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SOME PROPERTIES OF OPV-FRAMES

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Abstract . In quantum mechanics, a Hilbert space administer a structure (linear) for physical states and also provide an inner product to convey physically-measurable expectation values. Keeping this aspect in mind, in this paper, we prove some significant properties concerning operator valued frames (OPV-frames). Also, we construct OPV-frames with the help of some given operators. Finally, we consider perturbation of OPV-frames and discuss the stability of OPV-frames.

1. Introduction

A framework that can be used to describe wave functions in quantum mechanics is a Hilbert Space. Hilbert space administer a structure (linear) for physical states and also provide an inner product to convey physically- measurable expectation values. Wave functions are illustrated over a perticular number field. Assuming that there exists a despondence between the inner product of the Hilbert space and its elements, a wave function demands a scalar product as well as a Hilbert space. In this article, our aim is to study the concept of frame, introduced by Duffin and Schaeffer who while working on a problem of non-harmonic analysis introduced this notion, under the quantum mechinics (QM) setting. The concept of frame is a generalization of basis. It has been studied and developed rapidly in the past 35 odd years mainly due to its useful applications in various areas including signal and image processing. Frames were first introduced in 1952 by Duffin and Schaeffer [10] and were further developed and set in motion by Daubechies, Grossman and Meyer [9] in 1986. Since then, many generalizations like fusion frames, g-frames etc were introduced [2, 11, 18].

A generalization of the concept of vector-valued frame, that enables us to deal with the operators in a Hilbert space instead of its elements, was introduced by L.Găvruta [12]. The notion of OPV-frames, provides a more general way of series expansion of elements that is very similar to frame decomposition and have immense applications in quantum computing, packets encoding and many more. Kaftal [15] proved several

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crucial results concerning OPV-frames. Recently, Poumai at el.[16] worked on OPV-frames and highlighted its applications in quantum mechanics. For more details on OPV-frames one may refer to [14, 13, 16].

The present paper is organized as follows: To make the paper self-contained, we recall some basic definitions and results concerning OPV-frames in the remaining part of this section. In Section 2, we prove some properties of OPV-frames. Section 3 deals with the construction of an OPV-frame using operators. Finally, in Section 4, we consider perturbation of OPV-frames and obtain conditions for the stability of OPV-frames in the QM setting.

Throughout the paper, $\mathcal{H}, \mathcal{K}$ denote Hilbert spaces, $\mathbb{N}$ the set of all natural numbers and $\mathcal{I}$ an index set. The set of all bounded operators from $\mathcal{H}$ to $\mathcal{K}$ is denoted by $\mathcal{B}(\mathcal{H}, \mathcal{K})$ and identity operator on $\mathcal{H}$ is denoted by $I_{\mathcal{H}}$. We start by recalling the definition of an OPV-frame as follows:

Definition 1.1. A sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}} \subseteq \mathcal{B}(\mathcal{H}, \mathcal{K})$ is said to be an *operator valued frame (OPV-frame)* for $\mathcal{H}$ with range in $\mathcal{K}$ if there exist two positive constants A and B such that

$$AI_{\mathcal{H}} \leq \sum_{n\in\mathbb{N}} \mathcal{F}_n^* \mathcal{F}_n \leq BI_{\mathcal{H}},$$

or equivalently

$$A\|\mathfrak{h}\|^2 \leq \sum_{n\in\mathbb{N}} \|\mathcal{F}_n(\mathfrak{h})\|^2 \leq B\|\mathfrak{h}\|^2, \ \mathfrak{h} \in \mathcal{H}. \tag{1.1}$$

The constants A and B are known as lower and upper frame bounds for the OPV-frame $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$, respectively and the inequality (1.1) is known as the *frame inequality*. The sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is called a *Bessel OPV-sequence* for $\mathcal{H}$ with range in $\mathcal{K}$ if it satisfies the right hand side of the frame inequality (1.1). Also, $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is said to be a *Parseval OPV-frame* if $A = B = 1$ i.e., $\sum_{n\in\mathbb{N}} \mathcal{F}_n^* \mathcal{F}_n = I_{\mathcal{H}}$.

Let $\{e_n\}_{n\in\mathbb{N}}$ be the standard orthonormal basis of ℓ^2. For each $n \in \mathbb{N}$, define the partial isometry $|e_n\rangle_{\ell^2} : \mathcal{K} \to \mathcal{K} \otimes \ell^2$ such that

$$|e_n\rangle_{\ell^2}(k) = k \otimes e_n, \ k \in \mathcal{K},$$

and its adjoint $|e_n\rangle_{\ell^2}^* := {}_{\ell^2}\langle e_n| : \mathcal{K} \otimes \ell^2 \to \mathcal{K}$ is given by

$${}_{\ell^2}\langle e_n|(k \otimes c) = c_n k, \ k \in \mathcal{K}, c = \{c_n\}_{n\in\mathbb{N}} \in \ell^2.$$

Moreover, the *analysis operator* of an OPV-frame $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ for $\mathcal{H}$ with range in $\mathcal{K}$ is given by $\mathcal{T}_{\mathcal{F}} : \mathcal{H} \to \mathcal{K} \otimes \ell^2$ such that

$$\mathcal{T}_{\mathcal{F}}(\mathfrak{h}) = \sum_{n\in\mathbb{N}} \mathcal{F}_n(\mathfrak{h}) \otimes e_n, \ \mathfrak{h} \in \mathcal{H},$$

and its adjoint $\mathcal{T}_{\mathcal{F}}^*$, the *synthesis operator* is given by $\mathcal{T}_{\mathcal{F}}^* : \mathcal{K} \otimes \ell^2 \to \mathcal{K}$ such that

$$\mathcal{T}_{\mathcal{F}}^*(k \otimes c) = \sum_{n\in\mathbb{N}} c_n \mathcal{F}_n^*(k), \ k \in \mathcal{K}, c = \{c_n\}_{n\in\mathbb{N}} \in \ell^2.$$

Consequently, the *frame operator* of the OPV-frame $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is given by $\mathcal{S}_{\mathcal{F}} : \mathcal{H} \to \mathcal{H}$ such that $\mathcal{S}_{\mathcal{F}} = \mathcal{T}_{\mathcal{F}}^* \mathcal{T}_{\mathcal{F}} = \sum_{n\in\mathbb{N}} \mathcal{F}_n^* \mathcal{F}_n$. One can observe that the frame operator of an

OPV-frame is a bounded, self-adjoint, positive and invertible. For more details, one may refer to the following reference [16].

OPV-frames are further classified as follows:

Definition 1.2. An OPV-frame $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ for $\mathcal{H}$ with range in $\mathcal{K}$ is said to be

(1) a *Riesz OPV-frame* if $\mathcal{T}_{\mathcal{F}}(\mathcal{H}) = \mathcal{K}\otimes\ell^2$.
(2) an *orthonormal OPV-frame* if it is both Parseval and Riesz OPV-frame.

Remark 1.3. A more general definition of Riesz and orthonormal OPV-frames is given by Poumai [16] in the form of a theorem which states that an OPV-frame $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is a Riesz OPV-frame if and only if $\mathcal{F}_n\mathcal{S}_{\mathcal{F}}^{-1}\mathcal{F}_m^* = \delta_{m,n}I_{\mathcal{K}}$, where $\mathcal{S}_{\mathcal{F}}$ is the frame operator of the OPV-frame $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$.

2. Properties of operator valued frames

In this section, we shall discuss some of the properties of OPV-frames in the QM setting.
In the following result, we give a characterization of Riesz OPV-frames in terms of orthonormal OPV-frames. Moreprecisely, we prove the following result.

Theorem 2.1. *Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be an orthonormal OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ and $\mathcal{U}$ be a bounded invertible operator on $\mathcal{H}$. Then $\{\mathcal{F}_n\mathcal{U}\}_{n\in\mathbb{N}}$ forms a Riesz OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$.*

Proof. Since $\mathcal{U}:\mathcal{H}\to\mathcal{H}$ is bounded and invertible, there exist two positive constants C_1 and C_2 such that

$$C_1\|\mathfrak{h}\|^2 \le \|\mathcal{U}(\mathfrak{h})\|^2 \le C_2\|\mathfrak{h}\|^2,\ \mathfrak{h}\in\mathcal{H}.$$

For all $n\in\mathbb{N}$, let $\mathcal{G}_n := \mathcal{F}_n\mathcal{U}$. Then, we have

$$\begin{aligned}\sum_{n\in\mathbb{N}}\mathcal{G}_n^*\mathcal{G}_n &= \sum_{n\in\mathbb{N}}\mathcal{U}^*\mathcal{F}_n^*\mathcal{F}_n\mathcal{U}\\ &= \mathcal{U}^*\Big(\sum_{n\in\mathbb{N}}\mathcal{F}_n^*\mathcal{F}_n\Big)\mathcal{U}\\ &= \mathcal{U}^*\mathcal{U}.\end{aligned}$$

This gives

$$C_1I_{\mathcal{H}} \le \sum_{n\in\mathbb{N}}\mathcal{G}_n^*\mathcal{G}_n = \mathcal{U}^*\mathcal{U} \le C_2I_{\mathcal{H}}.$$

Let $\mathcal{S}_{\mathcal{G}} := \mathcal{U}^*\mathcal{U}$. Then for all $n,m\in\mathbb{N}$, we obtain

$$\mathcal{G}_n\mathcal{S}_{\mathcal{G}}^{-1}\mathcal{G}_m^* = \mathcal{F}_n\mathcal{U}\mathcal{U}^{-1}(\mathcal{U}^*)^{-1}\mathcal{U}^*\mathcal{F}_m^* = \mathcal{F}_n\mathcal{F}_m^* = \delta_{m,n}I_{\mathcal{K}}.$$

This implies that $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ forms a Riesz OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ and with frame operator $\mathcal{S}_{\mathcal{G}}$. ☐

Related to the converse of Theorem 2.1, we prove the following result.

Theorem 2.2. *Let $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ be a Riesz OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$. Then, there exists a bounded invertible operator $\mathcal{U}$ on $\mathcal{H}$ and an orthonormal OPV-frame $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ such that for each $n\in\mathbb{N}$, $\mathcal{G}_n = \mathcal{F}_n\mathcal{U}$.*

Proof. Let $\mathcal{S}_{\mathcal{G}}$ be the frame operator of $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$. Take $\mathcal{F}_n := \mathcal{G}_n\mathcal{S}_{\mathcal{G}}^{-1/2}, n \in \mathbb{N}$. Then

$$\begin{aligned}\sum_{n\in\mathbb{N}} \mathcal{F}_n^*\mathcal{F}_n &= \mathcal{S}_{\mathcal{G}}^{-1/2}\Big(\sum_{n\in\mathbb{N}} \mathcal{G}_n^*\mathcal{G}_n\Big)\mathcal{S}_{\mathcal{G}}^{-1/2} \\ &= \mathcal{S}_{\mathcal{G}}^{-1/2}\mathcal{S}_{\mathcal{G}}\mathcal{S}_{\mathcal{G}}^{-1/2} = I_{\mathcal{H}}.\end{aligned}$$

This shows that $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ forms a Parseval OPV-frame for $\mathcal{H}$. Let $\mathcal{S}_{\mathcal{F}}$ be the frame operator for $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$. Then, for all $n, m \in \mathbb{N}$, we obtain

$$\begin{aligned}\mathcal{F}_n\mathcal{F}_m^* &= \mathcal{G}_n\mathcal{S}_{\mathcal{G}}^{-1/2}\mathcal{S}_{\mathcal{G}}^{-1/2}\mathcal{G}_m^* \\ &= \mathcal{G}_n\mathcal{S}_{\mathcal{G}}^{-1}\mathcal{G}_m^* \\ &= \delta_{m,n}I_{\mathcal{K}}.\end{aligned}$$

Thus, $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ forms an orthonormal OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ such that $\mathcal{G}_n = \mathcal{F}_n\mathcal{U}$, where $\mathcal{U} := \mathcal{S}_{\mathcal{G}}^{1/2}$. □

In the following result, we give a precise relationship between two Riesz OPV-frames.

Lemma 2.3. *Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ and $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ be two Riesz OPV-frames for $\mathcal{H}$ with range in $\mathcal{K}$. Then there exists a bounded invertible operator T on $\mathcal{H}$ satisfying $\mathcal{F}_n = \mathcal{G}_nT$ or $\mathcal{G}_n = \mathcal{F}_nT$.*

Proof. In view of the proof of Theorem 2.1, there exist two bounded invertible operators $\mathcal{U}_1$ and $\mathcal{U}_2$ on $\mathcal{H}$ and an orthonormal OPV-frame $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$, $\mathcal{F}_n = \mathcal{E}_n\mathcal{U}_1$ and $\mathcal{G}_n = \mathcal{E}_n\mathcal{U}_2$. Then for all $n \in \mathbb{N}$, we have

$$\mathcal{F}_n = \mathcal{E}_n\mathcal{U}_2\mathcal{U}_2^{-1}\mathcal{U}_1 = \mathcal{G}_nT,$$

where $T := \mathcal{U}_2^{-1}\mathcal{U}_1$. □

Towards the converse of the above lemma, i.e., whether a Riesz OPV-frame composed with a bounded invertible operator turns out to be a Riesz OPV-frame, we prove the following result.

Theorem 2.4. *Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be a Riesz OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ and let $\mathcal{U}$ be a bounded invertible operator on $\mathcal{H}$. Then $\{\mathcal{F}_n\mathcal{U}\}_{n\in\mathbb{N}}$ forms a Riesz OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$.*

Proof. Let $\mathcal{S}_{\mathcal{F}}$ be the frame operator of the Riesz OPV-frame $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$. For all $n \in \mathbb{N}$, let $\mathcal{G}_n := \mathcal{F}_n\mathcal{U}$. Then, we compute

$$\begin{aligned}\sum_{n\in\mathbb{N}} \mathcal{G}_n^*\mathcal{G}_n &= \sum_{n\in\mathbb{N}} \mathcal{U}^*\mathcal{F}_n^*\mathcal{F}_n\mathcal{U} \\ &= \mathcal{U}^*\Big(\sum_{n\in\mathbb{N}} \mathcal{F}_n^*\mathcal{F}_n\Big)\mathcal{U} \\ &= \mathcal{U}^*\mathcal{S}_{\mathcal{F}}\mathcal{U}.\end{aligned}$$

Let $\mathcal{S}_{\mathcal{G}} := \mathcal{U}^*\mathcal{S}_{\mathcal{F}}\mathcal{U}$. Then, for all $n, m \in \mathbb{N}$, we obtain

$$\begin{aligned}\mathcal{G}_n\mathcal{S}_{\mathcal{G}}^{-1}\mathcal{G}_m^* &= \mathcal{F}_n\mathcal{U}\mathcal{S}_{\mathcal{G}}^{-1}\mathcal{U}^*\mathcal{F}_m^* \\ &= \mathcal{F}_n\mathcal{U}\mathcal{U}^{-1}\mathcal{S}_{\mathcal{F}}^{-1}(\mathcal{U}^*)^{-1}\mathcal{U}^*\mathcal{F}_m^* \\ &= \mathcal{F}_n\mathcal{S}_{\mathcal{F}}^{-1}\mathcal{F}_m^* \\ &= \delta_{m,n}I_{\mathcal{K}}.\end{aligned}$$

Thus, $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ forms a Riesz OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ with frame operator $\mathcal{S}_{\mathcal{G}}$. □

Next, ww give a result related to the relationship between two orthonormal OPV-frames.

Theorem 2.5. *Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ and $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ be two orthonormal OPV-frames for $\mathcal{H}$ with range in $\mathcal{K}$. Then there exists a unitary operator $\mathcal{U}$ on $\mathcal{H}$ such that $\mathcal{F}_n = \mathcal{G}_n\mathcal{U}$ and $\mathcal{T}_{\mathcal{F}} = \mathcal{T}_{\mathcal{G}}\mathcal{U}$ where $\mathcal{T}_{\mathcal{F}}$ and $\mathcal{T}_{\mathcal{G}}$ are analysis operators of $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ and $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$, respectively.*

Proof. Since $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ and $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ are orthonormal OPV-frames, we have

$$\mathcal{T}_{\mathcal{F}}^*\mathcal{T}_{\mathcal{F}} = I_{\mathcal{H}}, \quad \mathcal{T}_{\mathcal{F}}\mathcal{T}_{\mathcal{F}}^* = I_{\mathcal{K}\otimes\ell^2},$$
$$\mathcal{T}_{\mathcal{G}}^*\mathcal{T}_{\mathcal{G}} = I_{\mathcal{H}}, \quad \mathcal{T}_{\mathcal{G}}\mathcal{T}_{\mathcal{G}}^* = I_{\mathcal{K}\otimes\ell^2}.$$

Also, for all $n \in \mathbb{N}$, we compute

$$\mathcal{F}_n = {}_{\ell^2}\langle e_n|\mathcal{T}_{\mathcal{F}} = {}_{\ell^2}\langle e_n|\mathcal{T}_{\mathcal{G}}\mathcal{T}_{\mathcal{G}}^*\mathcal{T}_{\mathcal{F}} = \mathcal{G}_n\mathcal{T}_{\mathcal{G}}^*\mathcal{T}_{\mathcal{F}}.$$

This implies that $\mathcal{F}_n = \mathcal{G}_n\mathcal{U}$, where $\mathcal{U} := \mathcal{T}_{\mathcal{G}}^*\mathcal{T}_{\mathcal{F}}$ is a unitary operator.
Moreover, $\mathcal{T}_{\mathcal{F}} = \mathcal{T}_{\mathcal{G}}\mathcal{T}_{\mathcal{G}}^*\mathcal{T}_{\mathcal{F}} = \mathcal{T}_{\mathcal{G}}\mathcal{U}$. □

Concerning the converse of the above theorem, i.e., whether an orthonormal OPV-frame composed with a unitary operator turns out to be an orthonormal OPV-frame, we prove the following result.

Theorem 2.6. *Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be an orthonormal OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ and let $\mathcal{U}$ be an unitary operator on $\mathcal{H}$. Then, $\{\mathcal{F}_n\mathcal{U}\}_{n\in\mathbb{N}}$ forms an orthonormal OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$.*

Proof. For $n \in \mathbb{N}$, let $\mathcal{G}_n := \mathcal{F}_n\mathcal{U}$. Then, we have

$$\begin{aligned} \sum_{n\in\mathbb{N}} \mathcal{G}_n^*\mathcal{G}_n &= \mathcal{U}^*\Big(\sum_{n\in\mathbb{N}} \mathcal{F}_n^*\mathcal{F}_n\Big)\mathcal{U} \\ &= \mathcal{U}^*\mathcal{U} = I_{\mathcal{H}}. \end{aligned}$$

Also for $m, n \in \mathbb{N}$,

$$\mathcal{G}_n\mathcal{G}_m^* = \mathcal{F}_n\mathcal{U}\mathcal{U}^*\mathcal{F}_m^* = \mathcal{F}_n\mathcal{F}_m^* = \delta_{m,n}I_{\mathcal{H}}.$$

Thus, $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ forms an orthonormal OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$. □

Next, we prove that a Parseval OPV-frame composed with a bounded invertible operator turns out to be an OPV-frame.

Theorem 2.7. *Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be a Parseval OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ and let $\mathcal{U}$ be a bounded invertible operator on $\mathcal{H}$. Then $\{\mathcal{F}_n\mathcal{U}\}_{n\in\mathbb{N}}$ forms an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$.*

Proof. For all $n \in \mathbb{N}$, let $\mathcal{G}_n := \mathcal{F}_n\mathcal{U}$. Then, we have

$$\begin{aligned} \sum_{n\in\mathbb{N}} \mathcal{G}_n^*\mathcal{G}_n &= \mathcal{U}^*\Big(\sum_{n\in\mathbb{N}} \mathcal{F}_n^*\mathcal{F}_n\Big)\mathcal{U} \\ &= \mathcal{U}^*\mathcal{U}. \end{aligned}$$

It is easy to verify that $\mathcal{S}_{\mathcal{G}} := \mathcal{U}^*\mathcal{U}$ is a bounded and invertible operator and hence $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ forms an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ and with frame operator $\mathcal{S}_{\mathcal{G}}$. $\square$

In the following result, we prove that an OPV-frame composed with a bounded invertible operator again turns out to be an OPV-frame.

Theorem 2.8. *Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ and $\mathcal{U}$ be a bounded invertible operator on $\mathcal{H}$. Then $\{\mathcal{F}_n\mathcal{U}\}_{n\in\mathbb{N}}$ forms an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$.*

Proof. Let $\mathcal{S}_{\mathcal{F}}$ be the frame operator of $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$. For all $n \in \mathbb{N}$, let $\mathcal{G}_n := \mathcal{F}_n\mathcal{U}$. Then, we have

$$\begin{aligned}\sum_{n\in\mathbb{N}} \mathcal{G}_n^*\mathcal{G}_n &= \sum_{n\in\mathbb{N}} \mathcal{U}^*\mathcal{F}_n^*\mathcal{F}_n\mathcal{U} \\ &= \mathcal{U}^*\Big(\sum_{n\in\mathbb{N}} \mathcal{F}_n^*\mathcal{F}_n\Big)\mathcal{U} \\ &= \mathcal{U}^*\mathcal{S}_{\mathcal{F}}\mathcal{U}.\end{aligned}$$

Thus, $\mathcal{U}^*\mathcal{S}_{\mathcal{F}}\mathcal{U}$ is a bounded invertible operator on $\mathcal{H}$. Hence $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ forms an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ and with frame operator $\mathcal{S}_{\mathcal{G}} := \mathcal{U}^*\mathcal{S}_{\mathcal{F}}\mathcal{U}$. $\square$

Remark 2.9. In Theorem 2.8, if A and B be lower and upper frame bounds for the OPV-frame $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$, respectively, then, in particular, for $\mathcal{U} = \mathcal{S}_{\mathcal{F}}^{-1}$, $\{\mathcal{G}_n\}_{n\in\mathbb{N}} := \{\mathcal{F}_n\mathcal{S}_{\mathcal{F}}^{-1}\}_{n\in\mathbb{N}}$ forms an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ and with lower and upper frame bounds B^{-1} and A^{-1}, respectively. Also, the frame operator is given by $\mathcal{S}_{\mathcal{G}} := \mathcal{S}_{\mathcal{F}}^{-1}$. Indeed, since $AI_{\mathcal{H}} \leq \mathcal{S}_{\mathcal{F}} \leq BI_{\mathcal{H}}$, we have

$$B^{-1}\|\mathfrak{h}\|^2 \leq \langle \mathcal{S}_{\mathcal{F}}^{-1}(\mathfrak{h}), \mathfrak{h}\rangle \leq A^{-1}\|\mathfrak{h}\|^2, \ \mathfrak{h} \in \mathcal{H}.$$

Also, observe that

$$\langle \mathcal{S}_{\mathcal{F}}^{-1}(\mathfrak{h}), \mathfrak{h}\rangle = \Big\langle \sum_{n\in\mathbb{N}} \mathcal{G}_n^*\mathcal{G}_n(\mathfrak{h}), \mathfrak{h}\Big\rangle = \sum_{n\in\mathbb{N}} \|\mathcal{G}_n(\mathfrak{h})\|^2, \ \mathfrak{h} \in \mathcal{H}.$$

Thus, we obtain

$$B^{-1}\|\mathfrak{h}\|^2 \leq \sum_{n\in\mathbb{N}} \|\mathcal{G}_n(\mathfrak{h})\|^2 \leq A^{-1}\|\mathfrak{h}\|^2, \ \mathfrak{h} \in \mathcal{H}.$$

3. Construction of operator valued frames

In this section, we try to construct OPV-frames with the help of the partial isometries $\{|e_n\rangle_{\ell^2}\}_{n\in\mathbb{N}}$ and with a given operator. First, we show that one can construct an orthonormal OPV-frame from a given unitary operator.

Theorem 3.1. *Let $\mathcal{U} : \mathcal{H} \to \mathcal{K} \otimes \ell^2$ be a unitary operator. Then, $\{{}_{\ell^2}\langle e_n|\mathcal{U}\}_{n\in\mathbb{N}}$ forms an orthonormal OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$.*

Proof. For each $n \in \mathbb{N}$, let $\mathcal{F}_n := {}_{\ell^2}\langle e_n|\mathcal{U}$. Since $\mathcal{U}^*\mathcal{U} = I_{\mathcal{H}}$, we have

$$\begin{aligned}\sum_{n\in\mathbb{N}} \mathcal{F}_n^*\mathcal{F}_n &= \mathcal{U}^*\Big(\sum_{n\in\mathbb{N}} |e_n\rangle_{\ell^2\,\ell^2}\langle e_n|\Big)\mathcal{U} \\ &= \mathcal{U}^*\mathcal{U} = I_{\mathcal{H}}.\end{aligned}$$

Also, for $n, m \in \mathbb{N}$, we obtain

$$\mathcal{F}_n\mathcal{F}_m^* = {}_{\ell^2}\langle e_n|\mathcal{U}\mathcal{U}^*|e_m\rangle_{\ell^2} = {}_{\ell^2}\langle e_n||e_m\rangle_{\ell^2} = \delta_{m,n}I_{\mathcal{K}}.$$

Thus, $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ forms an orthonormal OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$. □

In the following result, we construct a Riesz OPV-frame using a bounded invertible operator.

Theorem 3.2. *Let $\mathcal{U} : \mathcal{H} \to \mathcal{K}\otimes\ell^2$ be a bounded invertible operator. Then, $\{{}_{\ell^2}\langle e_n|\mathcal{U}\}_{n\in\mathbb{N}}$ forms a Riesz OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$.*

Proof. Following the proof of Theorem 3.1, for each $n \in \mathbb{N}$, let $\mathcal{F}_n := {}_{\ell^2}\langle e_n|\mathcal{U}$. Then

$$\sum_{n\in\mathbb{N}} \mathcal{F}_n^*\mathcal{F}_n = \mathcal{U}^*\mathcal{U}.$$

Let $\mathcal{S}_{\mathcal{F}} := \mathcal{U}^*\mathcal{U}$. Then $\mathcal{S}_{\mathcal{F}}$ is an invertible operator.
Hence, for all $n, m \in \mathbb{N}$, we get

$$\begin{aligned}\mathcal{F}_n\mathcal{S}_{\mathcal{F}}^{-1}\mathcal{F}_m^* &= {}_{\ell^2}\langle e_n|\mathcal{U}\mathcal{U}^{-1}(\mathcal{U}^*)^{-1}\mathcal{U}^*|e_m\rangle_{\ell^2} \\ &= {}_{\ell^2}\langle e_n||e_m\rangle_{\ell^2} \\ &= \delta_{m,n}I_{\mathcal{K}}.\end{aligned}$$

Thus, $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ forms a Riesz OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$. □

Next, We construct an OPV-frame using a bounded injective operator.

Theorem 3.3. *Let $\mathcal{U} : \mathcal{H} \to \mathcal{K}\otimes\ell^2$ be a bounded injective operator. Then $\{{}_{\ell^2}\langle e_n|\mathcal{U}\}_{n\in\mathbb{N}}$ forms an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$.*

Proof. For each $n \in \mathbb{N}$, let $\mathcal{F}_n := {}_{\ell^2}\langle e_n|\mathcal{U}$. Then, we have

$$\sum_{n\in\mathbb{N}} \mathcal{F}_n^*\mathcal{F}_n = \mathcal{U}^*\mathcal{U},$$

which further gives that

$$\sum_{n\in\mathbb{N}} \|\mathcal{F}_n(\mathfrak{h})\|^2 = \|\mathcal{U}(\mathfrak{h})\|^2,\ \mathfrak{h} \in \mathcal{H}.$$

Now since $\mathcal{U}$ is a bounded injective function, there exists two positive constants C_1 and C_2 such that,

$$C_1\|\mathfrak{h}\|^2 \le \sum_{n\in\mathbb{N}} \|\mathcal{F}_n(\mathfrak{h})\|^2 = \|\mathcal{U}(\mathfrak{h})\|^2 \le C_2\|\mathfrak{h}\|^2,\ \mathfrak{h} \in \mathcal{H}.$$

Therefore, $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ forms an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$. □

Remark 3.4. Let $\mathcal{U} : \mathcal{H} \to \mathcal{K}\otimes\ell^2$ be a bounded operator. Then $\{{}_{\ell^2}\langle e_n|\mathcal{U}\}_{n\in\mathbb{N}}$ forms a Bessel OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$.

Indeed, for each $n \in \mathbb{N}$, let $\mathcal{F}_n := {}_{\ell^2}\langle e_n|\mathcal{U}$. Since $\mathcal{U}$ is a bounded operator, there exists a positive constant C such that

$$\sum_{n\in\mathbb{N}} \|\mathcal{F}_n(\mathfrak{h})\|^2 = \|\mathcal{U}(\mathfrak{h})\|^2 \le C\|\mathfrak{h}\|^2,\ \mathfrak{h} \in \mathcal{H}.$$

4. Perturbation of operator valued frames

In order to check the stability of a system, it is essential that we Perturb the system in various ways and check for the stability of such a perturbation. So perturbation is a very important tool in many areas of mathematics [1, 4, 5] . In frame theory, it started with some perturbation result of the type of Paley and Wiener. In this section, we prove that a perturbation of an OPV-frame is again an OPV-frame whenever the gap between them is small enough. We prove a result on perturbation of OPV-frames, inspired by the most fundamental form of the Paley-Wiener perturbation theorem given by Casazza and Christensen [1], as follows:

Theorem 4.1. *Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ with lower and upper frame bounds A and B respectively. Let $\{\mathcal{G}_n\}_{n\in\mathbb{N}} \subseteq \mathcal{B}(\mathcal{H},\mathcal{K})$ be a sequence of operators and there exist α, β, and $\gamma > 0$ such that for all finite subset $\mathcal{I} \subset \mathbb{N}$ and for each $k \in \mathcal{K}$*

$$\Big\| \sum_{i\in\mathcal{I}} c_i\Big(\mathcal{F}_i^*(k) - \mathcal{G}_i^*(k)\Big)\Big\| \leq \alpha\Big\|\sum_{i\in\mathcal{I}} c_i\mathcal{F}_i^*(k)\Big\| + \beta\Big\|\sum_{i\in\mathcal{I}} c_i\mathcal{G}_i^*(k)\Big\| + \gamma\|k\|\Big(\sum_{i\in\mathcal{I}}|c_i|^2\Big)^{1/2} \tag{4.1}$$

where $c = \{c_n\}_{n\in\mathbb{N}} \in \ell^2$ and $0 \leq \max\{\alpha + \gamma\sqrt{B}A^{-1}, \beta\} < 1$. Then, $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ also forms an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ and with lower and upper frame bounds $\frac{\left(1-(\alpha+\gamma\sqrt{B}A^{-1})\right)^2 A^2}{(1+\beta)^2 B}$ and $\frac{\left((1+\alpha)\sqrt{B}+\gamma\right)^2}{(1-\beta)^2}$ respectively.

Proof. Let $\mathcal{T}_\mathcal{F}$and $\mathcal{S}_\mathcal{F}$ be the analysis and frame operators of $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ respectively. Then, from the given condition (4.1), for $k \in \mathcal{K}$ and $c = \{c_n\}_{n\in\mathbb{N}} \in \ell^2$, we have

$$\begin{aligned} \Big\|\sum_{i\in\mathcal{I}} c_i\mathcal{G}_i^*(k)\Big\| &\leq \Big\|\sum_{i\in\mathcal{I}} c_i\Big(\mathcal{F}_i^*(k) - \mathcal{G}_i^*(k)\Big)\Big\| + \Big\|\sum_{i\in\mathcal{I}} c_i\mathcal{F}_i^*(k)\Big\| \\ &\leq (1+\alpha)\Big\|\sum_{i\in\mathcal{I}} c_i\mathcal{F}_i^*(k)\Big\| + \beta\Big\|\sum_{i\in\mathcal{I}} c_i\mathcal{G}_i^*(k)\Big\| + \gamma\|k\|\|c\|, \end{aligned}$$

which implies that

$$\begin{aligned} \Big\|\sum_{i\in\mathcal{I}} c_i\mathcal{G}_i^*(k)\Big\| &\leq \frac{(1+\alpha)\big\|\sum_{i\in\mathcal{I}} c_i\mathcal{F}_i^*(k)\big\| + \gamma\|k\|\|c\|}{1-\beta} \\ &\leq \frac{(1+\alpha)\big\|\mathcal{T}_\mathcal{F}^*(k\otimes c)\big\| + \gamma\|k\otimes c\|}{1-\beta} \\ &\leq \left(\frac{(1+\alpha)\big\|\mathcal{T}_\mathcal{F}^*\big\| + \gamma}{1-\beta}\right)\|k\otimes c\| \\ &\leq \left(\frac{(1+\alpha)\sqrt{B} + \gamma}{1-\beta}\right)\|k\otimes c\|. \end{aligned}$$

Now define $\mathcal{U}_\mathcal{G} : \mathcal{K}\otimes\ell^2 \to \mathcal{H}$ as

$$\mathcal{U}_\mathcal{G}(k\otimes c) = \sum_{n\in\mathbb{N}} c_n\mathcal{G}_n^*(k),\ k\in\mathcal{K}, c = \{c_n\} \in \ell^2.$$

It is easy to verify that $\mathcal{U}_{\mathcal{G}}$ is a well-defined bounded operator with $\|\mathcal{U}_{\mathcal{G}}\| \leq \frac{(1+\alpha)\sqrt{B}+\gamma}{(1-\beta)}$. Then, $\mathcal{T}_{\mathcal{G}} := \mathcal{U}_{\mathcal{G}}^{*} : \mathcal{H} \to \mathcal{K} \otimes \ell^2$ is given by

$$\mathcal{T}_{\mathcal{G}}(\mathfrak{h}) = \sum_{n\in\mathbb{N}} \mathcal{G}_n(\mathfrak{h}) \otimes e_n, \ \mathfrak{h} \in \mathcal{H}.$$

Also for $\mathfrak{h} \in \mathcal{H}$,

$$\sum_{n\in\mathbb{N}} \|\mathcal{G}_n(\mathfrak{h})\|^2 = \|\mathcal{T}_{\mathcal{G}}(\mathfrak{h})\|^2 \leq \left(\frac{(1+\alpha)\sqrt{B}+\gamma}{1-\beta}\right)^2 \|\mathfrak{h}\|^2. \tag{4.2}$$

Let $\mathcal{W} := \mathcal{T}_{\mathcal{G}}^{*}\mathcal{T}_{\mathcal{F}}\mathcal{S}_{\mathcal{F}}^{-1}$. Then for $\mathfrak{h} \in \mathcal{H}$, we have

$$\begin{aligned}
\|\mathfrak{h} - \mathcal{W}(\mathfrak{h})\| &= \|\mathcal{T}_{\mathcal{F}}^{*}\mathcal{T}_{\mathcal{F}}\mathcal{S}_{\mathcal{F}}^{-1}(\mathfrak{h}) - \mathcal{T}_{\mathcal{G}}^{*}\mathcal{T}_{\mathcal{F}}\mathcal{S}_{\mathcal{F}}^{-1}(\mathfrak{h})\| \\
&\leq \alpha\|\mathfrak{h}\| + \beta\|\mathcal{W}(\mathfrak{h})\| + \gamma\|\mathcal{T}_{\mathcal{F}}\mathcal{S}_{\mathcal{F}}^{-1}(\mathfrak{h})\| \\
&\leq \alpha\|\mathfrak{h}\| + \beta\|\mathcal{W}(\mathfrak{h})\| + \gamma\|\mathcal{T}_{\mathcal{F}}\|\|\mathcal{S}_{\mathcal{F}}^{-1}\|\|\mathfrak{h}\| \\
&\leq (\alpha + \gamma\sqrt{B}A^{-1})\|\mathfrak{h}\| + \beta\|\mathcal{W}(\mathfrak{h})\| \\
&< \|\mathfrak{h}\| + \|\mathcal{W}(\mathfrak{h})\|.
\end{aligned}$$

This gives that $\mathcal{W}$ is an invertible operator. Moreover,

$$\begin{aligned}
\|\mathcal{W}(\mathfrak{h})\| &= \|\mathfrak{h} - (\mathfrak{h} - \mathcal{W}(\mathfrak{h}))\| \\
&\geq \|\mathfrak{h}\| - \|\mathfrak{h} - \mathcal{W}(\mathfrak{h})\| \\
&\geq \|\mathfrak{h}\| - (\alpha + \gamma\sqrt{B}A^{-1})\|\mathfrak{h}\| - \beta\|\mathcal{W}(\mathfrak{h})\|.
\end{aligned}$$

$$\implies \|\mathfrak{h}\|\left(\frac{1-(\alpha+\gamma\sqrt{B}A^{-1})}{1+\beta}\right) \leq \|\mathcal{W}(\mathfrak{h})\|.$$

$$\implies \|\mathcal{W}^{-1}\| \leq \frac{1+\beta}{1-(\alpha+\gamma\sqrt{B}A^{-1})}.$$

Also for $\mathfrak{h} \in \mathcal{H}$, we have

$$\begin{aligned}
\|\mathfrak{h}\|^2 &= \|(\mathcal{W}^{*})^{-1}\mathcal{W}^{*}(\mathfrak{h})\|^2 \\
&\leq \|(\mathcal{W}^{*})^{-1}\|^2\|\mathcal{S}_{\mathcal{F}}^{-1}\mathcal{T}_{\mathcal{F}}^{*}\mathcal{T}_{\mathcal{G}}(\mathfrak{h})\|^2 \\
&\leq \|(\mathcal{W}^{*})^{-1}\|^2\|\mathcal{S}_{\mathcal{F}}^{-1}\|^2\|\mathcal{T}_{\mathcal{F}}^{*}\|^2\|\mathcal{T}_{\mathcal{G}}(\mathfrak{h})\|^2 \\
&\leq \left(\frac{1+\beta}{1-(\alpha+\gamma\sqrt{B}A^{-1})}\right)^2 (\sqrt{B}A^{-1})^2 \sum_{n\in\mathbb{N}} \|\mathcal{G}_n(\mathfrak{h})\|^2.
\end{aligned}$$

Thus,

$$\frac{\big(1-(\alpha+\gamma\sqrt{B}A^{-1})\big)^2 A^2}{(1+\beta)^2 B}\|\mathfrak{h}\|^2 \leq \sum_{n\in\mathbb{N}} \|\mathcal{G}_n(\mathfrak{h})\|^2. \tag{4.3}$$

Thus from (4.2) and (4.3), we conclude that $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ forms an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ and with lower and upper frame bounds $\frac{\big(1-(\alpha+\gamma\sqrt{B}A^{-1})\big)^2 A^2}{(1+\beta)^2 B}$ and $\frac{\big((1+\alpha)\sqrt{B}+\gamma\big)^2}{(1-\beta)^2}$ respectively. $\square$

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