{\frac{1}{2}}\ \frac{|y|}{30(\varsigma+2)(1+|y|)}\ d\varsigma, \tag{5.2}$$
$$y(\frac{2}{3}) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^{\frac{2}{3}} (\frac{2}{3} - \varsigma)^{\frac{1}{2}}\ \frac{y}{20e^{-\varsigma+2}(1+y)}\ d\varsigma, \tag{5.3}$$

Consider

$$\theta(\mathfrak{r}, u, v, w) = \frac{3+|u|+|v|+|w|}{3e^{\mathfrak{r}+1}\big(1+|u|+|v|+|w|\big)},$$

Obviously, θ is jointly continuous.
Indeed, for any $u_1.v_1, w_1, u_2.v_2, w_2 \in R$ and $\mathfrak{r} \in \mathbb{I}$

$$|\theta(\mathfrak{r}, u, v, w) - \theta(\mathfrak{r}, u_1, v_1, w_1)| \leq \frac{1}{3e}\big(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|\big)$$

As a result, condition (H_2) is met with $\|\psi\| = \frac{1}{3e}$. Also, we obtain

$$|\theta(\mathfrak{r}, u, v, w)| = \frac{1}{3e^{\mathfrak{r}+1}}\big(3+|u|+|v|+|w|\big),$$

Thus condition (H_4) is satisfied with $\theta(\mathfrak{r}, 0, 0, 0) = \frac{1}{e^{\mathfrak{r}+1}}$, $\psi(\mathfrak{r}) = \frac{1}{3e^{\mathfrak{r}+1}}$.
Set $F_1(\mathfrak{r}, x(\mathfrak{r})) = \frac{|x|}{30(\mathfrak{r}+2)(1+|x|)}$ and $F_2(\mathfrak{r}, y(\mathfrak{r})) = \frac{y}{10e^{-\mathfrak{r}+2}(1+y)}$

$$\begin{aligned}|F_1(\mathfrak{r}, x(\mathfrak{r})) - F_1(\mathfrak{r}, y(\mathfrak{r}))| &\leq \Big|\frac{|x|}{30(\mathfrak{r}+2)(1+|x|)} - \frac{|y|}{30(\mathfrak{r}+2)(1+|y|)}\Big| \\ &\leq \frac{1}{30}\,|x(\mathfrak{r}) - y(\mathfrak{r})|,\end{aligned}$$

and

$$\begin{aligned}|F_2(\mathfrak{r}, x(\mathfrak{r})) - F_2(\mathfrak{r}, y(\mathfrak{r}))| &\leq \Big|\frac{x}{20e^{-\mathfrak{r}+2}(1+x)} - \frac{y}{20e^{-\mathfrak{r}+2}(1+y)}\Big| \\ &\leq \frac{1}{20e}\,|x(\mathfrak{r}) - y(\mathfrak{r})|.\end{aligned}$$

Hence, the condition (H_1) is satisfied with $k_1 = \frac{1}{30}$ and $k_2 = \frac{1}{20e}$.
We shall check that condition 3.14, Indeed

$$\frac{\|\psi\|\ \tau}{\mathfrak{M}} + \frac{(\mathfrak{r}-\eta)k_1 + (\tau-\mathfrak{r})k_2}{(\tau-\eta)\Gamma(\gamma+1)} \approx 0.582908 < 1,$$

where $\mathfrak{M} = 1 - \|\psi\|\ \big(\frac{1}{\Gamma(\frac{5}{3})} + \frac{\Phi}{\Gamma(\frac{3}{2})}\big) = 0.2320544711$ is satisfied with $\delta = \frac{3}{2}$, $\zeta = \frac{4}{3}$, $\|\psi\| = \frac{1}{2e}$ and $\Phi = e$. It follows from Theorem 3.3 that the ISDP (5.1)-(5.3) has at least one solution on $\mathbb{I}$.

Example 5.2. Consider the following ISDP:

$$\frac{d^2}{d\mathfrak{r}^2}y(\mathfrak{r}) = \frac{e^{-\mathfrak{r}}}{e^{\mathfrak{r}}+8}\left(\frac{|y(\mathfrak{r})|}{1+|y(\mathfrak{r})} - \frac{|{}^cD^{\frac{4}{3}}y(\mathfrak{r})|}{1+|{}^cD^{\frac{4}{3}}y(\mathfrak{r})} - \frac{|\int_0^1 \ln(\mathfrak{r}+\varsigma){}^cD^{\frac{5}{3}}y(\varsigma)|\ d\varsigma}{1+|\int_0^1 \ln(\mathfrak{r}+\varsigma){}^cD^{\frac{5}{3}}y(\varsigma)|\ d\varsigma}\right) \quad (5.4)$$

$$y(\frac{3}{4}) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^{\frac{3}{4}} (\frac{3}{4}-\varsigma)^{\frac{1}{2}-1}\ \frac{|y(\varsigma)|}{20}\ d\varsigma, \quad (5.5)$$

$$y(\frac{1}{4}) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^{\frac{1}{4}} (\frac{1}{4}-\varsigma)^{\frac{1}{2}-1}\ \frac{e^{-y(\varsigma)}}{30}\ d\varsigma. \quad (5.6)$$

Set

$$\theta(\mathfrak{r}, u, v, w) = \frac{e^{-\mathfrak{r}}}{e^{\mathfrak{r}}+8}\left(\frac{|u(\mathfrak{r})|}{1+|u(\mathfrak{r})|} - \frac{|v(\mathfrak{r})|}{1+|v(\mathfrak{r})|} - \frac{|w(\mathfrak{r})|}{1+|w(\mathfrak{r})|}\right),$$

Obviously, θ is jointly continuous. Indeed, for any $u_1, v_1, w_1, u_2, v_2, w_2 \in \mathbb{R}$ and $\mathfrak{r} \in \mathbb{I}$

$$\begin{aligned} |\theta(\mathfrak{r}, u, v, w) - \theta(\mathfrak{r}, u_1, v_1, w_1)| &\leq \frac{e^{-\mathfrak{r}}}{e^{\mathfrak{r}} + 8}\big(|u - u_1| + |v - v_1| + |w - w_1|\big) \\ &\leq \frac{1}{9}\big(|u - u_1| + |v - v_1| + |w - w_1|\big). \end{aligned}$$

Hence the condition (H_2) holds with $\|\psi\| = \frac{1}{9}$.
Also, we obtain

$$\begin{aligned} &|\mathcal{F}(\mathfrak{r}, x(\mathfrak{r})) - \mathcal{F}(\mathfrak{r}, y(\mathfrak{r}))| \\ \leq\ & \frac{1}{\Gamma(\frac{1}{2})} \int_0^{\frac{3}{4}} (\frac{3}{4} - s)^{\frac{1}{2}} \frac{||x(\varsigma)| - |y(\varsigma)||}{20}\ d\varsigma + \frac{1}{\Gamma(\frac{1}{2})} \int_0^{\frac{1}{4}}\ (\frac{1}{4} - s)^{\frac{1}{2}}\ \frac{|e^{-x(\varsigma)} - e^{-y(\varsigma)}|}{30}\ d\varsigma \\ \leq\ & 0.0258587\ |x(\varsigma) - y(\varsigma)|. \end{aligned}$$

Then, lemma 3.2 satisfied, implying that $\mathcal{F}$ is lipschitz with constant $c = 0.0258587$.

We check out (3.14). Indeed

$$\big(c + \frac{\tau\|\psi\|}{\mathfrak{M}}\big) \approx 0.3069135234 < 1,$$

where

$$\mathfrak{M} = 1 - \|\psi\|\ \big(\frac{1}{\Gamma(\frac{5}{3})} + \frac{\Phi}{\Gamma(\frac{4}{3})}\big) = 0.7906721526,$$

is satisfied with $\delta = \frac{3}{2}$, $\zeta = \frac{4}{3}$, $c = 0.0258587$, $\|\psi\| = \frac{1}{9}$, $\Phi = \ln(2)$. It follows from Theorem 3.4 that the the ISDP (5.4)-(5.6) has a unique solution on $\mathbb{I}$.

6. Conclusion

We have established the existence and uniqueness of solutions for boundary value problems of implicit second-order differential equation. The required results are obtained using Krasnoselskii's fixed point theorem and Banach contraction principle. Finally, we have presented some examples to prove the validity of our theoretical conclusions . By varying the integral boundary condition parameters, our findings also have some additional implications for previously studied research. The discussion of Hyers-Ulam stability for implicit ψ-Caputo fractional differential equations via integral boundary conditions will be our future work.

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