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RATIONAL LIMIT CYCLES FOR A CLASS OF GENERALIZED ABEL'S POLYNOMIAL DIFFERENTIAL EQUATIONS

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***Abstract*.** In this manuscript, we deal with a class of generalized Abel's ordinary polynomial differential equations of the form

$$\frac{dy}{dt} = F(t)y^2 + G(t)y^n,$$

where $F(t)$, $G(t)$ are real polynomials with $G(t) \neq 0$ and $n \geqslant 3$. We prove that these Able differential equations have non-trivial rational limit cycles. We also discuss the relation between the existence of the non-trivial rational limit cycles and the degrees of real polynomials $F(t)$, $G(t)$.

1. Introduction and presentation of the main results

In this research, we want to study the existence of non-trivial rational limit cycles of a class of polynomial differential equations

$$\frac{dy}{dt} = F(t)y^2 + G(t)y^n, \tag{1.1}$$

where t, y are real variables and $G(t)$ and $F(t)$ are two real polynomials of degree r, s respectively with $G(t) \neq 0$.

The study of limit cycles of differential equations is one of the main problems in the qualitative theory of differential equations (see for instance [2, 10, 11, 12, 13]) in addition to that, the differential equations (1.1) are interesting because they happen in a lot of models of real phenomena (see for instance [3, 6]).

A periodic solution of equation (1.1) is a solution $y(t)$ defined in $[0, 1]$ such as $y(0) - y(1) = 0$, note that without loss of over-simplification we are supposing that

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the period is 1. A limit cycle of polynomial differential equations (1.1) is a periodic solution isolated in the set of all periodic solutions of a differential equation (1.1) see [7, 2, 14]. Note that $y(t) \equiv 0$ is always a periodic solution, which we shall call the trivial solution.

The limit cycle is called a polynomial limit cycle if the periodic solution $y(t)$ is a polynomial in the variable t. The limit cycles of these equations have been intensively investigated mainly when the functions $F(t)$ and $G(t)$ are periodic, or are polynomial see for example ([1], [4], [5], [9], [14]). So, in the current article we will concentrate on the type in which the limit cycles are rational.

In [9] J. Llibre and C. Valls, provide the differential equation (1.1) with $n = 3$ having two rational limit cycles non-trivial.

The present paper is concerned with the existence of non-trivial rational limit cycles for equations (1.1) with $n \geqslant 3$ of the expression $y(t) = \frac{p(t)}{q(t)}$ where p, q are real polynomials and $(q(t), p(t)) = 1$, we will also discuss the relation between the existence of non-trivial rational limit cycles and s, r the degrees of $F(t)$, $G(t)$ respectively.

Our main theorem concerning differential equations (1.1) is the following one.

Theorem 1.1. *The polynomial differential equations (1.1) has non-trivial rational limit cycle if s and r satisfies one of the following conditions*

$$\begin{cases} C_1 : r \geq (s+1)^{n-2} + s + 1 \\ C_2 : (1+s)^{n-2} \leq r \leq (s+1)^{n-2} + s \\ C_3 : 1 + s = r \geq 2, \end{cases}$$

where $s = \deg\ F(t)$ *and* $r = \deg\ G(t)$.

2. Proof of Theorem 1.1

We first prove an auxiliary results

Lemma 2.1. *The non-zero rational function* $y = \frac{p(t)}{q(t)}$ *with* $q(t)$ *non-constant is a periodic solution of polynomial differential equation (1.1) if and only if:*

1- $p(t) = c \in \mathbb{R}^*$

2- $q(0) = q(1)$ *and* $q(t) \neq 0$ *for all* $t \in [0, 1]$

3- $(q'(t) + cF(t))\, q^{n-2}(t) + c^{n-1}G(t) = 0$

Proof. For the direct implication we note that is if $y = \frac{p(t)}{q(t)}$ a periodic solution of polynomial differential equation (1.1) then obviously $q(t)$ has no zero in $[0, 1]$ .Let $f(t, y) = yq(t) - p(t)$, then along the curve $\{f(t, y) = 0\}$, the derivative of $f(t, y)$ in t is zero. i.e.,

$$\frac{df(t,y)}{dt} = yq'(t) + \left(F(t)y^2 + G(t)y^n\right) q(t) - p'(t) = 0.$$

Notice that $f(t, y)$ is irreducible, so there exists a polynomial $k(t, y)$ so that

$$yq'(t) + \left(F(t)y^2 + G(t)y^n\right) q(t) - p'(t) = f(t, y)k(t, y). \tag{2.1}$$

The left-hand side of formula (2.1) is a polynomial of degree n in y and $f(t, y)$ is a polynomial of degree 1 in y, we get that the highest degree in y in $k(t, y)$ is $(n - 1)$ and so it can put that

$$k(t, y) = k_0(t) + k_1(t)y + k_2(t)y^2 + ... + k_{n-1}(t)y^{n-1},$$

where $k_0, k_1, k_2, ..., k_{n-1}$ are polynomials in t. Via Comparing the coefficients $y^m, m = 0, 1, 2, ..., n$, in the expression (2.1), we find

$$\begin{cases} k_0(t)p(t) = p'(t) \\ k_0(t)q(t) - k_1(t)p(t) = q'(t) \\ k_1(t)q(t) - k_2(t)p(t) = F(t)q(t) \\ k_2(t)q(t) - k_3(t)p(t) = 0 \\ k_3(t)q(t) - k_4(t)p(t) = 0 \\ .... \\ k_{n-2}(t)q(t) - k_{n-1}(t)p(t) = 0 \\ k_{n-1}(t)q(x) = q(t)G(t). \end{cases}$$

From the first relation we get that $\frac{p(t)}{p'(t)}$. This implies that $p(t)$ is a constant and we denote by $p(t) = c \in \mathbb{R}$, if $c = 0$, then $\frac{p(t)}{p'(t)} = 0$. This is not possible and so $c \neq 0$.Furthermore, $y = \frac{c}{q(t)}$, is a periodic solution Denote by $y = \frac{c}{q(t)} = \varphi(t, y_0)$ the solution of polynomial differential equation (1.1) such that $\varphi(0, y_0) = y_0$ and since $\varphi(t, y_0)$ a periodic solution of polynomial differential equation (1.1), so $\varphi(1, y_0) = y_0$ thus $q(0) = q(1)$, the condition (2) holds.

From the second relation we get that;

$$k_1(t) = -\frac{1}{c}q'(t).$$

From the third relation we get that:

$$k_2(t) = -\frac{1}{c^2}q(t)q'(t) - \frac{1}{c}q(t)F(t).$$

From the fourth till the ante-penultimate relation we get :

$$k_{n-2}(t) = \frac{1}{c}k_{n-3}(t)q(t) = \frac{1}{c^2}k_{n-4}(t)q^2(t) = \frac{1}{c^3}k_{n-5}(t)q^3(t) = ... = \frac{1}{c^{n-4}}k_2(t)q^{n-4}(t),$$

and from the last relation we have $G(t) = k_{n-1}(t)$. Substituting them in the penultimate relation we get

$$(p'(t) + cF(t))\, q^{n-2}(t) + c^{n-1}G(t) = 0. \tag{2.2}$$

and the direct inclusion is proved.

For the reverse implication we note that if on the contrary, if all the three conditions (1), (2), (3) hold, then one can easily check that the rational function $y = \frac{c}{q(t)}$ is a periodic solution of polynomial differential equation (1.1). □

To decide a periodic solution of polynomial differential equation (1.1) is a hyperbolic limit cycle, we need the following lemma, see [5].

Lemma 2.2. *Consider the differential equation* $\frac{dy}{dt} = \mathcal{F}(t, y)$,*where* $\mathcal{F}(t, y)$ *is a function of class* C^2 *in* $\mathbb{R}^2$.*Assume that* $y = \varphi(t)$ *is a periodic orbit of this equation, then it is a hyperbolic limit cycle if and only if* $\mathcal{D}_1(1) \neq 0$, *where*

$$\mathcal{D}_1(1) = \int_0^x \frac{\partial \mathcal{F}(t, \varphi(t))}{\partial y} dt.$$

Corollary 2.3. If $y = \frac{1}{q(t)}$ is a periodic solution of polynomial differential equation (1.1), then it is a hyperbolic limit cycle if and only if
$\int_0^1 (n-2)\frac{F(t)}{q(t)}dt \neq 0$.

Proof. We use Lemma 2.2 to deal with rational limit cycles, we have

$$\begin{aligned}
\mathcal{D}_1(1) &= \int_0^1 \frac{\partial\left(F(t)y^2 + G(t)y^n\right)}{\partial y}\left(\eta, \frac{1}{q(\eta)}\right) d\eta \\
&= \int_0^1 \left(2yF(t) + nG(t)y^{n-1}\right)\left(\eta, \frac{1}{q(\eta)}\right) d\eta \\
&= \int_0^1 \left(\frac{2F(t)}{q(t)} + nG(t)\left(\frac{1}{q(t)}\right)^{n-1}\right) dt
\end{aligned}$$

By Lemma 2.1, $q(0) = q(1)$ and $G(t) = -\left(q'(t) + F(t)\right) q^{n-2}(t)$, so

$$\begin{aligned}
\mathcal{D}_1(1) &= \left(\int_0^1 -\frac{nq'(t)}{q(t)} - \frac{(n-2)\,F(t)}{q(t)}\right) dt \\
&= -\int_0^1 (n-2)\frac{F(t)}{q(t)}dt,
\end{aligned}$$

the corollary is proved. □

2.1. Proof of Theorem 1.1. If polynomial differential equation (1.1) has a non trivial limit cycle $y = \frac{1}{q(t)}$, then by Lemma (2.1), $q(t)$ satisfies that:

$$\left(F(t) + q'(t)\right) q^{n-2}(t) + G(t) = 0. \tag{2.3}$$

We recall that $\deg F(t) = s$, $\deg G(t) = r$.

The solution $y = \frac{1}{q(t)}$ is a limit cycle, so $q(0) = q(1)$, which implies that $\deg q(t) \geq 2$, else $\deg q(t) = l$, $q(t)$ is a constant, which is a contradiction with that $y = \frac{1}{q(t)}$ is a non trivial rational limit cycle.

1- if $\deg q(t) = l > 1 + s$ then $\deg q'(t) = l - 1 > s$.

$r = \deg q^{n-2}(t) + \deg q'(t) = l^{n-2} + l - 1 \geq (s+1)^{n-2} + s + 1$, the condition C_1 holds.

2- if $\deg q(t) = l = 1 + s$ then $s \geq 1$.

$(1+s)^{n-2} = \deg q^{n-2}(t) \leq r \leq \deg q^{n-2}(t) + \deg q'(t) = (s+1)^{n-2} + s$, the condition C_2 holds; or $r = 1 + s \geq 2$, the condition C_3 holds.

3- if $\deg q(t) = l < 1 + s$ then $\deg q'(t) = l - 1 < s$.

$r = \deg q^{n-2}(t) + \deg q'(t) = l^{n-2} + l - 1$, with implies that $(1+s)^{n-2} \leq r \leq (s+1)^{n-2} + s$, the condition C_2 holds.

In any case, one of the conditions (C_1), (C_1), (C_1) holds, so if s, r and n does not satisfies these conditions, differential equations (1.1) has no non trivial rational limit cycles.

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