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APPROXIMATION OF A FUNCTION HAVING BOUNDED DERIVATIVES UPTO THE SECOND ORDER BY SINE-COSINE WAVELET EXPANSION AND ITS APPLICATIONS

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Abstract. In this paper, sine-cosine wavelet has been introduced and the approximation errors of the function $f(t)$ whose first and second derivatives are bounded have been estimated using this wavelet and it is used to solve some linear differential equations. Solution obtained by this method is compared with Euler's method and with exact solution. We observe that the solution obtained by this method is better than the solution given by the Euler's method which shows the usefulness of this method.

1. Introduction

Different types of orthogonal functions are widely used in approximation theory. In which some of the well known basic orthogonal functions are the walsh function, block-pulse functions, Laguerre, Chebyshev, Legendre polynomials, Jacobi polynomials, etc. Also, in the Fourier Analysis, we use sine-cosine functions as orthogonal functions. Due to the continuous nature of the orthogonal polynomials and sine-cosine functions, the approximations obtained using these types of function are better than the approximation obtained by piecewise constant functions. Various problems of the dynamical systems have been solved by using these orthogonal functions. When we use these techniques to solve these types of problems then the problems are changed into system of algebraic equations. This is the beauty of this technique.

The authors like Chen and Hsiao 1975 [1], Chen et al. [15], Hwang and shish [2], Chang and Wang [3], Horng and Chou [4], Razzaghi and Razzaghi [5], Paraskevopoulos

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et al. [12] have discussed about walsh functions, block-pulse functions, Lagurre polynomial with series expansion, Legendre polynomials, Chebyshev polynomials, Fourier and Bessel series respectively. Our choice of orthogonal polynomials depend upon the types of signals which we have to analyse. All the orthogonal polynomial discussed so for having support on the interval of the form $[\gamma, \delta]$. Due to the global support on the interval $[\gamma, \delta]$, these types of orthogonal polynomials are not very useful for analyzing the signals which are changing in very short interval of time such as transient signals.

To overcome the difficulities in analyzing the signals as discussed above, the wavelet theory plays a very important role. In 1996, Gu and Jiang [10] developed the Haar wavelet operational matrix of integration. Razzaghi and Yousefi [13] derived the Legendre wavelet operational matrix of integration. Razzaghi and Razzaghi [6] have discussed instabilities in the solution of heat conduction problem using Taylor series and alternative approaches, Schechter [14] have explained the variation method in engineering, Mohan and Datta [11] have analyzed identification via Fourier series for a class of lumped and distributed parameter system, Endow [9] provided optimal control via Fourier series operational matrix of integration, Chen and Hsiao [7] have studied the Haar wavelet method for solving lumped and distributed-parameter systems, Chen and Tsay [8] have analyzed the Walsh operational matrices for fractional calculus and their application to distributed parameter systems. Now a days, many authors use these types of operational matrices to solve different types of problem of mathematical analysis.

Chen and Hsiao [1] solved the variational problems using the walsh series and the solution obtained by them was discontinuous due to the discontinuous nature of walsh function. Chang and Wang [3], Hwang et al. [16] obtained the continuous solution of some problems given in Chen et al. [15]using the Legendre, Chebyshev, and Laguerre polynomials.

In this present paper, we have discussed the sine-cosine wavelet and its operational matrix of integration. This technique has many advantages in solving several problems in engineering. This operational matrix of integration can be used to solve different types of problems of mathematical analysis as other kinds of orthogonal polynomials. This paper is groomed as follows.
Section 1 is introductry. In Section 2, some basic definitions have been discussed along with sine-cosine wavelet. Section 3 and Section 4 contains the theorems and their proofs respectively. Section 5 is designing to discuss the numerical accuracy of the theorems with the help of numerical example. Section 6, brief procedure for forming operational matrix of integration is introduced. Section 7 is related to the solution of differential equations using this technique. Finally, in Section 8 some conclusions have been given.

2. Preliminaries

2.1. Sine-Cosine wavelets. Sine-Cosine wavelets $\psi_{n,m}(t) = \psi(n, k, m, t)$ have four arguments, n, k, m & t, where $n = 0, 1, 2, 3, ...2^k - 1$, $k = 0, 1, 2, ...$, and m is a non negative integer, and t is normalized time and they are defined as

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k+1}{2}} f_m(2^k t - n), & \dfrac{n}{2^k} \leq \text{t} < \dfrac{n+1}{2^k}; \\ 0, & otherwise. \end{cases}$$

with

$$f_m(t) = \begin{cases} \dfrac{1}{\sqrt{2}}, & m = 0; \\ \cos(2m\pi t), & m = 1, 2, ..., L; \\ \sin 2(m-L)\pi t), & m = L+1, L+2, ..., 2L. \end{cases}$$

Here L is any positive integer. We can easily see that the system of sine-cosine wavelets form an orthonormal set.

2.2. Function expansion. A function $f(t)$ defined over [0 1) can be expanded as follows

$$f(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(t),$$

where

$$c_{n,m} = (f(t), \psi_{n,m}(t)). \tag{2.1}$$

If we cut-off the above infinite series in the form of partial sums $S_{2^k,\ 2L+1}$, then we can write,

$$S_{2^k,\ 2L+1} = \sum_{n=0}^{2^k-1}\sum_{m=0}^{2L} c_{n,m}\psi_{n,m}(t) = C^T\psi(t).$$

Here C and $\psi(t)$ are the column vectors of order $2^k(2L+1) \times 1$ which are given by

$$C = [c_{0,0}, c_{0,1}, ...c_{0,2L}, c_{1,0}, ...c_{1,2L}, ..., c_{2^k-1,0}, ...c_{2^k-1,2L}]^T,$$

and

$$\psi(t) = [\psi_{0,0}(t), \psi_{0,1}(t), ...\psi_{0,2L}(t), \psi_{1,0}(t), ...\psi_{1,2L}(t), ..., \psi_{2^k-1,0}(t), ...\psi_{2^k-1,2L}(t)]^T.$$

2.3. Wavelet approximation. We define $||f||_2 = \left[\int_0^1 |f(t)|^2 dt\right]^{\frac{1}{2}}$. The Wavelet Approximation $E_{2^{k-1},M}(f)$ of f by $S_{2^{k-1},M}$ of its sine-cosine expansion under the norm $||.||_2$ is defined by

$$E_{2^{k-1},M}(f) = \inf_{S_{2^{k-1}},M} ||f - S_{2^{k-1},M}||_2, (Zygmund[13]).$$

If $E_{2^{k-1},M}(f) \to 0$ as $k \to \infty$, $M \to \infty$ then $E_{2^{k-1},M}(f)$ is called the best wavelet approximation of f (Zygmund[13]).

3. Main Results

Theorem 3.1. *Let $f(t)$ be a function belonging to $L^2[0,1]$ such that $f'(t)$ is bounded i.e $|f'(t)| \le N_1$, $\forall\ t\ \in [0,1]$. Let sine-cosine wavelet expansion of f is*

$$f(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(t). \tag{3.1}$$

Then the sine-cosine wavelet approximation error $E^{(1)}_{2^k,2L+1}(f)$ of f is given by $S_{2^k,2L+1}$ is

$$E^{(1)}_{2^k,2L+1}(f) = inf||f - S_{2^k,2L+1}||_2 = \begin{cases} O\left(\frac{1}{2^k}\right), & m = 0; \\ O\left(\frac{1}{2^k\sqrt{2L+1}}\right), & m > 0 \end{cases}$$

Theorem 3.2. *Let f be a function belonging to $L^2[0,1]$ such that $f''(t)$ is bounded i.e $|f''(t)| \leq N_2$, $\forall \quad t \in [0,1]$. Let sine-cosine wavelet expansion of f is given by (3.1). Suppose $(2^k, 2L+1)^{th}$ partial sums of the series (3.1) is*

$$S_{2^k,2L+1} = \sum_{n=0}^{2^k-1} \sum_{m=0}^{2L} c_{n,m}\psi_{n,m}. \tag{3.2}$$

Then the sine-cosine wavelet approximation error of f by $S_{2^k,2L+1}$ is given by

$$E^{(2)}_{2^k,2L+1}(f) = inf||f - S_{2^k,2L+1}||_2 = O\left(\frac{1}{2^{2k}}\frac{1}{(2L+1)^{\frac{3}{2}}}\right). \tag{3.3}$$

4. Proof of Theorem 3.1

Now,

$$\psi_{n,0}(t) = \begin{cases} 2^{\frac{k+1}{2}} f_0(2^k t - n), & \frac{n}{2^k} \leq t < \frac{n+1}{2^k}; \\ 0, & otherwise. \end{cases}$$

$$\psi_{n,0}(t) = \begin{cases} 2^{\frac{k}{2}}, & \frac{n}{2^k} \leq t < \frac{n+1}{2^k}; \\ 0, & otherwise. \end{cases}$$

Since

$$f(t) = \sum_{n=0}^{\infty} c_{n,0}\psi_{n,0} \quad \text{and} \quad S_{2^k,0} = \sum_{n=0}^{2^k-1} c_{n,0}\psi_{n,0}.$$

So,

$$\begin{aligned} c_{n,0} &= \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} f(t)2^{\frac{k}{2}}dt = 2^{\frac{k}{2}} \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} f(t)dt \\ &= 2^{\frac{k}{2}} \int_0^{\frac{1}{2^k}} f\left(\frac{n}{2^k} + u\right) du \quad \text{Here } (t = \frac{n}{2^k} + u) \\ &= 2^{\frac{k}{2}} \int_0^{\frac{1}{2^k}} \left[f\left(\frac{n}{2^k}\right) + \frac{u}{1!}f'\left(\frac{n}{2^k} + \theta u\right)\right] du; \quad 0 < \theta < 1. \end{aligned}$$

$$\begin{aligned} &= 2^{\frac{k}{2}} \int_0^{\frac{1}{2^k}} \left[f\left(\frac{n}{2^k}\right) + \frac{u}{1!}f'\left(\frac{n}{2^k} + \theta u\right)\right] du \\ &= 2^{\frac{k}{2}} \left[\frac{1}{2^k} f\left(\frac{n}{2^k}\right) + \int_0^{\frac{1}{2^k}} uf'\left(\frac{n}{2^k} + \theta u\right) du\right]. \end{aligned}$$

Therefore,

$$c_{n,0}^2 \tag{4.1}$$

$$=2^k\left[\frac{1}{2^{2k}}f^2\left(\frac{n}{2^k}\right)+\left(\int_0^{\frac{1}{2^k}}uf'\left(\frac{n}{2^k}+\theta u\right)du\right)^2+\frac{2}{2^k}f\left(\frac{n}{2^k}\right)\int_0^{\frac{1}{2^k}}uf'\left(\frac{n}{2^k}+\theta u\right)du\right].$$

$$=\frac{1}{2^k}f^2\left(\frac{n}{2^k}\right)+2^k\left(\int_0^{\frac{1}{2^k}}uf'\left(\frac{n}{2^k}+\theta u\right)du\right)^2+2f\left(\frac{n}{2^k}\right)\int_0^{\frac{1}{2^k}}uf'\left(\frac{n}{2^k}+\theta u\right)du. \tag{4.2}$$

Next,

$$\begin{aligned}
||f||_2^2 &= \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}}|f(t)|^2dt\\
&= \int_0^{\frac{1}{2^k}}\left|f\left(u+\frac{n}{2^k}\right)\right|^2du\\
&= \int_0^{\frac{1}{2^k}}\left|f\left(u+\frac{n}{2^k}\right)\right|^2du\\
&= \int_0^{\frac{1}{2^k}}f^2\left(u+\frac{n}{2^k}\right)du\\
&= \int_0^{\frac{1}{2^k}}\left[f\left(\frac{n}{2^k}\right)+uf'\left(\frac{n}{2^k}+\theta u\right)\right]^2du\\
&= \int_0^{\frac{1}{2^k}}\left[f^2\left(\frac{n}{2^k}\right)+u^2f'^2\left(\frac{n}{2^k}+\theta u\right)+2f\left(\frac{n}{2^k}\right)uf'\left(\frac{n}{2^k}+\theta u\right)\right]du\\
&= \frac{1}{2^k}f^2\left(\frac{n}{2^k}\right)+\int_0^{\frac{1}{2^k}}u^2f'^2\left(\frac{n}{2^k}+\theta u\right)du+2f\left(\frac{n}{2^k}\right)\int_0^{\frac{1}{2^k}}uf'\left(\frac{n}{2^k}+\theta u\right)du.
\end{aligned} \tag{4.3}$$

Now from equations (4.2) and (4.3), we have

$$\begin{aligned}
||e_{n,0}||_2^2 &= ||f||^2-c_{n,0}^2\\
&= \frac{1}{2^k}f^2\left(\frac{n}{2^k}\right)+\int_0^{\frac{1}{2^k}}u^2f'^2\left(\frac{n}{2^k}+\theta u\right)du+2f\left(\frac{n}{2^k}\right)\int_0^{\frac{1}{2^k}}uf'\left(\frac{n}{2^k}+\theta u\right)du\\
&\quad -\frac{1}{2^k}f^2\left(\frac{n}{2^k}\right)-2^k\left(\int_0^{\frac{1}{2^k}}uf'\left(\frac{n}{2^k}+\theta u\right)du\right)^2-\\
&\qquad 2f\left(\frac{n}{2^k}\right)\int_0^{\frac{1}{2^k}}uf'\left(\frac{n}{2^k}+\theta u\right)du
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\frac{1}{2^k}} u^2 f'^2\left(\frac{n}{2^k}+\theta u\right)du - 2^k\left[\int_0^{\frac{1}{2^k}} uf'\left(\frac{n}{2^k}+\theta u\right)du\right]^2 \\
&\leq \int_0^{\frac{1}{2^k}} u^2\left|f'\left(\frac{n}{2^k}+\theta u\right)\right|^2 du + 2^k\left[\int_0^{\frac{1}{2^k}} u\left|f'\left(\frac{n}{2^k}+\theta u\right)\right|du\right]^2 \\
&\leq N_1^2\int_0^{\frac{1}{2^k}} u^2du + 2^kN_1^2\left[\int_0^{\frac{1}{2^k}} udu\right]^2, \ \left(\because \left|f'\left(\frac{n}{2^k}+\theta u\right)\right| \leq N_1\right) \\
&= \frac{N_1^2}{3}\frac{1}{2^{3k}} + \frac{N_1^2 2^k}{4}\frac{1}{2^{4k}} = \frac{7N_1^2}{12}\frac{1}{2^{3k}}.
\end{aligned}
$$

Hence,

$$||e_{n,0}||_2^2 = ||f||_2^2 - c_{n,0}^2 \leq \frac{7N_1^2}{12}\frac{1}{2^{3k}}.$$

Therefore,

$$||f - S_{2^k,0}||_2^2 = \sum_{n=0}^{2^k-1}||e_{n,0}||_2^2 \leq \sum_{n=0}^{2^k-1}\left(\frac{7N_1^2}{12}\right)\frac{1}{2^{3k}} = \frac{7N_1^2}{12}\frac{1}{2^{3k}}2^k = \frac{7N_1^2}{12}\frac{1}{2^{2k}},$$

and so

$$||f - S_{2^k,0}||_2 = O\left(\frac{1}{2^k}\right).$$

Thus,

$$E_{2^k,0} = inf||f - S_{2^k,0}||_2 = O\left(\frac{1}{2^k}\right).$$

Now, for $m = 1, 2, ..., L$, we have

$$
\begin{aligned}
c_{n,m} &= \int_{-\infty}^{\infty} f(t)\psi_{n,m}(t)dt \\
&= \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} f(t)2^{\frac{k+1}{2}}f_m(2^kt-n)dt \\
&= 2^{\frac{k+1}{2}}\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} f(t)f_m(2^kt-n)\,dt \\
&= 2^{\frac{k+1}{2}}\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} f(t)cos2m\pi(2^kt-n)\,dt, \ for, m = 1, 2, ...L \\
&= 2^{\frac{k+1}{2}}\left(f(t)\frac{sin2m\pi(2^kt-n)}{2m\pi(2^k)}\right)_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} - \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} f'(t)\frac{sin2m\pi(2^kt-n)}{2m\pi(2^k)}dt \\
&= 2^{\frac{k+1}{2}}\times 0 - 2^{\frac{k+1}{2}}\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} f'(t)\frac{sin2m\pi(2^kt-n)}{2m\pi(2^k)}dt \\
&= -\frac{2^{\frac{k+1}{2}}}{2m\pi(2^k)}\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} f'(t)\sin 2m\pi(2^kt-n)dt
\end{aligned}
$$

$$= -\frac{1}{\sqrt{2}m\pi(2^k)}\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} f'(t)\sin 2m\pi(2^kt-n)dt.$$

Let $2m\pi(2^kt-n)=u$ and simplifying the above equation, we get

$$c_{n,m} = -\frac{1}{2\sqrt{2}m^2\pi^2(2^{\frac{3k}{2}})}\int_0^{2m\pi} f'\left(\frac{u+2m\pi n}{2^{k+1}m\pi}\right)\sin(u)du.$$

Taking modulus on both side of above equation, we get

$$\begin{aligned}
|c_{n,m}| &= \left|-\frac{1}{2\sqrt{2}m^2\pi^2(2^{\frac{3k}{2}})}\int_0^{2m\pi} f'\left(\frac{u+2m\pi n}{2^{k+1}m\pi}\right)\sin(u)du\right| \\
&\leq \frac{1}{2\sqrt{2}m^2\pi^2(2^{\frac{3k}{2}})}\int_0^{2m\pi} \left|f'\left(\frac{u+2m\pi n}{2^{k+1}m\pi}\right)\right| |sin(u)|du \\
&\leq \frac{1}{2\sqrt{2}m^2\pi^2(2^{\frac{3k}{2}})} N_1 \int_0^{2m\pi} |sin(u)|du \\
&\leq \frac{N_1}{\sqrt{2}m^2\pi^2(2^{\frac{3k}{2}})}. \qquad (4.4)
\end{aligned}$$

Again, for $m=L+1, L+2, ..., 2L$, we have

$$\begin{aligned}
c_{n,m} &= \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} f(t)2^{\frac{k+1}{2}} f_m(2^kt-n)dt \\
&= 2^{\frac{k+1}{2}}\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} f(t)\, sin[2(m-L)\pi(2^kt-n)]dt \\
&= 2^{\frac{k+1}{2}}\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} f(t)\, sin[2^{k+1}(m-L)\pi.t-2(m-L)n\pi]dt \\
&= 2^{\frac{k+1}{2}}\int_0^{2(m-L)n\pi} f\left[\frac{u+2(m-L)n\pi}{2^{k+1}(m-L)\pi}\right] sin(u)\frac{du}{2^{k+1}(m-L)\pi}.
\end{aligned}$$

Let $2(m-L)(2^kt-n)\pi=u$ and simplifying the above equation, we get

$$\begin{aligned}
&c_{n,m} \\
&= \frac{2^{\frac{k+1}{2}}}{2^{k+1}(m-L)\pi}\int_0^{2(m-L)n\pi} f\left[\frac{u+2(m-L)n\pi}{2^{k+1}(m-L)\pi}\right] sin(u)\, du \\
&= \frac{\sqrt{2}}{2\times 2^{\frac{k}{2}}(m-L)\pi}\int_0^{2(m-L)\pi} f\left[\frac{u+2(m-L)n\pi}{2^{k+1}(m-L)\pi}\right] sin(u)\, du \\
&= \frac{1}{2^{\frac{k+1}{2}}(m-L)\pi}\left[0+\int_0^{2(m-L)\pi} f'\left(\frac{u+2(m-L)n\pi}{2^{k+1}(m-L)\pi}\right)\frac{1}{2^{k+1}(m-L)\pi}cos(u)\, du\right] \\
&= \frac{1}{2^{\frac{3(k+1)}{2}}(m-L)^2\pi^2}\left[\int_0^{2(m-L)\pi} f'\left(\frac{u+2(m-L)n\pi}{2^{k+1}(m-L)\pi}\right)cos(u)\, du\right].
\end{aligned}$$

Taking the modulus on both sides of above, we get

$$\begin{aligned} |c_{n,m}| &\leq \frac{N_1}{2^{\frac{3(k+1)}{2}}} \frac{1}{(m-L)^2\pi^2} \left[2(m-L)\pi\right] \\ &= \frac{2N_1}{2^{\frac{3(k+1)}{2}}(m-L)\pi}. \end{aligned} \tag{4.5}$$

Since

$$\begin{aligned} & ||f - S_{2^k,2L+1}||_2^2 \\ = & \sum_{n=0}^{2^k-1} \sum_{m=2L+1}^{\infty} |c_{n,m}|^2 \\ = & \sum_{n=0}^{2^k-1} \left[\sum_{m=2L+1}^{3L} |c_{n,m}|^2 + \sum_{m=3L+1}^{4L} |c_{n,m}|^2 + \sum_{m=4L+1}^{5L} |c_{n,m}|^2 + \sum_{m=5L+1}^{6L} |c_{n,m}|^2 + ... \right] \end{aligned}$$

$$\begin{aligned} = & \sum_{n=0}^{2^k-1} \left[\frac{1}{2L+1} + \frac{1}{2L+1} \right] \frac{1}{2} \frac{N_1^2}{2^{3k}\pi^2} \quad \text{(using the equations(4.4)and(4.5))} \\ = & \frac{1}{2} \frac{N_1^2}{2^{2k}\pi^2} \frac{2}{2L+1} \\ = & \frac{N_1^2}{\pi^2\, 2^{2k}\,(2L+1)}. \end{aligned}$$

Thus

$$\|f - S_{2^k,2L+1}\|_2 = O\left(\frac{1}{2^k\sqrt{2L+1}}\right).$$

4.1. Proof of Theorem 3.2. By equation (3.1), it follows that

$$\begin{aligned} c_{n,m} &= \frac{-\sqrt{2}}{2m\pi 2^{\frac{k}{2}}} \int_0^{2m\pi} f'\left(\frac{u+2m\pi}{2^{k+1}m\pi}\right) \frac{1}{2^{k+1}m\pi} \sin(u)du \\ &= \frac{-\sqrt{2}}{2m\pi 2^{\frac{k}{2}}} \frac{1}{2^{k+1}m\pi} \int_0^{2m\pi} f'\left(\frac{u+2m\pi}{2^{k+1}m\pi}\right) sin(u)du \\ &= \frac{-\sqrt{2}}{2m\pi 2^{\frac{k}{2}}} I_1 \end{aligned} \tag{4.6}$$

where,

$$\begin{aligned} I_1 &= \int_0^{2m\pi} f'\left(\frac{u+2m\pi}{2^{k+1}m\pi}\right) sin(u)\,du \\ &= \int_0^{2m\pi} f''\left(\frac{u+2m\pi}{2^{k+1}m\pi}\right) \frac{1}{2^{k+1}m\pi} cos(u)du \\ &= \frac{1}{2^{k+1}m\pi} \int_0^{2m\pi} f''\left(\frac{u+2m\pi}{2^{k+1}m\pi}\right) cos(u)du. \end{aligned}$$

Putting the value of I_1 in equation (4.6), we have

$$\begin{aligned} c_{n,m} &= \frac{-\sqrt{2}}{2m\pi 2^{\frac{k}{2}}} \frac{1}{2^{k+1}m\pi} \frac{1}{2^{k+1}m\pi} \int_0^{2m\pi} f''\left(\frac{u+2m\pi}{2^{k+1}m\pi}\right) cos(u)du \\ &= \frac{1}{2} \frac{\sqrt{2}}{2\ \sqrt{2}\ \sqrt{2}m^3\pi^3\ 2^{\frac{k}{2}+k+k}} \int_0^{2m\pi} f''\left(\frac{u+2m\pi}{2^{k+1}m\pi}\right) cos(u)du. \end{aligned}$$

and so

$$\begin{aligned} |c_{n,m}| &\leq \frac{N_2}{4\sqrt{2}m^3\pi^3\ 2^{\frac{5k}{2}}} 2m\pi \\ &= \frac{N_2}{2\sqrt{2}m^2\pi^2\ 2^{\frac{5k}{2}}} \quad \text{for } m = 1, 2, ..., L. \end{aligned} \tag{4.7}$$

Now, for $m = L+1, ..., 2L$, we have

$$\begin{aligned} c_{n,m} &= \frac{1}{2^{\frac{3(k+1)}{2}}} \frac{1}{(m-L)^2\pi^2} \int_0^{2(m-L)\pi} f'\left(\frac{u+2(m-L)n\pi}{2^{k+1}(m-L)\pi}\right) cos(u)du \\ &= \frac{1}{2^{\frac{3(k+1)}{2}}(m-L)^2\pi^2} \frac{1}{2^{k+1}(m-L)\pi} \\ &\qquad \left[0 - \int_0^{2(m-L)\pi} f''\left(\frac{u+2(m-L)n\pi}{2^{k+1}(m-L)\pi}\right) sin(u)du\right] \\ &= -\frac{1}{2^{\frac{5k}{2}} 2^{\frac{5}{2}} (m-L)^3\pi^3} \int_0^{2(m-L)\pi} f''\left(\frac{u+2(m-L)n\pi}{2^{k+1}(m-L)\pi}\right) sin(u)du. \end{aligned} \tag{4.8}$$

Again,

$$\begin{aligned} |c_{n,m}| &\leq -\frac{N_2}{2^{\frac{5k}{2}} 2^{\frac{5}{2}} \pi^3} \frac{1}{(m-L)^3} 2(m-L)\pi \\ &= -\frac{N_2}{2^{\frac{5k}{2}} 2^{\frac{3}{2}} \pi^2 (m-L)^2}. \end{aligned} \tag{4.9}$$

By using the equations (4.7) and (4.9), we have

$$\begin{aligned} ||f - S_{2^k,2L+1}||_2^2 &= \sum_{n=0}^{2^k-1} \sum_{m=2L+1}^{\infty} |c_{n,m}|^2 \\ &\leq \sum_{n=0}^{2^k-1} \sum_{m=2L+1}^{\infty} \frac{N_2^2}{8m^4\pi^4\ 2^{5k}} \\ &= \frac{N_2^2}{8\pi^4\ 2^{5k}} 2^k \sum_{m=2L+1}^{\infty} \frac{1}{m^4} \\ &\leq \frac{N_2^2}{8\pi^4\ 2^{4k}} \frac{1}{(2L+1)^3}. \end{aligned}$$

Hence

$$||f - S_{2^k,2L+1}||_2 = O\left(\frac{1}{2^{2k}} \frac{1}{(2L+1)^{\frac{3}{2}}}\right).$$

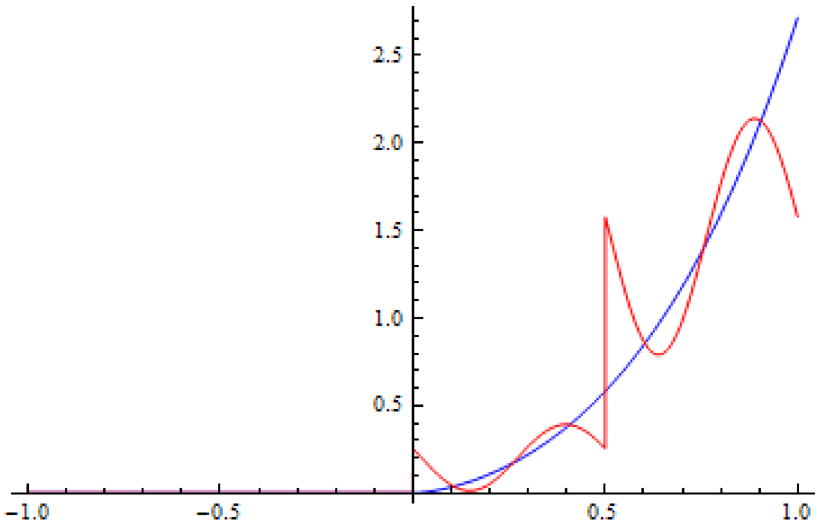
5. Numerical justification of calculated approximation

In this section, approximation of the function $f(t) = e^t t^{\frac{3}{2}}$ over interval [0,1) for $k = 1$, $L = 1$; $k = 1$, $L = 2$ and also for $k = 2$, $L = 2$ have been calculated and it is shown in the Figures 1, 2, and 3.

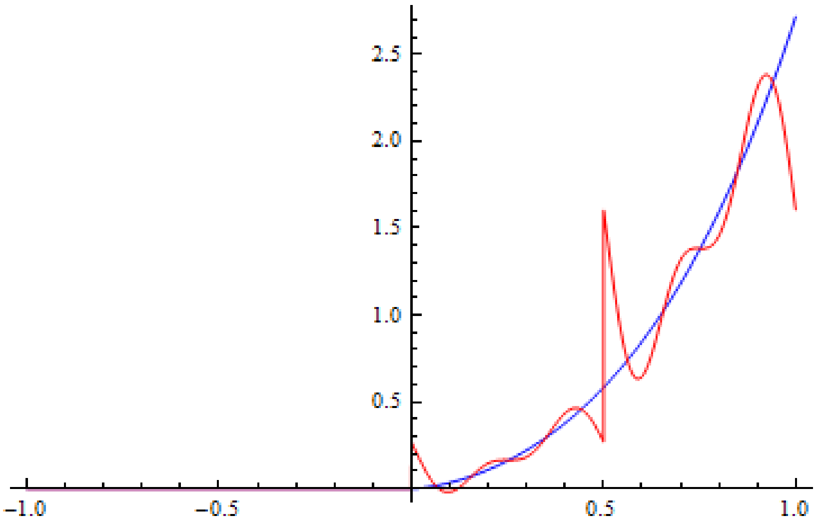
$$S_{2^1,3}(t) =$$
$$\begin{cases} 0.203243702 + 0.05235858\, cos(4\pi t) - 0.1818458\, sin(4\pi t), \;\; 0 \le t < \frac{1}{2}; \\ 1.46642633 + 0.1113816\, cos(2\pi(2t-1)) - 0.665594\, sin(2\pi(2t-1)), \;\; \frac{1}{2} \le t < 1. \end{cases}$$

$$S_{2^1,5}(t) = \begin{cases} 0.203243702 + 0.05235858\, cos(4\pi t) + 0.01361042\, cos(8\pi t) \\ -0.1818458\, sin(4\pi t) - 0.0926076\, sin(8\pi t), \;\; 0 \le t < \frac{1}{2}; \\ 1.46642633 + 0.1113816\, cos(2\pi(2t-1)) + 0.028162\, cos(4\pi(2t-1)) \\ -0.665594\, sin(2\pi(2t-1)) - 0.338076\, sin(4\pi(2t-1)), \;\; \frac{1}{2} \le t < 1. \end{cases}$$

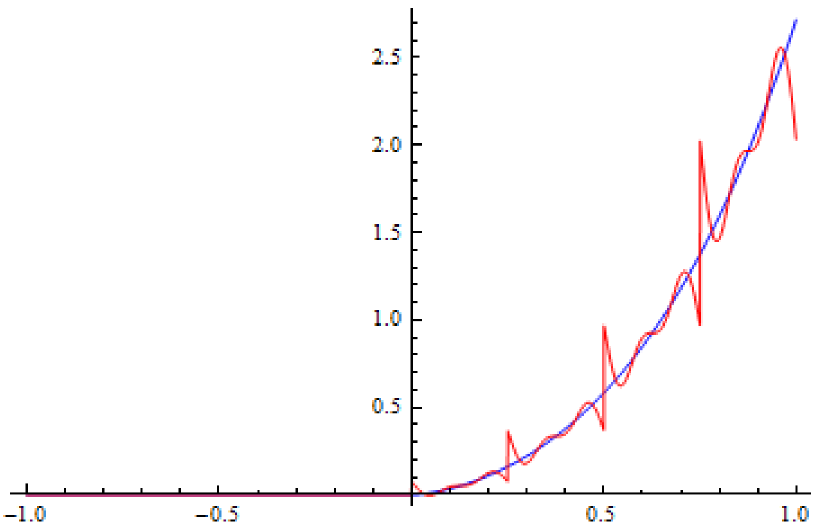
$$S_{2^2,5}(t) =$$
$$\begin{cases} 0.0299295 \times 2 + 0.00424397 \times 2^{\frac{3}{2}}\, cos(8\pi t) + 0.00111471 \times 2^{\frac{3}{2}}\, cos(16\pi t) \\ -0.0182709 \times 2^{\frac{3}{2}}\, sin(8\pi t) - 0.00909785 \times 2^{\frac{3}{2}}\, sin(16\pi t), \;\; ,0 \le t < \frac{1}{4}; \\ 0.173314 \times 2 + 0.00538005 \times 2^{\frac{3}{2}}\, cos(8\pi t - 2\pi) + 0.00135053 \times 2^{\frac{3}{2}}\, cos(16\pi t - 4\pi) \\ -0.0472126 \times 2^{\frac{3}{2}}\, sin(8\pi t - 2\pi) - 0.0237283 \times 2^{\frac{3}{2}}\, sin(16\pi t - 4\pi), \;\; \frac{1}{4} \le t < \frac{1}{2}; \\ 0.470853 \times 2 + 0.00799845 \times 2^{\frac{3}{2}}\, cos(8\pi t - 4\pi) + 0.00200566 \times 2^{\frac{3}{2}}\, cos(16\pi t - 8\pi) \\ -0.0886332 \times 2^{\frac{3}{2}}\, sin(8\pi t - 4\pi) - 0.0445084 \times 2^{\frac{3}{2}}\, sin(16\pi t - 8\pi), \;\; \frac{1}{2} \le t < \frac{3}{4}; \\ 0.995576 \times 2 + 0.011915 \times 2^{\frac{3}{2}}\, cos(8\pi t - 6\pi) + 0.0029869 \times 2^{\frac{3}{2}}\, cos(16\pi t - 12\pi) \\ -0.150422 \times 2^{\frac{3}{2}}\, sin(8\pi t - 6\pi) - 0.0754908 \times 2^{\frac{3}{2}}\, sin(16\pi t - 12\pi), \;\; \frac{3}{4} \le t < 1. \end{cases}$$



[Fig 1: Graph of $S_{2^1,3}(t)$ and function $f(t)$]



[Fig 2: Graph of $S_{2^1,5}(t)$ and function $f(t)$]



[Fig 3: Graph of $S_{2^2,5}(t)$ and function $f(t)$]

6. Sine-Cosine wavelet Operational matrix of integration

Here, the operational matrix of integration derived. We find matrix P with $k = 1$ and $L = 4$.
The 18 basis function are given by,

$$\begin{cases} \psi_{0,0}=\sqrt{2} \\ \psi_{0,1}=2cos(4\pi t) \\ \psi_{0,2}=2cos(8\pi t) \\ \psi_{0,3}=2cos(12\pi t) \\ \psi_{0,4}=2cos(16\pi t) \\ \psi_{0,5}=2sin(4\pi t) \\ \psi_{0,6}=2sin(8\pi t) \\ \psi_{0,7}=2sin(12\pi t) \\ \psi_{0,8}=2sin(16\pi t) \end{cases} \text{for } 0 \leq t < \tfrac{1}{2} \quad (6.1) \quad \text{and} \quad \begin{cases} \psi_{1,0}=\sqrt{2} \\ \psi_{1,1}=2cos2\pi(2t-1) \\ \psi_{1,2}=2cos4\pi(2t-1) \\ \psi_{1,3}=2cos6\pi(2t-1) \\ \psi_{1,4}=2cos8\pi(2t-1) \\ \psi_{1,5}=2sin2\pi(2t-1) \\ \psi_{1,6}=2sin4\pi(2t-1) \\ \psi_{1,7}=2sin6\pi(2t-1) \\ \psi_{1,8}=2sin8\pi(2t-1) \end{cases} \text{for } \frac{1}{2} \leq t < 1.$$

(6.2)

By integrating equation (6.1) from 0 to t and using in equation 2.1 we get,

$$\begin{cases} \int_0^t \psi_{0,0}(t)dt = \left[\frac{1}{4},0,0,0,0,\frac{-1}{2\sqrt{2}\pi},\frac{-1}{4\sqrt{2}\pi},\frac{-1}{6\sqrt{2}\pi},\frac{-1}{8\sqrt{2}\pi},\frac{1}{2},0,0,0,0,0,0,0,0\right]^T \psi_{18}(t), \\ \int_0^t \psi_{0,1}(t)dt = \left[0,0,0,0,0,\frac{1}{4\pi},0,0,0,0,0,0,0,0,0,0,0,0\right]^T \psi_{18}(t), \\ \int_0^t \psi_{0,2}(t)dt = \left[0,0,0,0,0,0,\frac{1}{8\pi},0,0,0,0,0,0,0,0,0,0,0\right]^T \psi_{18}(t), \\ \int_0^t \psi_{0,3}(t)dt = \left[0,0,0,0,0,0,0,\frac{1}{12\pi},0,0,0,0,0,0,0,0,0,0\right]^T \psi_{18}(t), \\ \int_0^t \psi_{0,4}(t)dt = \left[0,0,0,0,0,0,0,0,\frac{1}{16\pi},0,0,0,0,0,0,0,0,0\right]^T \psi_{18}(t), \\ \int_0^t \psi_{0,5}(t)dt = \left[\frac{1}{2\sqrt{2}\pi},-\frac{1}{4\pi},0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0\right]^T \psi_{18}(t), \\ \int_0^t \psi_{0,6}(t)dt = \left[\frac{1}{4\sqrt{2}\pi},0,-\frac{1}{8\pi},0,0,0,0,0,0,0,0,0,0,0,0,0,0,0\right]^T \psi_{18}(t), \\ \int_0^t \psi_{0,7}(t)dt = \left[\frac{1}{6\sqrt{2}\pi},0,0,-\frac{1}{12\pi},0,0,0,0,0,0,0,0,0,0,0,0,0,0\right]^T \psi_{18}(t), \\ \int_0^t \psi_{0,8}(t)dt = \left[\frac{1}{8\sqrt{2}\pi},0,0,0,-\frac{1}{16\pi},0,0,0,0,0,0,0,0,0,0,0,0,0\right]^T \psi_{18}(t). \end{cases}$$

And similarly for (6.2),

$$\begin{cases} \int_0^t \psi_{1,0}(t)dt = \left[0,0,0,0,0,0,0,0,0,\frac{1}{4},0,0,0,0,\frac{-1}{2\sqrt{2}\pi},\frac{-1}{4\sqrt{2}\pi},\frac{-1}{6\sqrt{2}\pi},\frac{-1}{8\sqrt{2}\pi}\right]^T \psi_{18}(t), \\ \int_0^t \psi_{1,1}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{4\pi},0,0,0\right]^T \psi_{18}(t), \\ \int_0^t \psi_{1,2}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{8\pi},0,0\right]^T \psi_{18}(t), \\ \int_0^t \psi_{1,3}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{12\pi},0\right]^T \psi_{18}(t), \\ \int_0^t \psi_{1,4}(t)dt = \left[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{16\pi}\right]^T \psi_{18}(t), \\ \int_0^t \psi_{1,5}(t)dt = \left[0,0,0,0,0,0,0,0,0,\frac{1}{2\sqrt{2}\pi},-\frac{1}{4\pi},0,0,0,0,0,0,0\right]^T \psi_{18}(t), \\ \int_0^t \psi_{1,6}(t)dt = \left[0,0,0,0,0,0,0,0,0,\frac{1}{4\sqrt{2}\pi},0,-\frac{1}{8\pi},0,0,0,0,0,0\right]^T \psi_{18}(t), \\ \int_0^t \psi_{1,7}(t)dt = \left[0,0,0,0,0,0,0,0,0,\frac{1}{6\sqrt{2}\pi},0,0,-\frac{1}{12\pi},0,0,0,0,0\right]^T \psi_{18}(t), \\ \int_0^t \psi_{1,8}(t)dt = \left[0,0,0,0,0,0,0,0,0,\frac{1}{8\sqrt{2}\pi},0,0,0,-\frac{1}{16\pi},0,0,0,0\right]^T \psi_{18}(t). \end{cases}$$

Thus, we can write

$$\int_0^t \psi_{18}(t)dt = P_{18\times 18}\psi_{18}(t).$$

where,

$$\begin{aligned}\psi_{18}(t) &= [\psi_{0,0}, \psi_{0,1}, \psi_{0,2}, \psi_{0,3}, \psi_{0,4}, \psi_{0,5}, \psi_{0,6}, \psi_{0,7}, \psi_{0,8}, \psi_{1,0},\\ &\qquad \psi_{1,1}, \psi_{1,2}, \psi_{1,3}, \psi_{1,4}, \psi_{1,5}, \psi_{1,6}, \psi_{1,7}, \psi_{1,8}]^T.\end{aligned}$$

and, $P_{18\times 18}$ be an operational matrix of integration which is given as,

$$P_{18\times 18}= \begin{bmatrix} A & B \\ O & A \end{bmatrix}.$$

Here

$$A_{9\times 9} = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{-1}{2\sqrt{2}\pi} & \frac{-1}{4\sqrt{2}\pi} & \frac{-1}{6\sqrt{2}\pi} & \frac{-1}{8\sqrt{2}\pi} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4\pi} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8\pi} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{12\pi} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16\pi} \\ \frac{1}{2\sqrt{2}\pi} & -\frac{1}{4\pi} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4\sqrt{2}\pi} & 0 & -\frac{1}{8\pi} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6\sqrt{2}\pi} & 0 & 0 & -\frac{1}{12\pi} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{8\sqrt{2}\pi} & 0 & 0 & 0 & -\frac{1}{16\pi} & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_{9\times 9} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$O_{9\times 9} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

7. Numerical Examples

Example 7.1. Application to first order differential equation.
Consider the differential equation,

$$y'(t) + y(t) = 1,\ y(0) = 0. \tag{7.1}$$

Exact solution of equation (7.1) is given by,

$$y(t) = (-e^{-t} + 1).$$

Now, we will solve this equation by sine-cosine wavelet method for $k = 1$, $L = 4$. Let

$$y(t) = C^T \psi(t),$$

on integrating both sides of the above equation on (0 t) and simplifying, we get

$$\int_0^t \psi(x)dx = P\psi(x).$$

Expanding $\alpha(t) = 1$, in the form basis function we have, $\alpha(t) = d^T \psi(t)$.
Where, $d = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$.
Putting $y(t) = C^T \psi(t)$ in equation (7.1), and integrating between 0 to t, and simplifying, we get,

$$C = (P^T + I)^{-1} P^T d.$$

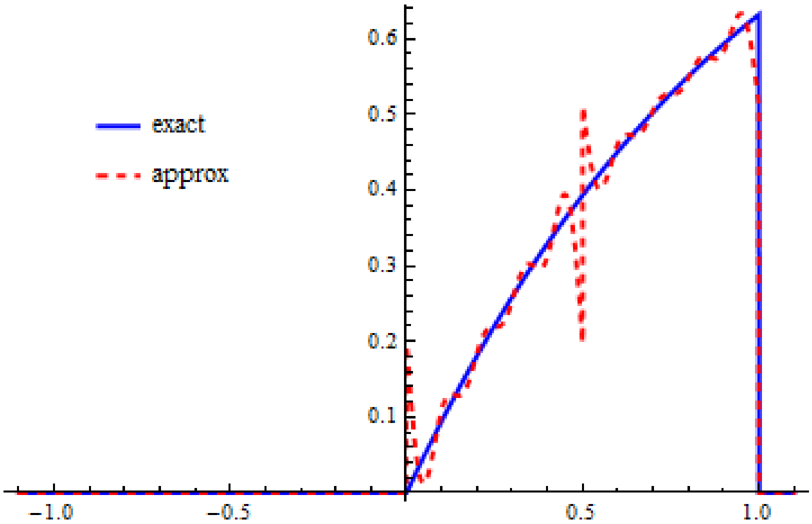
This is a system of 18 algebraic equations,which is solvable for C. After solving we obtain the unknown vector C as,

$$C = \begin{pmatrix} 0.14942700 \\ -0.0049629 \\ -0.0012466 \\ -0.0005545 \\ -0.0003120 \\ -0.0623661 \\ -0.0313309 \\ -0.0209056 \\ -0.0156840 \\ 0.36934212 \\ -0.0030058 \\ -0.0007550 \\ -0.0003358 \\ -0.0001889 \\ -0.0377727 \\ -0.0189759 \\ -0.0126617 \\ -0.0094992 \end{pmatrix}.$$

In table (1), the exact solution is compared with the solutions obtained by sine-cosine wavelet method and by Euler method with step size 0.1.

t	Approximate Solution $y(t) = C^T\psi(t)$ for $k = 1$, $L = 4$	Exact Solution y(t)=$(-e^{-t}+1)$	By Eulers Method with step size $h = 0.1$
0.0	0.18564	0.00000	0.00000
0.1	0.09714	0.09516	0.10000
0.2	0.17197	0.18126	0.19000
0.3	0.24308	0.25918	0.27100
0.4	0.30215	0.32968	0.34390
0.5	0.51138	0.39346	0.40951
0.6	0.45828	0.45118	0.46855
0.7	0.50318	0.50341	0.52170
0.8	0.54584	0.55067	0.56953
0.9	0.58129	0.59343	0.61257

Table 1. Comparison between approximate solution, exact solution and by Eulers method for $K = 1$, $L = 4$.



Comparison between Approximate solution and Exact solution through Graph.

Example 7.2. Solution of the Legendre differential equation.
In this example we have solved the Legendre differential equation for $n = 1$ with initial conditions

$$(x^2 - 1)y'' - 2xy' + 2y = 0,\ \ y(0) = 0, y'(0) = 1. \tag{7.2}$$

The exact solution for equation (7.2) is $y(t) = t$. Here we will solve the equation (7.2) by sine-cosine wavelet method.

To do this,let

$$y'' = C^T \psi(t). \tag{7.3}$$

integrating equation (7.3) between 0 to t and after simplifying, we get

$$y(t) = C^T P^2 \psi(t) + d^T P \psi(t). \tag{7.4}$$

using equations (7.2),(7.3) & (7.4) we get

$$C^T = 2d^T(\tilde{E} - P)(\tilde{F} + 2P^2 - 2P\tilde{E})^{-1}. \tag{7.5}$$

The equation (7.5), again shows the collection of 18 algebraic equations,and it is solvable for C^T.
Here we find $\tilde{E}$ and $\tilde{F}$.
For $\tilde{E}$,

$$\begin{cases} \int_0^{\frac{1}{2}} \times\sqrt{2}dx = \frac{1}{4\sqrt{2}}, \\ \int_0^{\frac{1}{2}} \times 2\cos 4\pi x dx = 0, \\ \int_0^{\frac{1}{2}} \times 2\cos 8\pi x dx = 0, \\ \int_0^{\frac{1}{2}} \times 2\cos 12\pi x dx = 0, \\ \int_0^{\frac{1}{2}} \times 2\cos 16\pi x dx = 0, \\ \int_0^{\frac{1}{2}} \times 2\sin 4\pi x dx = -\frac{1}{4\pi}, \\ \int_0^{\frac{1}{2}} \times 2\sin 8\pi x dx = -\frac{1}{8\pi}, \\ \int_0^{\frac{1}{2}} \times 2\sin 12\pi x dx = -\frac{1}{12\pi}, \\ \int_0^{\frac{1}{2}} \times 2\sin 16\pi x dx = -\frac{1}{16\pi}, \\ \int_0^{\frac{1}{2}} (x^2-1) \times \sqrt{2}dx = -\frac{11}{12\sqrt{2}}. \end{cases} \quad \textit{and} \quad \begin{cases} \int_{\frac{1}{2}}^{1} \times\sqrt{2}dx = \frac{3}{4\sqrt{2}}, \\ \int_{\frac{1}{2}}^{1} \times 2\cos 2\pi(2x-1) dx = 0, \\ \int_{\frac{1}{2}}^{1} \times 2\cos 4\pi(2x-1) dx = 0, \\ \int_{\frac{1}{2}}^{1} \times 2\cos 6\pi(2x-1) dx = 0, \\ \int_{\frac{1}{2}}^{1} \times 2\cos 8\pi(2x-1) dx = 0, \\ \int_{\frac{1}{2}}^{1} \times 2\sin 2\pi(2x-1) dx = -\frac{1}{4\pi}, \\ \int_{\frac{1}{2}}^{1} \times 2\sin 4\pi(2x-1) dx = -\frac{1}{8\pi}, \\ \int_{\frac{1}{2}}^{1} \times 2\sin 6\pi(2x-1) dx = -\frac{1}{12\pi}, \\ \int_{\frac{1}{2}}^{1} \times 2\sin 8\pi(2x-1) dx = -\frac{1}{16\pi}. \end{cases}$$

$$\tilde{E}_{18\times 18} = \begin{bmatrix} X & O \\ O & Y \end{bmatrix}.$$

where X ,O and Y are $(2L+1)\times(2L+1)$ matrices given by

$$X_{9\times 9} = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{2}\pi} & -\frac{1}{4\sqrt{2}\pi} & -\frac{1}{6\sqrt{2}\pi} & -\frac{1}{8\sqrt{2}\pi} \\ 0 & \frac{1}{4} & 0 & 0 & 0 & -\frac{1}{8\pi} & -\frac{1}{3\pi} & -\frac{3}{16\pi} & -\frac{1}{12\pi} \\ 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{6\pi} & -\frac{1}{16\pi} & -\frac{1}{4\pi} & -\frac{1}{8\pi} \\ 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{16\pi} & \frac{1}{4\pi} & 0 & -\frac{1}{4\pi} \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{12\pi} & \frac{1}{8\pi} & \frac{1}{4\pi} & 0 \\ -\frac{1}{2\sqrt{2}\pi} & -\frac{1}{8\pi} & -\frac{1}{12\pi} & \frac{1}{16\pi} & \frac{1}{12\pi} & \frac{1}{4} & 0 & 0 & 0 \\ -\frac{1}{4\sqrt{2}\pi} & -\frac{1}{3\pi} & -\frac{1}{16\pi} & \frac{1}{4\pi} & \frac{1}{8\pi} & -\frac{1}{4\pi} & \frac{1}{4} & 0 & 0 \\ 0 & -\frac{3}{16\pi} & -\frac{1}{4\pi} & 0 & \frac{1}{4\pi} & -\frac{1}{16\pi} & 0 & \frac{1}{4} - \frac{1}{6\sqrt{2}\pi} & 0 \\ -\frac{1}{8\sqrt{2}\pi} & -\frac{1}{12\pi} & -\frac{1}{8\pi} & -\frac{1}{4\pi} & 0 & 0 & 0 & 0 & \frac{1}{4} \end{bmatrix},$$

$$O_{9\times 9} = \begin{bmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{bmatrix},$$

and

$$Y_{9\times 9} = \begin{bmatrix} \frac{3}{4} & 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{2}\pi} & -\frac{1}{4\sqrt{2}\pi} & -\frac{1}{6\sqrt{2}\pi} & -\frac{1}{8\sqrt{2}\pi} \\ 0 & \frac{3}{4} & 0 & 0 & 0 & -\frac{1}{8\pi} & -\frac{1}{3\pi} & -\frac{3}{16\pi} & 0 \\ 0 & 0 & \frac{3}{4} & 0 & 0 & \frac{1}{6\pi} & -\frac{1}{16\pi} & -\frac{1}{4\pi} & -\frac{1}{8\pi} \\ 0 & 0 & 0 & \frac{3}{4} & 0 & \frac{1}{16\pi} & \frac{1}{4\pi} & 0 & -\frac{1}{4\pi} \\ 0 & 0 & 0 & 0 & \frac{3}{4} & \frac{1}{12\pi} & \frac{1}{8\pi} & \frac{1}{4\pi} & 0 \\ -\frac{1}{2\sqrt{2}\pi} & -\frac{1}{8\pi} & \frac{1}{6\pi} & \frac{1}{16\pi} & \frac{1}{12\pi} & \frac{3}{4} & 0 & 0 & 0 \\ -\frac{1}{4\sqrt{2}\pi} & -\frac{1}{4\pi} & -\frac{1}{16\pi} & \frac{1}{4\pi} & \frac{1}{8\pi} & 0 & \frac{3}{4} & 0 & 0 \\ -\frac{1}{6\sqrt{2}\pi} & -\frac{3}{16\pi} & -\frac{1}{4\pi} & 0 & \frac{1}{4\pi} & 0 & 0 & \frac{3}{4} & 0 \\ -\frac{1}{8\sqrt{2}\pi} & -\frac{1}{12\pi} & -\frac{1}{8\pi} & -\frac{1}{4\pi} & 0 & 0 & 0 & 0 & \frac{3}{4} \end{bmatrix}.$$

For $\tilde{F}$,

$$\begin{cases} \int_0^{\frac{1}{2}}(x^2-1)\times\sqrt{2}dx = -\frac{11}{12\sqrt{2}}, \\ \int_0^{\frac{1}{2}}(x^2-1)\times 2\cos 4\pi x dx = \frac{1}{8\pi^2}, \\ \int_0^{\frac{1}{2}}(x^2-1)\times 2\cos 8\pi x dx = \frac{1}{32\pi^2}, \\ \int_0^{\frac{1}{2}}(x^2-1)\times 2\cos 12\pi x dx = \frac{1}{72\pi^2}, \\ \int_0^{\frac{1}{2}}(x^2-1)\times 2\cos 16\pi x dx = \frac{1}{128\pi^2}, \\ \int_0^{\frac{1}{2}}(x^2-1)\times 2\sin 4\pi x dx = -\frac{1}{8\pi}, \\ \int_0^{\frac{1}{2}}(x^2-1)\times 2\sin 8\pi x dx = -\frac{1}{16\pi}, \\ \int_0^{\frac{1}{2}}(x^2-1)\times 2\sin 12\pi x dx = -\frac{1}{24\pi}, \\ \int_0^{\frac{1}{2}}(x^2-1)\times 2\sin 16\pi x dx = -\frac{1}{32\pi}. \end{cases} \quad \text{and} \quad \begin{cases} \int_{\frac{1}{2}}^{1}(x^2-1)\times\sqrt{2}dx = -\frac{5}{12\sqrt{2}}, \\ \int_{\frac{1}{2}}^{1}(x^2-1)\times 2\cos 2\pi(2x-1)dx = \frac{1}{8\pi^2}, \\ \int_{\frac{1}{2}}^{1}(x^2-1)\times 2\cos 4\pi(2x-1)dx = \frac{1}{32\pi^2}, \\ \int_{\frac{1}{2}}^{1}(x^2-1)\times 2\cos 6\pi(2x-1)dx = \frac{1}{72\pi^2}, \\ \int_{\frac{1}{2}}^{1}(x^2-1)\times 2\cos 8\pi(2x-1)dx = \frac{1}{128\pi^2}, \\ \int_{\frac{1}{2}}^{1}(x^2-1)\times 2\sin 2\pi(2x-1)dx = -\frac{3}{8\pi}, \\ \int_{\frac{1}{2}}^{1}(x^2-1)\times 2\sin 4\pi(2x-1)dx = -\frac{3}{16\pi}, \\ \int_{\frac{1}{2}}^{1}(x^2-1)\times 2\sin 6\pi(2x-1)dx = -\frac{1}{8\pi}, \\ \int_{\frac{1}{2}}^{1}(x^2-1)\times 2\sin 8\pi(2x-1)dx = -\frac{3}{32\pi}. \end{cases}$$

$$\tilde{F}_{18\times 18} = \begin{bmatrix} S & O \\ O & T \end{bmatrix}.$$

Where S ,O and T are $(2L+1)\times(2L+1)$ matrices given by

$$S_{9\times 9} = \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & Z \end{bmatrix}.$$

Here,

$$A_{3\times3}=\begin{bmatrix} -\frac{11}{12} & \frac{1}{4\sqrt{2}\pi^2} & \frac{1}{16\sqrt{2}\pi^2} \\ \frac{1}{4\sqrt{2}\pi^2} & -\frac{11}{12} & \frac{1}{72\pi^2} \\ \frac{1}{16\sqrt{2}\pi^2} & \frac{5}{36\pi^2} & -\frac{11}{12}+\frac{1}{128\pi^2} \end{bmatrix}, B_{3\times3}=\begin{bmatrix} \frac{1}{36\sqrt{2}\pi^2} & \frac{1}{64\sqrt{2}\pi^2} & -\frac{1}{4\sqrt{2}\pi} \\ \frac{5}{128\pi^2} & \frac{1}{72\pi^2} & -\frac{1}{16\pi} \\ \frac{1}{8\pi^2} & \frac{1}{32\pi^2} & \frac{1}{12\pi} \end{bmatrix},$$

$$C_{3\times3}=\begin{bmatrix} -\frac{1}{8\sqrt{2}\pi} & -\frac{1}{12\sqrt{2}\pi} & -\frac{1}{16\sqrt{2}\pi} \\ -\frac{1}{28\pi^2}-\frac{1}{6\pi} & -\frac{3}{32\pi} & -\frac{1}{24\pi} \\ -\frac{1}{32\pi} & -\frac{1}{8\pi} & -\frac{1}{16\pi} \end{bmatrix}, D_{3\times3}=\begin{bmatrix} \frac{1}{36\sqrt{2}\pi^2} & \frac{1}{32\pi^2} & \frac{1}{8\pi^2} \\ \frac{1}{64\sqrt{2}\pi^2} & \frac{1}{72\pi^2} & \frac{1}{32\pi^2} \\ -\frac{1}{4\sqrt{2}\pi} & -\frac{1}{16\pi} & -\frac{1}{24\pi} \end{bmatrix},$$

$$E_{3\times3}=\begin{bmatrix} -\frac{11}{12} & \frac{1}{8\pi^2} & \frac{1}{32\pi} \\ \frac{1}{8\pi^2} & -\frac{11}{12} & \frac{1}{24\pi} \\ \frac{1}{32\pi} & \frac{1}{24\pi} & -\frac{11}{12}-\frac{1}{32\pi^2} \end{bmatrix}, F_{3\times3}=\begin{bmatrix} \frac{1}{8\pi} & 0 & -\frac{1}{8\pi} \\ \frac{1}{16\pi} & \frac{1}{8\pi} & 0 \\ \frac{1}{9\pi^2} & \frac{3}{128\pi^2} & \frac{1}{72\pi^2} \end{bmatrix},$$

$$G_{3\times3}=\begin{bmatrix} -\frac{1}{8\sqrt{2}\pi} & -\frac{1}{6\pi} & -\frac{1}{32\pi} \\ 0 & -\frac{3}{32\pi} & -\frac{1}{8\pi} \\ -\frac{1}{16\sqrt{2}\pi} & -\frac{1}{24\pi} & -\frac{1}{16\pi} \end{bmatrix}, H_{3\times3}=\begin{bmatrix} \frac{1}{8\pi} & \frac{1}{16\pi} & \frac{1}{9\pi^2}-\frac{1}{8\pi} \\ 0 & \frac{1}{8\pi} & \frac{3}{128\pi^2}-\frac{1}{32\pi} \\ -\frac{1}{8\pi} & 0 & \frac{1}{72\pi^2} \end{bmatrix},$$

$$Z_{3\times3}=\begin{bmatrix} -\frac{11}{12}-\frac{1}{128\pi^2} & \frac{1}{8\pi^2} & \frac{1}{32\pi^2} \\ \frac{1}{8\pi^2} & -\frac{11}{12}-\frac{1}{12\sqrt{2}\pi} & \frac{1}{8\pi^2} \\ \frac{1}{32\pi^2} & \frac{1}{8\pi^2} & -\frac{11}{12} \end{bmatrix}.$$

$$O_{9\times9}=\begin{bmatrix} 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \end{bmatrix}.$$

$$T_{9\times9}=\begin{bmatrix} J & K & L \\ M & N & Y \\ U & Q & R \end{bmatrix}.$$

Where

$$J_{3\times3}=\begin{bmatrix} -\frac{5}{12} & \frac{1}{4\sqrt{2}\pi^2} & \frac{1}{16\sqrt{2}\pi^2} \\ \frac{1}{4\sqrt{2}\pi^2} & -\frac{5}{12}+\frac{1}{32\pi^2} & \frac{5}{36\pi^2} \\ \frac{1}{4\sqrt{2}\pi^2} & \frac{1}{8\pi^2} & -\frac{5}{12}+\frac{1}{128\pi^2} \end{bmatrix}, K_{3\times3}=\begin{bmatrix} \frac{1}{36\sqrt{2}\pi^2} & \frac{1}{64\sqrt{2}\pi^2} & -\frac{3}{4\sqrt{2}\pi} \\ \frac{5}{128\pi^2} & \frac{1}{72\pi^2} & -\frac{3}{32\pi} \\ \frac{1}{8\pi^2} & \frac{1}{32\pi^2} & \frac{11}{32\pi} \end{bmatrix},$$

$$L_{3\times3}=\begin{bmatrix} -\frac{3}{8\sqrt{2}\pi} & -\frac{1}{4\sqrt{2}\pi} & -\frac{3}{16\sqrt{2}\pi} \\ -\frac{1}{2\pi} & -\frac{15}{32\pi} & -\frac{1}{32\pi} \\ -\frac{3}{32\pi} & -\frac{3}{8\pi} & -\frac{3}{16\pi} \end{bmatrix}, M_{3\times3}=\begin{bmatrix} \frac{1}{36\sqrt{2}\pi^2} & \frac{5}{128\pi^2} & \frac{1}{8\pi^2} \\ \frac{1}{64\sqrt{2}\pi^2} & \frac{1}{72\pi^2} & \frac{1}{32\pi^2} \\ -\frac{3}{4\sqrt{2}\pi} & -\frac{3}{16\pi} & \frac{1}{4\pi} \end{bmatrix},$$

$$N_{3\times3}=\begin{bmatrix} -\frac{5}{12} & \frac{1}{8\pi^2} & \frac{3}{32\pi} \\ \frac{1}{8\pi^2} & -\frac{5}{12} & \frac{1}{8\pi} \\ \frac{3}{32pi} & \frac{1}{8\pi} & -\frac{5}{12\pi} \end{bmatrix}, Y_{3\times3}=\begin{bmatrix} \frac{3}{8\pi} & 0 & -\frac{3}{8\pi} \\ \frac{3}{16\pi} & \frac{3}{8\pi} & 0 \\ \frac{1}{9\pi^2} & \frac{3}{128\pi^2} & 0 \end{bmatrix},$$

$$U_{3\times 3} = \begin{bmatrix} -\frac{3}{8\sqrt{2}\pi} & -\frac{3}{8\pi} & -\frac{3}{32\pi} \\ -\frac{1}{4\sqrt{2}\pi} & -\frac{9}{32\pi} & -\frac{3}{8\pi} \\ -\frac{3}{16\sqrt{2}\pi} & -\frac{1}{8\pi} & -\frac{3}{16\pi} \end{bmatrix}, Q_{3\times 3} = \begin{bmatrix} \frac{3}{8\pi} & \frac{3}{16\pi} & \frac{1}{9\pi^2} \\ 0 & \frac{3}{8\pi} & \frac{3}{128\pi^2} \\ -\frac{3}{8\pi} & 0 & \frac{1}{72\pi^2} \end{bmatrix},$$

$$R_{3\times 3} = \begin{bmatrix} -\frac{5}{12\pi} - \frac{1}{128\pi^2} & \frac{1}{8\pi^2} & \frac{1}{32\pi^2} \\ \frac{1}{8\pi^2} & -\frac{5}{12} & \frac{1}{8\pi^2} \\ \frac{1}{32\pi^2} & \frac{1}{8\pi^2} & -\frac{5}{12} \end{bmatrix}.$$

After solving equation (7.5), we get

$$C^T = (0, 0, 0, 0, 0, 0, 0, 0, 0). \tag{7.6}$$

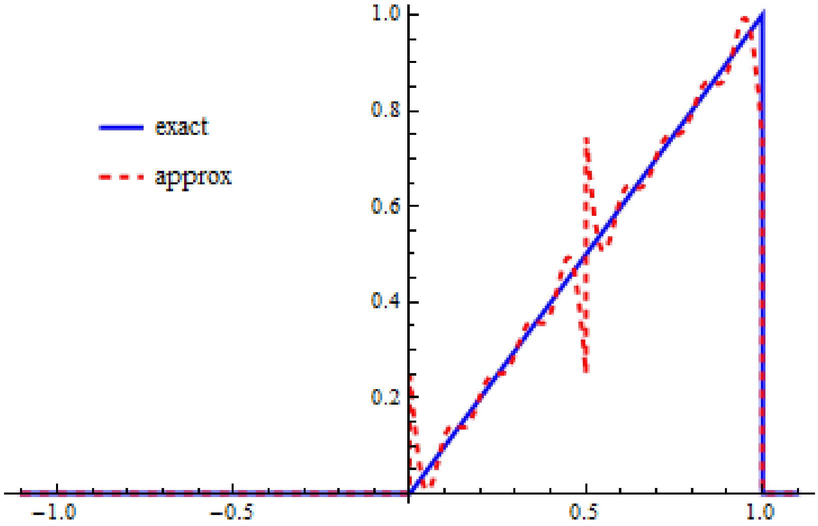
From equations (7.4) and (7.6), we get

$$y(t) = d^T P \psi(t).$$

Thus approximate solution & exact solution is given below in table (2).

t	Approximate Solution	Exact Solution
0.0	0.250000	0.000000
0.1	0.120885	0.100000
0.2	0.205066	0.200000
0.3	0.294934	0.300000
0.4	0.379115	0.400000
0.5	0.750000	0.500000
0.6	0.620885	0.600000
0.7	0.705066	0.700000
0.8	0.794934	0.800000
0.9	0.879115	0.900000

TABLE 2. Comparison between approximate solution & exact solution.



Comparison between approximate solution & exact solution through graph.

8. Conclusions

(i) From theorems (3.1) and (3.2), it follows that

$$E^{(1)}_{2^k,2L+1} = ||f - S_{2^k,2L+1}||_2 = \begin{cases} O\left(\frac{1}{2^k}\right), & m = 0; \\ O\left(\frac{1}{2^k\sqrt{2L+1}}\right), & m > 0. \end{cases}$$

and

$$E^{(2)}_{2^k,2L+1} = ||f - S_{2^k,2L+1}||_2 = O\left(\frac{1}{2^{2k}}\frac{1}{(2L+1)^{\frac{3}{2}}}\right).$$

Since $E^{(1)}_{2^k,2L+1}$ and $E^{(2)}_{2^k,2L+1} \to 0$ as $k, L \to \infty$. Therefore, the approximations obtained in theorems (3.1) and (3.2) are best possible in wavelet analysis.
(ii) Approximations calculated in theorems (3.1) and (3.2) are verified by an example in section (5) which is significant part of the paper.
(iii) These approximations are used in solving the different types of differential equations and solutions obtained by such method are compared to the exact solutions. We observe that the exact solution and approximate solution is very close to each other which is shown in the graphs.

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