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APPROXIMATIONS IN HÖLDER'S CLASS AND SOLUTION OF BESSEL'S DIFFERENTIAL EQUATIONS BY EXTENDED HAAR WAVELET

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Abstract. In this paper, extended Haar wavelet has been introduced in the interval $[0, \lambda)$, $\lambda > 0$. It reduces to classical Haar wavelet for $\lambda = 1$. The orthonormality of extended Haar wavelets has been discussed. The convergence analysis of an extended Haar wavelet series of a function f belonging to Hölder's classes $H^{\alpha}[0, \lambda)$ & $H_2^{\alpha}[0, \lambda)$ have been studied. Consequently, the approximations of function f belonging to the generalised Hölder's class have been estimated. The solutions of Bessel's differential equation of order zero have been obtained by the extended Haar operational matrix method for $\lambda = 1$ & 2. These solutions for $\lambda = 1$ & 2 are compared with their exact solutions. It is observed that the extended Haar wavelet solutions and their exact solutions are almost the same. This validates the adopted procedure for solutions of Bessel's differential equation by an extended Haar operational matrix. This is a significant achievement in wavelet analysis.

1. Introduction

The approximations of a function f belonging to $H^{\alpha}[0, 1)$, $H_2^{\alpha}[0, 1)$, $H^{\alpha,w}[0, 1)$, have been determined by researchers Lal and Kumar [8], Debnath [7], Meyer [11], Zygmund [14], Lee and Yueh [9], Aznam and Chowdhury [4] and many others. Working in the same direction, recently Ahmedov et al. [1], Baleanu et al. [5], Umer Saeed et al. [13] have developed the estimators of function by Haar wavelet technique. Generally, the Haar wavelet is defined in the interval [0,1). In nature, there are several actual problems which are associated with the interval $[0, \lambda)$, $\lambda > 0$ instead of [0,1). To deal with such a problem, the classical Haar wavelet and Haar operational matrix are not applicable. Hence Haar wavelet and Haar operational matrix are to be defined in the interval $[0, \lambda)$. These are not considered till now. This short-comming in wavelet analysis motivates

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us to extend the definition of the classical Haar wavelet and its operational matrix in the interval $[0, \lambda)$. It is remarkable to note that the extended Haar wavelet and its operational matrix reduces to classical Haar wavelet if $\lambda = 1$.
In precise, the objectives of this research paper are mentioned as follows.
(1) To define Haar wavelet in the interval $[0, \lambda)$.
(2) To evaluate the approximations of function f belonging to classes $H^{\alpha}[0, \lambda)$, $H_2^{\alpha}[0, \lambda)$ and $H^{\alpha,w}[0, \lambda)$.
(3) To obtain the extended Haar operational matrix in the interval $[0, \lambda)$.
(4) To solve Bessel's differential equation of order zero by extended Haar operational matrix method.
The remaining part of this paper is organised as follows: in Section 2, some fundamental definitions and properties of the Haar wavelet and the Haar scalling function in $[0, \lambda)$ have been mentioned. The extended Haar wavelet series, extended Haar wavelet approximations of function in Hölder's class, and orthonormality of the extended Haar wavelet are discussed. In Section 3, the multiresolution analysis of the wavelet series has been studeid in the interval $[0,\lambda)$. In Section 4, the convergence analysis of the extended Haar wavelet series has been discussed. In Section 5, the approximations of function $f \in H^{\alpha}[0, \lambda)$, $H_2^{\alpha}[0, \lambda)$ and $H^{\alpha,w}[0, \lambda)$ by extended Haar wavelet have been estimated, and their proofs and verification are given. In Section 6, the extended Haar operational matrix has been developed in the interval $[0,\lambda)$, and it is used to obtain the solution of Bessel's differential equation for different values of j in the intervals [0,1) and [0,2). Section 7 is designated for the conclusions of this research paper.

2. Definitions and Preliminaries

In this section, related definitions and preliminaries of this research paper are mentioned as follows:

2.1. Haar Wavelet and Haar Scaling Function. Haar wavelet $\psi^{(\lambda)}$ and Haar scaling function $\phi^{(\lambda)}$ in the interval $[0, \lambda)$, $\lambda > 0$ are defined by

$$\psi^{(\lambda)} = \frac{1}{\sqrt{m}}(\chi_{[0,\frac{\lambda}{2})} - \chi_{[\frac{\lambda}{2},\lambda)}), \quad \phi^{(\lambda)} = \frac{1}{\sqrt{m}}\chi_{[0,\lambda)}. \tag{2.1}$$

where $m = 2^j$ $(j = 0, 1, 2, ..., J,\ J$ being a positive integer) represents the level of the Haar wavelet $\psi^{(\lambda)}$.
The family $\psi_{j,k}^{(\lambda)}$ of extended Haar wavelets are defined by

$$\psi_{j,k}^{(\lambda)}(t) = \frac{1}{\sqrt{m}} \begin{cases} 2^{\frac{j}{2}}, & if \quad t \in \left[\frac{k}{2^j}\lambda, \frac{k+\frac{1}{2}}{2^j}\lambda\right), \\ \\ -2^{\frac{j}{2}}, & if \quad t \in \left[\frac{k+\frac{1}{2}}{2^j}\lambda, \frac{k+1}{2^j}\lambda\right), \\ 0, & \text{otherwise}, \end{cases} \tag{2.2}$$

where $k = 0, 1, 2, ..., 2^j - 1$.
The family $\phi_{j,k}^{(\lambda)}$ of extended Haar scaling functions are defined by

$$\phi_{j,k}^{(\lambda)}(t) = \frac{1}{\sqrt{m}} \begin{cases} 2^{\frac{j}{2}}, & if \quad t \in [\frac{k}{2^j}\lambda, \frac{k+1}{2^j}\lambda), \\ 0, & \text{otherwise}. \end{cases} \tag{2.3}$$

Here $\left\langle \frac{\sqrt{m}}{\sqrt{\lambda}}\psi_{j,k}^{(\lambda)}, \frac{\sqrt{m}}{\sqrt{\lambda}}\psi_{j,k}^{(\lambda)}\right\rangle = 1$ and $\left\langle \frac{\sqrt{m}}{\sqrt{\lambda}}\phi_{j,k}^{(\lambda)}, \frac{\sqrt{m}}{\sqrt{\lambda}}\phi_{j,k}^{(\lambda)}\right\rangle = 1$.

2.2. Extended Haar Wavelet Series. The extended Haar wavelet of a function $f \in L^2[0,\lambda)$ is

$$f(t) = \sum_{k=0}^{2^N-1} d_{N,k}\phi_{N,k}^{(\lambda)}(t) + \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} c_{j,k}\psi_{j,k}^{(\lambda)}(t) \tag{2.4}$$

$$= (S_{2^N}f)(t) + (R_{2^N}f)(t),$$

where $d_{N,k} = \frac{m}{\lambda}\left\langle f, \phi_{N,k}^{(\lambda)}\right\rangle$, $c_{j,k} = \frac{m}{\lambda}\left\langle f, \psi_{j,k}^{(\lambda)}\right\rangle$.

2.3. Wavelet Approximation. The wavelet approximation of function f by $(S_N f)$ under $||\cdot||_2$ is denoted by $E_N(f)$ and defined by

$$E_N(f) = \min_{(S_N f)} ||f - (S_N f)||_2, \tag{2.5}$$

$E_N(f)$ is said to best approximation of the function f if $E_N(f) \to 0$ as $N \to \infty$ (Zygmund [14]).
Let $(S_N f)(t) = d_{0,0}\phi_{0,0}^{(\lambda)}(t) + \sum_{j=0}^{N-1}\sum_{k=0}^{2^j-1} c_{j,k}\psi_{j,k}^{(\lambda)}(t)$, then the estimator $E_N^{(1)}(f)$ is defined by $E_N^{(1)}(f) = ||f - (S_N f)||_2$.
For $(S_{2^N}f)(t) = \sum_{k=0}^{2^N-1} d_{N,k}\phi_{N,k}^{(\lambda)}(t)$, the estimator $E_{2^N}^{(2)}(f)$ is defined by $E_{2^N}^{(2)}(f) = ||f - (S_{2^N}f)||_2$, whereas if $f \in H^{\alpha,\omega}[0,\lambda)$, then the estimator $E_N^{(3)}(f)$ is defined by $E_N^{(3)}(f) = ||f - (S_N f)||_2$.

2.4. Orthonormality of Extended Haar Wavelets. In this section, the proposition of orthonormality of extended Haar wavelets has been discussed.

Proposition 2.1. $\{\frac{\sqrt{m}}{\sqrt{\lambda}}\psi_{j,k}^{(\lambda)}\}_{j,k\in\mathbb{Z}}$ forms an orthonormal set. i.e.

$$\left\langle \frac{\sqrt{m}}{\sqrt{\lambda}}\psi_{j,k}^{(\lambda)}, \frac{\sqrt{m}}{\sqrt{\lambda}}\psi_{j',k'}^{(\lambda)}\right\rangle = \begin{cases} 1, & if \;\; j = j' \text{ and } \; k = k', \\ 0, & \text{otherwise.} \end{cases}$$

Proof. **Case 1:** $j = j'$ and $k = k'$

$$\begin{aligned}
\left\langle \frac{\sqrt{m}}{\sqrt{\lambda}}\psi^{(\lambda)}_{j,k}, \frac{\sqrt{m}}{\sqrt{\lambda}}\psi^{(\lambda)}_{j,k} \right\rangle &= \int_0^\lambda \frac{\sqrt{m}}{\sqrt{\lambda}}\psi^{(\lambda)}_{j,k}(t)\overline{\frac{\sqrt{m}}{\sqrt{\lambda}}\psi^{(\lambda)}_{j,k}}(t)dt \\
&= \frac{m}{\lambda}\int_0^\lambda \left(\psi^{(\lambda)}_{j,k}(t)\right)^2 dt \\
&= \frac{m}{\lambda}\left[\int_{\frac{k\lambda}{2^j}}^{\frac{k+\frac{1}{2}}{2^j}\lambda} \left(\frac{2^{\frac{j}{2}}}{\sqrt{m}}\right)^2 dt + \int_{\frac{k+\frac{1}{2}}{2^j}\lambda}^{\frac{k+1}{2^j}\lambda} \left(\frac{-2^{\frac{j}{2}}}{\sqrt{m}}\right)^2 dt\right] \\
&= \frac{m}{\lambda}\left[\int_{\frac{k\lambda}{2^j}}^{\frac{k+\frac{1}{2}}{2^j}\lambda} \frac{2^j}{m}dt + \int_{\frac{k+\frac{1}{2}}{2^j}\lambda}^{\frac{k+1}{2^j}\lambda} \frac{2^j}{m}dt\right] \\
&= \frac{2^j}{\lambda}\int_{\frac{k\lambda}{2^j}}^{\frac{k+1}{2^j}\lambda} dt = \frac{2^j}{\lambda}\left(\frac{k+1}{2^j}\lambda - \frac{k\lambda}{2^j}\right) = 1.
\end{aligned}$$

Case 2: $j \neq j'$ and $k = k'$

$$\begin{aligned}
\left\langle \frac{\sqrt{m}}{\sqrt{\lambda}}\psi^{(\lambda)}_{j,k}, \frac{\sqrt{m}}{\sqrt{\lambda}}\psi^{(\lambda)}_{j',k} \right\rangle &= \int_0^\lambda \frac{\sqrt{m}}{\sqrt{\lambda}}\psi^{(\lambda)}_{j,k}(t)\overline{\frac{\sqrt{m}}{\sqrt{\lambda}}\psi^{(\lambda)}_{j',k}}(t)dt \\
&= \frac{m}{\lambda}\int_0^\lambda \psi^{(\lambda)}_{j,k}(t)\psi^{(\lambda)}_{j',k}(t)dt
\end{aligned}$$

$\psi^{(\lambda)}_{j,k}$ is defined in $[\frac{k}{2^j}\lambda, \frac{k+1}{2^j}\lambda) \subset [0,\lambda)$ and $\psi_{j',k}$ is defined in $[\frac{k}{2^{j'}}\lambda, \frac{k+1}{2^{j'}}\lambda) \subset [0,\lambda)$
If $j \neq j'$, then the intervals $[\frac{k}{2^j}\lambda, \frac{k+1}{2^j}\lambda)$ and $[\frac{k}{2^{j'}}\lambda, \frac{k+1}{2^{j'}}\lambda)$ are disjoint. i.e.

$$\left[\frac{k}{2^j}\lambda, \frac{k+1}{2^j}\lambda\right) \cap \left[\frac{k}{2^{j'}}\lambda, \frac{k+1}{2^{j'}}\lambda\right) = \phi.$$

Therefore $\left\langle \frac{\sqrt{m}}{\sqrt{\lambda}}\psi^{(\lambda)}_{j,k}, \frac{\sqrt{m}}{\sqrt{\lambda}}\psi^{(\lambda)}_{j',k} \right\rangle = 0$ if $j \neq j'$ $\forall\ k, k' \in \mathbb{Z}$ (set of integer) .

Case 3: $k \neq k'$

$$\begin{aligned}
\left\langle \frac{\sqrt{m}}{\sqrt{\lambda}}\psi^{(\lambda)}_{j,k}, \frac{\sqrt{m}}{\sqrt{\lambda}}\psi^{(\lambda)}_{j,k'} \right\rangle &= \int_0^\lambda \frac{\sqrt{m}}{\sqrt{\lambda}}\psi^{(\lambda)}_{j,k}(t)\overline{\frac{\sqrt{m}}{\sqrt{\lambda}}\psi^{(\lambda)}_{j,k'}}(t)dt \\
&= \frac{m}{\lambda}\int_0^\lambda \psi^{(\lambda)}_{j,k}(t)\psi^{(\lambda)}_{j,k'}(t)dt
\end{aligned}$$

$\psi^{(\lambda)}_{j,k}$ is defined in $[\frac{k}{2^j}\lambda, \frac{k+1}{2^j}\lambda) \subset [0,\lambda)$ and $\psi^{(\lambda)}_{j,k'}$ is defined in $[\frac{k'}{2^j}\lambda, \frac{k'+1}{2^j}\lambda) \subset [0,\lambda)$
If $k \neq k'$, then the intervals $[\frac{k}{2^j}\lambda, \frac{k+1}{2^j}\lambda)$ and $[\frac{k'}{2^j}\lambda, \frac{k'+1}{2^j}\lambda)$ are disjoint i.e.

$$\left[\frac{k}{2^j}\lambda, \frac{k+1}{2^j}\lambda\right) \cap \left[\frac{k'}{2^j}\lambda, \frac{k'+1}{2^j}\lambda\right) = \phi.$$

Therefore $\left\langle \frac{\sqrt{m}}{\sqrt{\lambda}}\psi^{(\lambda)}_{j,k}, \frac{\sqrt{m}}{\sqrt{\lambda}}\psi^{(\lambda)}_{j,k'} \right\rangle = 0$ if $k \neq k'$ $\forall\ j, j' \in \mathbb{Z}$. $\square$

3. Multiresolution Analysis

$$\text{Let}\ \ \phi^{(\lambda)} = \frac{1}{\sqrt{m}}\chi_{[0,\lambda)},\ \ \psi^{(\lambda)} = \frac{1}{\sqrt{m}}(\chi_{[0,\frac{\lambda}{2})} - \chi_{[\frac{\lambda}{2},\lambda)}),$$

$$\psi_{j,k}^{(\lambda)}(t) = \frac{2^{\frac{j}{2}}}{\sqrt{m}}\psi^{(\lambda)}(2^j t - k\lambda),\ \ W_j^{(\lambda)} = \underset{L^2[0,\lambda)}{\text{clos}} < \psi_{j,k}^{(\lambda)} : k \in \mathbb{Z} >,\ j \in \mathbb{Z},$$

$$\phi_{j,k}^{(\lambda)}(t) = \frac{2^{\frac{j}{2}}}{\sqrt{m}}\phi^{(\lambda)}(2^j t - k\lambda),\ \ V_j^{(\lambda)} = \underset{L^2[0,\lambda)}{\text{clos}} < \phi_{j,k}^{(\lambda)} : k \in \mathbb{Z} >,\ j \in \mathbb{Z}.$$

A sequence of closed subspaces $V_J^{(\lambda)}$ of $L^2[0,\lambda)$ satisfies the following properties:
(i) $V_J^{(\lambda)} \subset V_{J+1}^{(\lambda)}$,
(ii) $f(t) \in V_J^{(\lambda)} \iff f(2t) \in V_{J+1}^{(\lambda)}$,
(iii) $f(t) \in V_0^{(\lambda)} \iff f(t+1) \in V_0^{(\lambda)}$,
(iv) $\cup_{j=-\infty}^{\infty} V_J^{(\lambda)}$ is dense in $L^2[0,\lambda)$ and $\cap_{j=-\infty}^{\infty} V_J^{(\lambda)} = \{0\}$,
(v) Suppose a function $\phi^{(\lambda)} \in V_0^{(\lambda)}$ exists such that the collection
$\{\phi(t-\lambda k); k \in \mathbb{Z}\}$ is a Riesz basis of $V_0^{(\lambda)}$.
Then $\{V_J^{(\lambda)}\}_{j=-\infty}^{\infty}$ is a multiresolution analysis of $L^2[0,\lambda)$.
Then this family of subspaces of $L^2[0,\lambda)$ gives a direct sum decomposition of $L^2[0,\lambda)$ in the sence that every $f \in L^2[0,\lambda)$ has a unique decomposition
$f(t) = ... + g_{-1}(t) + g_0(t) + g_1(t) + g_2(t) + ...,$
where $g_j \in W_j^{(\lambda)}$ and we describe this by writing

$$\begin{aligned} L^2[0,\lambda) &= \sum_{j\in\mathbb{Z}} W_j^{(\lambda)} = ... + W_{-1}^{(\lambda)} + W_0^{(\lambda)} + W_1^{(\lambda)} + ..., \\ &= ... + W_{-1}^{(\lambda)} + W_0^{(\lambda)} + ... + W_{N-1}^{(\lambda)} + W_N^{(\lambda)} + W_{N+1}^{(\lambda)} + ..., \\ &= V_N^{(\lambda)} + \sum_{j=N}^{\infty} W_j^{(\lambda)},\ \ V_N^{(\lambda)} = ... + W_{-1}^{(\lambda)} + W_0^{(\lambda)} + ... + W_{N-1}^{(\lambda)}. \\ \text{Then}\ \ f(t) &= \sum_{k=0}^{2^N-1} d_{N,k}\phi_{N,k}^{(\lambda)}(t) + \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} c_{j,k}\psi_{j,k}^{(\lambda)}(t). \end{aligned} \tag{3.1}$$

4. Convergence of Extended Haar Wavelet Series

In this section, it is discussed that the extended Haar wavelet expansion of a function $f \in L^2[0,\lambda)$ converges uniformly to f.

4.1. Lemma. To show the convergence, the following Lemma is required.

Lemma 4.1. *If the extended Haar wavelet expansion of a function* $f \in L^2[0,\lambda)$ *converges uniformly, then the extended Haar wavelet expansion converges to the function* f *in* $[0,\lambda)$.

Proof. Let

$$f(t)=\sum_{k=0}^{2^N-1} d_{N,k}\phi^{(\lambda)}_{N,k}(t)+\sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} c_{j,k}\psi^{(\lambda)}_{j,k}(t) \tag{4.1}$$

Multiplying both side of (4.1) by $\phi^{(\lambda)}_{N,p}$ and then integrating between $\left[\frac{p}{2^N}\lambda, \frac{p+1}{2^N}\lambda\right)$,

$$\begin{aligned}
\int_{\frac{p\lambda}{2^N}}^{\frac{(p+1)\lambda}{2^N}} f(t)\phi^{(\lambda)}_{N,p}(t)dt &= \sum_{k=0}^{2^N-1} d_{N,k}\int_{\frac{p\lambda}{2^N}}^{\frac{(p+1)\lambda}{2^N}} \phi^{(\lambda)}_{N,k}(t)\phi^{(\lambda)}_{N,p}(t)dt\\
&+ \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} c_{j,k}\int_{\frac{p\lambda}{2^N}}^{\frac{(p+1)\lambda}{2^N}} \psi^{(\lambda)}_{j,k}(t)\phi^{(\lambda)}_{N,p}(t)dt\\
&= d_{N,p}\int_{\frac{p\lambda}{2^N}}^{\frac{(p+1)\lambda}{2^N}} |\phi^{(\lambda)}_{N,p}(t)|^2dt+0\\
&= d_{N,p}\int_{\frac{p\lambda}{2^N}}^{\frac{(p+1)\lambda}{2^N}} \left(\frac{2^{\frac{N}{2}}}{\sqrt{m}}\phi^{(\lambda)}(2^Nt-p\lambda)\right)^2 dt,\\
&= d_{N,p}\int_0^{\lambda} \frac{2^N}{m}(\phi^{(\lambda)}(u))^2\frac{du}{2^N},\ u=2^Nt-p\lambda\\
&= \frac{\lambda}{m}d_{N,p}
\end{aligned}$$

Therefore, $d_{N,k}=\dfrac{m}{\lambda}<f,\phi^{(\lambda)}_{N,k}>, k=1,2,...,2^N-1.$ (4.2)

Next, multiplying both side of (4.1) by $\psi^{(\lambda)}_{l,q}$ and then integrating between $\left[\frac{q}{2^l}\lambda, \frac{q+1}{2^l}\lambda\right)$,

$$\begin{aligned}
\int_{\frac{q\lambda}{2^l}}^{\frac{(q+1)\lambda}{2^l}} f(t)\psi^{(\lambda)}_{l,q}(t)dt &= \sum_{k=0}^{2^N-1} d_{N,k}\int_{\frac{q\lambda}{2^l}}^{\frac{(q+1)\lambda}{2^l}} \phi^{(\lambda)}_{N,k}(t)\phi^{(\lambda)}_{l,q}(t)dt\\
&+ \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} c_{j,k}\int_{\frac{q\lambda}{2^l}}^{\frac{(q+1)\lambda}{2^l}} \psi^{(\lambda)}_{j,k}(t)\psi^{(\lambda)}_{l,q}(t)dt\\
&= 0+c_{l,q}\int_{\frac{q\lambda}{2^l}}^{\frac{(q+1)\lambda}{2^l}} |\psi^{(\lambda)}_{l,q}(t)|^2dt\\
&= c_{l,q}\int_{\frac{q\lambda}{2^l}}^{\frac{(q+1)\lambda}{2^l}} \left(\frac{2^{\frac{l}{2}}}{\sqrt{m}}\psi^{(\lambda)}(2^lt-q\lambda)\right)^2 dt
\end{aligned}$$

$$
\begin{aligned}
&= c_{l,q}\int_0^\lambda \frac{2^l}{m}(\psi^{(\lambda)}(u))^2\frac{du}{2^l},\ u=2^l t-q\lambda\\
&= \frac{\lambda}{m}c_{l,q}.\\
\text{Therefore,}\quad c_{j,k} &= \frac{m}{\lambda}< f,\psi^{(\lambda)}_{j,k}>,\ k=1,2,...,2^N-1. \qquad (4.3)\\
\text{Then}\quad f(t) &= \sum_{k=0}^{2^N-1}\frac{m}{\lambda}< f,\phi^{(\lambda)}_{N,k}>\phi^{(\lambda)}_{N,k}(t)+\sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1}\frac{m}{\lambda}< f,\psi^{(\lambda)}_{j,k}>\psi^{(\lambda)}_{j,k}(t).
\end{aligned}
$$

$$
\text{Let}\quad g(t)=\sum_{k=0}^{2^N-1}\frac{m}{\lambda}< f,\phi^{(\lambda)}_{N,k}>\phi^{(\lambda)}_{N,k}(t)+\sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1}\frac{m}{\lambda}< f,\psi^{(\lambda)}_{j,k}>\psi^{(\lambda)}_{j,k}(t) \qquad (4.4)
$$

Multiplying both side of (4.4) by $\phi^{(\lambda)}_{N,p}$ and then integrating between $\left[\frac{p}{2^N}\lambda,\frac{p+1}{2^N}\lambda\right)$,

$$
\begin{aligned}
< g,\phi^{(\lambda)}_{N,p}> &= \sum_{k=0}^{2^N-1}\frac{m}{\lambda}< f,\phi^{(\lambda)}_{N,k}>< \phi^{(\lambda)}_{N,k},\phi^{(\lambda)}_{N,p}>\\
&+ \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1}\frac{m}{\lambda}< f,\psi^{(\lambda)}_{j,k}>< \psi^{(\lambda)}_{j,k},\phi^{(\lambda)}_{N,k}>\\
&= < f,\phi^{(\lambda)}_{N,p}>+0\\
\text{Therefore,}\quad < f,\phi^{(\lambda)}_{N,k}> &= < g,\phi^{(\lambda)}_{N,k}>. \qquad (4.5)
\end{aligned}
$$

Also, multiplying both side of (4.4) by $\psi^{(\lambda)}_{l,q}$ and then integrating between $\left[\frac{q}{2^l}\lambda,\frac{q+1}{2^l}\lambda\right)$,

$$
\begin{aligned}
< g,\psi^{(\lambda)}_{l,q}> &= \sum_{k=0}^{2^N-1}\frac{m}{\lambda}< f,\phi^{(\lambda)}_{N,k}>< \phi^{(\lambda)}_{N,k},\psi^{(\lambda)}_{l,q}>\\
&+ \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1}\frac{m}{\lambda}< f,\psi^{(\lambda)}_{j,k}>< \psi^{(\lambda)}_{j,k},\psi^{(\lambda)}_{l,q}>\\
&= 0+< f,\psi^{(\lambda)}_{l,q}>\\
\text{Therefore,}\quad < f,\psi^{(\lambda)}_{j,k}> &= < g,\psi^{(\lambda)}_{j,k}>. \qquad (4.6)
\end{aligned}
$$

By Eqns. (4.5) and (4.6), $f=g\ \ \forall\ t\in[0,\lambda)$.

Hence, wavelet series (4.1) converges to f in $[0,\lambda)$. □

4.2. Convergence theorems of extended Haar wavelet series. In this section, two theorems for convergence of extended Haar wavelet series of function f in $H^\alpha[0,\lambda)$ & $H_2^\alpha[0,\lambda)$ have been established as follows:

Theorem 4.2. *If $f \in H^\alpha[0,\lambda)$ for $\frac{1}{2} < \alpha \leq 1$, then its extended Haar wavelet series*

$$f(t) = \sum_{k=0}^{2^N-1} \frac{m}{\lambda} < f, \phi_{N,k}^{(\lambda)} > \phi_{N,k}^{(\lambda)}(t) + \sum_{j=N}^{\infty} \sum_{k=0}^{2^j-1} \frac{m}{\lambda} < f, \psi_{j,k}^{(\lambda)} > \psi_{j,k}^{(\lambda)}(t) \tag{4.7}$$

converges uniformly to $f(t)$ in $[0,\lambda)$.

Proof. Following Eqn.(3.1) and values of $d_{N,k}$ & $c_{j,k}$ from previous lemma,

$$\begin{aligned}
f(t) &= \sum_{k=0}^{2^N-1} \frac{m}{\lambda} < f, \phi_{N,k}^{(\lambda)} > \phi_{N,k}^{(\lambda)}(t) + \sum_{j=N}^{\infty} \sum_{k=0}^{2^j-1} \frac{m}{\lambda} < f, \psi_{j,k}^{(\lambda)} > \psi_{j,k}^{(\lambda)}(t). \\
||f||_2 &\leq \sum_{k=0}^{2^N-1} |\frac{m}{\lambda} < f, \phi_{N,k}^{(\lambda)} > |||\phi_{N,k}^{(\lambda)}||_2 + \sum_{j=N}^{\infty} \sum_{k=0}^{2^j-1} |\frac{m}{\lambda} < f, \psi_{j,k}^{(\lambda)} > |||\psi_{j,k}^{(\lambda)}||_2 \\
&= \sqrt{\frac{m}{\lambda}} \sum_{k=0}^{2^N-1} | < f, \phi_{N,k}^{(\lambda)} > | + \sqrt{\frac{m}{\lambda}} \sum_{j=N}^{\infty} \sum_{k=0}^{2^j-1} | < f, \psi_{j,k}^{(\lambda)} > |. \qquad (4.8)
\end{aligned}$$

$$\begin{aligned}
\text{Now} \quad < f, \phi_{N,k}^{(\lambda)} > &= \int_{\frac{k\lambda}{2^N}}^{\frac{(k+1)\lambda}{2^N}} \left(f(t) - f\left(\frac{k\lambda}{2^N}\right)\right) \phi_{N,k}(t)dt + f\left(\frac{k\lambda}{2^N}\right) \int_{\frac{k\lambda}{2^N}}^{\frac{(k+1)\lambda}{2^N}} \phi_{N,k}(t)dt \\
| < f, \phi_{N,k}^{(\lambda)} > | &\leq \left(\frac{\lambda}{2^N}\right)^\alpha \int_{\frac{k\lambda}{2^N}}^{\frac{(k+1)\lambda}{2^N}} 2^{\frac{N}{2}} \phi^{(\lambda)}(2^N t - k\lambda)dt \\
&+ \left|f\left(\frac{k\lambda}{2^N}\right)\right| \int_{\frac{k\lambda}{2^N}}^{\frac{(k+1)\lambda}{2^N}} 2^{\frac{N}{2}} \phi^{(\lambda)}(2^N t - k\lambda)dt, \ f \in H^\alpha[0,\lambda) \\
&= \left(\frac{\lambda}{2^N}\right)^\alpha \frac{1}{2^{\frac{N}{2}}} \int_0^\lambda \phi^{(\lambda)}(u)du + \left|f\left(\frac{k\lambda}{2^N}\right)\right| \frac{1}{2^{\frac{N}{2}}} \int_0^\lambda \phi^{(\lambda)}(u)du, \ u = 2^N t - k\lambda \\
&= \left(\frac{\lambda}{2^N}\right)^\alpha \frac{1}{2^{\frac{N}{2}}} \frac{\lambda}{\sqrt{m}} + \left|f\left(\frac{k\lambda}{2^N}\right)\right| \frac{1}{2^{\frac{N}{2}}} \frac{\lambda}{\sqrt{m}} \\
&= \frac{\lambda}{\sqrt{m}} \left(\frac{\sqrt{m}}{2^{\frac{N}{2}}} \left(\frac{\lambda}{2^N}\right)^\alpha + \frac{\sqrt{m}}{2^{\frac{N}{2}}} \left|f\left(\frac{k\lambda}{2^N}\right)\right|\right). \qquad (4.9) \\
\text{Next,} \quad \left\langle f, \psi_{j,k}^{(\lambda)} \right\rangle &= \int_{\frac{k\lambda}{2^j}}^{\frac{(k+1)\lambda}{2^j}} f(t) \psi_{j,k}^{(\lambda)}(t)dt \\
&= \int_{\frac{k\lambda}{2^j}}^{\frac{(k+\frac{1}{2})\lambda}{2^j}} f(t) \frac{2^{\frac{j}{2}}}{\sqrt{m}} dt - \int_{\frac{(k+\frac{1}{2})\lambda}{2^j}}^{\frac{(k+1)\lambda}{2^j}} f(t) \frac{2^{\frac{j}{2}}}{\sqrt{m}} dt \\
&= \int_{\frac{k\lambda}{2^j}}^{\frac{(k+\frac{1}{2})\lambda}{2^j}} f(t) \frac{2^{\frac{j}{2}}}{\sqrt{m}} dt - \int_{\frac{k\lambda}{2^j}}^{\frac{(k+\frac{1}{2})\lambda}{2^j}} f\left(u + \frac{\lambda}{2^{j+1}}\right) \frac{2^{\frac{j}{2}}}{\sqrt{m}} du, \quad t = u + \frac{\lambda}{2^{j+1}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2^{\frac{j}{2}}}{\sqrt{m}} \int_{\frac{k\lambda}{2^j}}^{\frac{(k+\frac{1}{2})\lambda}{2^j}} \left(f(t) - f\left(t + \frac{\lambda}{2^{j+1}}\right)\right) dt \\
\left|\left\langle f, \psi_{j,k}^{(\lambda)} \right\rangle\right| &\leq \frac{2^{\frac{j}{2}}}{\sqrt{m}} \left(\frac{\lambda}{2^{j+1}}\right)^{\alpha} \int_{\frac{k\lambda}{2^j}}^{\frac{(k+\frac{1}{2})\lambda}{2^j}} dt, \quad f \in H^{\alpha}[0,\lambda) \\
&= \frac{2^{\frac{j}{2}}}{\sqrt{m}} \left(\frac{\lambda}{2^{j+1}}\right)^{\alpha+1}. \qquad (4.10)
\end{aligned}
$$

From Eqns. (4.8) , (4.9) and (4.10),

$$
\begin{aligned}
\|f\|_2 &\leq \sqrt{\frac{m}{\lambda}} \sum_{k=0}^{2^N-1} \frac{1}{\sqrt{m} 2^{\frac{N}{2}}} \left(\left(\frac{\lambda}{2^N}\right)^{\alpha} + \left| f\left(\frac{k\lambda}{2^N}\right)\right| \right) \\
&+ \sqrt{\frac{\lambda}{m}} \sum_{j=N}^{\infty} \sum_{k=0}^{2^j-1} \frac{2^{\frac{j}{2}}\sqrt{m}}{\lambda} \left(\frac{\lambda}{2^{j+1}}\right)^{\alpha+1} \\
&= \frac{1}{\sqrt{\lambda} 2^{\frac{N}{2}}} \sum_{k=0}^{2^N-1} \left(\left(\frac{\lambda}{2^N}\right)^{\alpha} + \left| f\left(\frac{k\lambda}{2^N}\right)\right| \right) + \frac{1}{\sqrt{\lambda}} \sum_{j=N}^{\infty} 2^{\frac{3j}{2}} \left(\frac{\lambda}{2^{j+1}}\right)^{\alpha+1} \\
&= \frac{1}{\sqrt{\lambda} 2^{\frac{N}{2}}} \left(2^N \left(\frac{\lambda}{2^N}\right)^{\alpha} + \sum_{k=0}^{2^N-1} \left| f\left(\frac{k\lambda}{2^N}\right)\right| \right) + \lambda^{\alpha+\frac{1}{2}} \sum_{j=N}^{\infty} \frac{1}{2^{(j+1)(\alpha+1)-\frac{3j}{2}}} \\
&\leq \frac{\lambda^{\alpha-\frac{1}{2}}}{2^{N(\alpha-\frac{1}{2})}} + \frac{1}{\sqrt{\lambda} 2^{\frac{N}{2}}} C_f + \frac{\lambda^{\alpha+\frac{1}{2}}}{2^{N(\alpha-\frac{1}{2})+\alpha+1}} < \infty, \quad \alpha > \frac{1}{2}, \qquad (4.11) \\
&\quad \text{where } C_f = \sum_{k=0}^{2^N-1} \left| f\left(\frac{k\lambda}{2^N}\right)\right|.
\end{aligned}
$$

Therefore, the series (4.7) converges to $f(t)$ uniformly in [0,λ). □

Theorem 4.3. *If $f \in H_2^{\alpha}[0,\lambda)$ for $\frac{1}{2} < \alpha \leq 1$, then its extended Haar wavelet series*

$$
f(t) = \sum_{k=0}^{2^N-1} \frac{m}{\lambda} < f, \phi_{N,k}^{(\lambda)} > \phi_{N,k}^{(\lambda)}(t) + \sum_{j=N}^{\infty} \sum_{k=0}^{2^j-1} \frac{m}{\lambda} < f, \psi_{j,k}^{(\lambda)} > \psi_{j,k}^{(\lambda)}(t) \qquad (4.12)
$$

converges uniformly to $f(t)$ in $[0,\lambda)$.

Proof. Following the proof of Theorem 4.2,

$$
\begin{aligned}
| < f, \phi_{N,k}^{(\lambda)} > | &\leq \left| \int_{\frac{k\lambda}{2^N}}^{\frac{(k+1)\lambda}{2^N}} \left(f(t) - f\left(\frac{k\lambda}{2^N}\right)\right) \phi_{N,k}(t) dt \right| + \left| f\left(\frac{k\lambda}{2^N}\right) \int_{\frac{k\lambda}{2^N}}^{\frac{(k+1)\lambda}{2^N}} \phi_{N,k}(t) dt \right| \\
&\leq \left(\int_{\frac{k\lambda}{2^j}}^{\frac{(k+\frac{1}{2})\lambda}{2^j}} \left| f(t) - f\left(t + \frac{\lambda}{2^{j+1}}\right)\right|^2 dt \right)^{\frac{1}{2}} \left(\int_{\frac{k\lambda}{2^j}}^{\frac{(k+\frac{1}{2})\lambda}{2^j}} \|\psi_{j,k}^{(\lambda)}\|^2 dt \right)^{\frac{1}{2}} \\
&+ \left| f\left(\frac{k\lambda}{2^N}\right)\right| \frac{1}{2^{\frac{N}{2}}} \frac{\lambda}{\sqrt{m}}, \text{ by Cauchy Schwarz's inequality.}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left(\frac{\lambda}{2^N}\right)^{\alpha}\left(\frac{\lambda}{m}\frac{\lambda}{2^N}\right)^{\frac{1}{2}} + \left|f\left(\frac{k\lambda}{2^N}\right)\right|\frac{1}{2^{\frac{N}{2}}}\frac{\lambda}{\sqrt{m}},\ f\in H_2^{\alpha}[0,\lambda)\\
&= \left(\frac{\lambda}{2^N}\right)^{\alpha}\frac{1}{2^{\frac{N}{2}}}\frac{\lambda}{\sqrt{m}} + \left|f\left(\frac{k\lambda}{2^N}\right)\right|\frac{1}{2^{\frac{N}{2}}}\frac{\lambda}{\sqrt{m}}\\
&= \frac{\lambda}{\sqrt{m}2^{\frac{N}{2}}}\left(\left(\frac{\lambda}{2^N}\right)^{\alpha} + \left|f\left(\frac{k\lambda}{2^N}\right)\right|\right). \qquad (4.13)
\end{aligned}
$$

$$
\begin{aligned}
\text{Next, }\left|\left\langle f,\psi_{j,k}^{(\lambda)}\right\rangle\right| &\leq \int_{\frac{k\lambda}{2^j}}^{\frac{(k+\frac{1}{2})\lambda}{2^j}}\left|f(t)-f\left(t+\frac{\lambda}{2^{j+1}}\right)\right|||\psi_{j,k}^{(\lambda)}||dt\\
&\leq \left(\frac{\lambda}{2^{j+1}}\right)^{\alpha}\left(\frac{\lambda}{m}\frac{\lambda}{2^{j+1}}\right)^{\frac{1}{2}},\quad f\in H_2^{\alpha}[0,\lambda). \qquad (4.14)
\end{aligned}
$$

From Eqns. (4.8) , (4.13) and (4.14),

$$
\begin{aligned}
||f||_2 \quad &\leq \quad \sqrt{\frac{m}{\lambda}}\sum_{k=0}^{2^N-1}\frac{1}{\sqrt{m}2^{\frac{N}{2}}}\left(\left(\frac{\lambda}{2^N}\right)^{\alpha}+\left|f\left(\frac{k\lambda}{2^N}\right)\right|\right)\\
&+ \quad \sqrt{\frac{m}{\lambda}}\sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1}\left(\frac{\lambda}{2^{j+1}}\right)^{\alpha}\left(\frac{\lambda}{m}\frac{\lambda}{2^{j+1}}\right)^{\frac{1}{2}}\\
&= \quad \frac{\lambda^{\alpha-\frac{1}{2}}}{2^{N(\alpha-\frac{1}{2})}}+\frac{1}{\sqrt{\lambda}2^{\frac{N}{2}}}C_f+\frac{1}{\sqrt{\lambda}}\sum_{j=N}^{\infty}2^j\left(\frac{\lambda}{2^{j+1}}\right)^{\alpha+1}\\
&\leq \quad \frac{\lambda^{\alpha-\frac{1}{2}}}{2^{N(\alpha-\frac{1}{2})}}+\frac{1}{\sqrt{\lambda}2^{\frac{N}{2}}}C_f+\frac{\lambda^{\alpha+\frac{1}{2}}}{2^{N(\alpha-\frac{1}{2})+\alpha+\frac{1}{2}}}<\infty,\quad \alpha>\frac{1}{2} \qquad (4.15)
\end{aligned}
$$

Therefore, the series (4.12) converges to $f(t)$ uniformly in [0,λ). □

5. Approximation of function by partial sums of its Extended Haar Wavelet Series

Any measurement is considered as approximation. A function $f \in L^2[0,\lambda)$ has been expressed into extended Haar wavelet series as Eqn. (2.4). The $(2^N)^{th}$ partial sum of the wavelet series (2.4) is $(S_{2^N}f)(t) = \sum_{k=0}^{2^N-1} d_{N,k}\phi_{N,k}^{(\lambda)}(t)$. The approximation of f by $(S_{2^N}f)$ under $||\cdot||_2$ is denoted by $E_{2^N}(f) = ||f-(S_{2^N}f)||_2$. Since wavelet series (2.4) is convergent and it is already discussed in previous section 3, therefore it is natural to find out the bound of $||f-(S_{2^N}f)||_2$. To estimate the bound of $||f-(S_{2^N}f)||_2$, the following estimators have been developed as follows:

Theorem 5.1. *If a function $f \in H^{\alpha}[0,\lambda)$ i.e. $f(x+t)-f(x) = O(|t|^{\alpha})$, $x+t$, $x \in [0,\lambda)$, $0<\alpha\leq 1$, and its wavelet expansion by extended Haar wavelet $\psi_{j,k}^{(\lambda)}$ and extended Haar* *scaling function* $\phi_{j,k}^{(\lambda)}$ *is given by*

$$\begin{aligned} f(t) &= d_{0,0}\phi_{0,0}^{(\lambda)}(t)+\sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1}c_{j,k}\psi_{j,k}^{(\lambda)}(t) \qquad (5.1)\\ &= \left(d_{0,0}\phi_{0,0}^{(\lambda)}(t)+\sum_{j=0}^{N-1}\sum_{k=0}^{2^j-1}c_{j,k}\psi_{j,k}^{(\lambda)}(t)\right)+\sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1}c_{j,k}\psi_{j,k}^{(\lambda)}(t)\\ &= (S_Nf)(t)+(R_Nf)(t) \end{aligned}$$

then the extended Haar wavelet approximation $E_N^{(1)}(f)$ *of* f *by* (S_Nf) *satisfies*

$$E_N^{(1)}(f) = \min_{(S_Nf)}||f-(S_Nf)||_2 = O\left(\frac{\lambda^{\alpha+\frac{1}{2}}}{2^{(N+1)\alpha}}\right). \qquad (5.2)$$

Proof. Since

$$\begin{aligned} f(t)-(S_Nf)(t) &= \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1}c_{j,k}\psi_{j,k}^{(\lambda)}(t).\\ \text{Therefore}\quad (f(t)-(S_Nf)(t))^2 &= \left(\sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1}c_{j,k}\psi_{j,k}^{(\lambda)}(t)\right)^2\\ &= \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1}c_{j,k}^2\left(\psi_{j,k}^{(\lambda)}(t)\right)^2\\ &+ 2\sum\sum_{N\le j\ne j'\le\infty}\sum_{k=0}^{2^j-1}\sum_{k'=0}^{2^{j'}-1}c_{j,k}\psi_{j,k}^{(\lambda)}(t)c_{j',k'}\psi_{j',k'}^{(\lambda)}(t).\\ \text{Now}\quad ||f-(S_Nf)||_2^2 &= \int_0^{\lambda}\left(\sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1}c_{j,k}\psi_{j,k}^{(\lambda)}(t)\right)^2dt\\ &= \int_0^{\lambda}(\sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1}c_{j,k}^2\left(\psi_{j,k}^{(\lambda)}(t)\right)^2dt\\ &+ 2\int_0^{\lambda}\sum\sum_{N\le j\ne j'\le\infty}\sum_{k=0}^{2^j-1}\sum_{k'=0}^{2^{j'}-1}c_{j,k}\psi_{j,k}^{(\lambda)}(t)c_{j',k'}\psi_{j',k'}^{(\lambda)}(t)dt \end{aligned}$$

$$= \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} c_{j,k}^2\frac{\lambda}{m}+0, \text{ by orthonormality of } \psi_{j,k}^{(\lambda)}(t)$$

$$= \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} c_{j,k}^2\frac{\lambda}{m} \tag{5.3}$$

By Eqns. (5.3) and (4.10),

$$||f-(S_Nf)||_2^2 \leq \frac{m}{\lambda}\sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1}\frac{2^j}{m}\left(\frac{\lambda}{2^{j+1}}\right)^{2(\alpha+1)}$$

$$= \frac{\lambda^{2\alpha+1}}{2^{2\alpha+2}}\sum_{j=N}^{\infty}\frac{1}{2^{2j\alpha}}$$

$$= \frac{\lambda^{2\alpha+1}}{2^{2\alpha+2}}\frac{\left(\frac{1}{2^{2N\alpha}}\right)}{\left(1-\frac{1}{2^{2\alpha}}\right)}$$

$$||f-(S_Nf)||_2 \leq \frac{\lambda^{\alpha+\frac{1}{2}}}{2^{\alpha+1}2^{N\alpha}}\sqrt{\frac{1}{1-\frac{1}{2^{2\alpha}}}}$$

$$\text{Therefore } E_N^{(1)}(f) = \min_{(S_Nf)}||f-(S_Nf)||_2 = O\left(\frac{\lambda^{\alpha+\frac{1}{2}}}{2^{(N+1)\alpha}}\right). \tag{5.4}$$

This completes the proof of Theorem 5.1.

□

Note. To validate the extended Haar wavelet series (5.1) and Theorem of this section, the following example has been considered:

$$\text{Let } f(t) = t, \ t\in[0,\lambda),$$

$$||f||_2^2 = \int_0^{\lambda} t^2dt = \frac{\lambda^3}{3}.$$

$$f(t) = d_{0,0}\phi_{0,0}^{(\lambda)}(t)+\sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1}c_{j,k}\psi_{j,k}^{(\lambda)}(t)$$

$$< f,\phi_{0,0}^{(\lambda)} > = \int_0^{\lambda} t\phi_{0,0}^{(\lambda)}(t)dt = \frac{\lambda^2}{2\sqrt{m}}$$

$$\text{Next, } < f,\psi_{j,k}^{(\lambda)} > = \int_0^{\lambda} t\psi_{j,k}^{(\lambda)}(t)dt$$

$$= \int_{\frac{k\lambda}{2^j}}^{\frac{(k+\frac{1}{2})\lambda}{2^j}} t\frac{2^{\frac{j}{2}}}{\sqrt{m}}dt - \int_{\frac{(k+\frac{1}{2})\lambda}{2^j}}^{\frac{(k+1)\lambda}{2^j}} t\frac{2^{\frac{j}{2}}}{\sqrt{m}}dt$$

$$
\begin{aligned}
&= \frac{2^{\frac{j}{2}}}{2\sqrt{m}} \frac{\lambda^2}{2^{2j}} \left[2\left(k^2+k+\frac{1}{4}\right) - 2k^2 - 2k - 1\right] \\
&= \frac{2^{\frac{j}{2}}}{2\sqrt{m}} \frac{\lambda^2}{2^{2j}} \left(-\frac{1}{2}\right) \\
&= -\frac{\lambda^2}{(4\sqrt{m})2^{\frac{3j}{2}}}. \qquad (5.5)
\end{aligned}
$$

Therefore, $$d_{0,0}^2 ||\phi_{0,0}^{(\lambda)}||_2^2 + \sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1} c_{j,k}^2 ||\psi_{j,k}^{(\lambda)}||_2^2$$

$$
\begin{aligned}
&= \left(\frac{m}{\lambda} < f, \phi_{0,0}^{(\lambda)} >\right)^2 \frac{\lambda}{m} + \sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1} \left(\frac{m}{\lambda} < f, \psi_{j,k}^{(\lambda)} >\right)^2 \frac{\lambda}{m} \\
&= \frac{m}{\lambda}\frac{\lambda^4}{4m} + \sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1} \frac{m}{\lambda}\frac{\lambda^4}{16m(2^{3j})} \\
&= \frac{\lambda^3}{4} + \frac{\lambda^3}{16}\sum_{j=0}^{\infty}\frac{1}{2^{2j}} = \frac{\lambda^3}{3}.
\end{aligned}
$$

Hence the series (5.1) is verified for $f(t) = t$.

$$
\begin{aligned}
\text{Again let } f(t) &= t, \ \forall\, t \in [0, \lambda). \\
\text{Since, } |f(x+t) - f(x)| &= |(x+t) - x| = |t|. \text{ Therefore, } f \in H^1[0, \lambda). \\
\text{Now } ||f - (S_N f)||_2^2 &= \frac{m}{\lambda}\sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} | < f, \psi_{j,k}^{(\lambda)} > |^2 \\
&= \frac{m}{\lambda}\sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} \frac{\lambda^4}{16m(2^{3j})} \\
&= \frac{\lambda^3}{16}\sum_{j=N}^{\infty}\frac{1}{2^{2j}} = \frac{\lambda^3}{3(2^{2N+2})}
\end{aligned}
$$

$$||f - (S_N f)||_2 = \frac{\lambda^{\frac{3}{2}}}{\sqrt{3}2^{N+1}} = O\left(\frac{\lambda^{\alpha+\frac{1}{2}}}{2^{(N+1)\alpha}}\right), \text{ if } \alpha = 1.$$

Hence, Theorem 5.1 is verified.

Theorem 5.2. *If a function* $f \in H_2^{\alpha}[0, \lambda)$ *i.e.* $\left(\int_0^{\lambda}(f(x+t) - f(x))^2\right)^{\frac{1}{2}} = O(|t|^{\alpha})$,

$$\text{and } f(t) = \sum_{k=0}^{2^N-1} d_{N,k}\phi_{N,k}^{(\lambda)}(t) + \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} c_{j,k}\psi_{j,k}^{(\lambda)}(t) = (S_{2^N} f)(t) + (R_{2^N} f)(t), \qquad (5.6)$$

then the extended Haar wavelet approximation $E^{(2)}_{2^N}(f)$ of f by $(S_{2^N}f)$ satisfies

$$E^{(2)}_{2^N}(f) = \min_{(S_{2^N}f)} ||f-(S_{2^N}f)||_2 = O\left(\frac{\lambda^{\alpha+\frac{1}{2}}}{2^{(N+1)\alpha+\frac{1}{2}}}\right).$$

Proof. Following the proof of Theorem 5.1,

$$||f-(S_{2^N}f)||_2^2 = \frac{m}{\lambda}\sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1}\left(\left\langle f,\psi^{(\lambda)}_{j,k}\right\rangle\right)^2 \tag{5.7}$$

By Eqns. (5.7) and (4.14),

$$\begin{aligned}
||f-(S_{2^N}f)||_2^2 &\leq \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1}\frac{\lambda^{2\alpha+1}}{2^{(j+1)(2\alpha+1)}}\\
&= \lambda^{2\alpha+1}\sum_{j=N}^{\infty}\frac{1}{2^{2(j+1)\alpha+1}}\\
&= \lambda^{2\alpha+1}\left(\frac{\frac{1}{2^{2(N+1)\alpha+1}}}{(1-\frac{1}{2^{2\alpha}})}\right).
\end{aligned}$$

$$\text{Therefore } E^{(2)}_{2^N}(f) = O\left(\frac{\lambda^{\alpha+\frac{1}{2}}}{2^{(N+1)\alpha+\frac{1}{2}}}\right). \tag{5.8}$$

This completes the proof of Theorem 5.2.

Note. To validate the extended Haar wavelet series (5.6) and Theorem of this section, the following example has been considered:

$$\begin{aligned}
\text{Let } f(t) &= t^2,\ t\in[0,\lambda).\\
||f||_2^2 &= \int_0^{\lambda}(t^2)^2dt = \frac{\lambda^5}{5}.\\
\text{Since } f(t) &= \sum_{k=0}^{2^N-1} d_{N,k}\phi^{(\lambda)}_{N,k}(t) + \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} c_{j,k}\psi^{(\lambda)}_{j,k}(t)\\
||f||_2^2 &= \sum_{k=0}^{2^N-1} d^2_{N,k}||\phi^{(\lambda)}_{N,k}||_2^2 + \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} c^2_{j,k}||\psi^{(\lambda)}_{j,k}||_2^2.\\
d_{N,k} &= \frac{m}{\lambda} < f,\phi^{(\lambda)}_{N,k} >\\
&= \frac{m}{\lambda}\int_{\frac{k\lambda}{2^N}}^{\frac{(k+1)\lambda}{2^N}} t^2 2^{\frac{N}{2}}\phi^{(\lambda)}(2^Nt-k\lambda)dt
\end{aligned}$$

$$\begin{aligned}
&= \frac{m}{\lambda}\int_0^{\lambda}\left(\frac{u+k\lambda}{2^N}\right)^2\frac{1}{2^{\frac{N}{2}}}\phi^{(\lambda)}(u)du,\ u=2^Nt-k\lambda.\\
&= \frac{m}{\lambda}\frac{1}{2^{\frac{5N}{2}}}\int_0^{\lambda}\frac{1}{\sqrt{m}}(u^2+2k\lambda+k^2\lambda^2)du\\
&= \frac{\sqrt{m}\lambda^2}{2^{\frac{5N}{2}}}\left[\frac{1}{3}+k+k^2\right].\\
c_{j,k} &= \frac{m}{\lambda}<f,\psi_{j,k}^{(\lambda)}>=\frac{m}{\lambda}\int_0^{\lambda}t^2\psi_{j,k}^{(\lambda)}(t)dt\\
&= \frac{m}{\lambda}\int_{\frac{k\lambda}{2^j}}^{\frac{(k+\frac{1}{2})\lambda}{2^j}}t^2\frac{2^{\frac{j}{2}}}{\sqrt{m}}dt-\frac{m}{\lambda}\int_{\frac{(k+\frac{1}{2})\lambda}{2^j}}^{\frac{(k+1)\lambda}{2^j}}t^2\frac{2^{\frac{j}{2}}}{\sqrt{m}}dt\\
&= \frac{\sqrt{m}2^{\frac{j}{2}}}{2}\frac{\lambda^2}{2^{3j}}\left[2\left(k+\frac{1}{2}\right)^2-k^3-(k+1)^3\right]\\
&= -\frac{\sqrt{m}\lambda^2}{(4)2^{\frac{5j}{2}}}(2k+1). \qquad (5.9)
\end{aligned}$$

Therefore,
$$\begin{aligned}
&\sum_{k=0}^{2^N-1}d_{N,k}^2||\phi_{N,k}^{(\lambda)}||_2^2+\sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1}c_{j,k}^2||\psi_{j,k}^{(\lambda)}||_2^2\\
&= \sum_{k=0}^{2^N-1}\frac{m\lambda^4}{2^{5N}}\left(\frac{1}{3}+k+k^2\right)^2\frac{\lambda}{m}\\
&+ \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1}\frac{m\lambda^4}{(16)2^{5j}}(1+2k)^2\frac{\lambda}{m}\\
&= \frac{\lambda^5}{2^{5N}}\left[\frac{2^N}{45}-\frac{2^{3N}}{9}+\frac{2^{5N}}{5}\right]+\frac{\lambda^5}{16}\left[\frac{16}{9(2^{2N})}-\frac{16}{45(2^{4N})}\right]\\
&= \lambda^5\left[\frac{1}{45(2^{4N})}-\frac{1}{9(2^{2N})}+\frac{1}{5}-\frac{1}{45(2^{4N})}+\frac{1}{9(2^{2N})}\right]=\frac{\lambda^5}{5}.
\end{aligned}$$

Hence the series (5.6) is verified for $f(t)=t^2$.

Again let $f(t)=t^2,\ t\in[0,\lambda)$.

Then
$$\begin{aligned}
&\left(\int_0^{\lambda}|f(x+t)-f(x)|^2dx\right)^{\frac{1}{2}}\\
&= \left(\int_0^{\lambda}|(x+t)^2-(x^2)|^2dx\right)^{\frac{1}{2}} \qquad (5.10)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_0^\lambda (t^2+2tx)^2 dx\right)^{\frac{1}{2}} \\
&= \left(\int_0^\lambda (t^4+4t^2x^2+4t^3x) dx\right)^{\frac{1}{2}} \\
&= \left(t^4\lambda + \frac{4}{3}t^2\lambda^3 + 2t^3\lambda^2\right)^{\frac{1}{2}} \\
&\leq \left(t^2\lambda^3 + \frac{4}{3}t^2\lambda^3 + 2t^2\lambda^3\right)^{\frac{1}{2}} \\
&= \left(\frac{13}{3}|t|^2\lambda^3\right)^{\frac{1}{2}} = |t|\left(\frac{13}{3}\lambda^3\right)^{\frac{1}{2}}.
\end{aligned}
$$

Therefore, $f \in H_2^1[0,\lambda)$.

$$
\begin{aligned}
\text{Now } \ ||f-(S_{2^N}f)||_2^2 &= \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} c_{j,k}^2 ||\psi_{j,k}^{(\lambda)}||_2^2 \\
&= \frac{\lambda}{m}\sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} \frac{m\lambda^2}{16(2^{5j})}(2k+1)^2, \quad \text{by Eqn. (5.9)} \\
&= \frac{\lambda^3}{16}\sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} \frac{1}{2^{5j}}(4k^2+4k+1) \\
&= \frac{\lambda^3}{16}\sum_{j=N}^{\infty} \frac{1}{2^{5j}}\left[\frac{2}{3}2^j(2^j-1)(2^{j+1}-1)+2^{j+1}(2^j-1)+2^j\right] \\
&= \frac{\lambda^3}{16}\sum_{j=N}^{\infty} \frac{1}{3}\left[\frac{4}{2^{2j}}-\frac{1}{2^{4j}}\right] \\
&= \frac{\lambda^3}{3}\left[\frac{1}{3.2^{2N}}-\frac{1}{15.2^{4N}}\right] \\
||f-(S_{2^N}f)||_2 &= \frac{\lambda^{\frac{3}{2}}}{3.2^N}\sqrt{1-\frac{1}{5.2^{2N}}} \\
&\leq \frac{\lambda^{\frac{3}{2}}}{2^{N+\frac{3}{2}}} = O\left(\frac{\lambda^{\alpha+\frac{1}{2}}}{2^{(N+1)\alpha+\frac{1}{2}}}\right), \text{ if } \alpha = 1.
\end{aligned}
$$

Hence, Theorem 5.2 is verified. □

Theorem 5.3. *Let $\omega(t)$ be a positive monotonic increasing function of t such that $|t|^\alpha\omega(|t|) \to 0$ as $t \to 0^+$. If a function $f \in H^{\alpha,\omega}[0,\lambda)$ i.e. $f(x+t)-f(x) = O(|t|^\alpha\omega(|t|))$,* *and*

$$\begin{aligned} f(t) &= d_{0,0}\phi_{0,0}^{(\lambda)}(t) + \sum_{j=0}^{N-1}\sum_{k=0}^{2^j-1} c_{j,k}\psi_{j,k}^{(\lambda)}(t) + \sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} c_{j,k}\psi_{j,k}^{(\lambda)}(t) \\ &= (S_N f)(t) + (R_N f)(t), \end{aligned}$$

then the extended Haar wavelet approximation $E_N^{(3)}(f)$ of f by $(S_N f)$ satisfies

$$E_N^{(3)}(f) = \underset{(S_N f)}{min} ||f - (S_N f)||_2 = O\left(\frac{\lambda^{\alpha+\frac{1}{2}}}{2^{(N+1)\alpha}}\omega\left(\frac{\lambda}{2^N}\right)\right). \tag{5.11}$$

Proof. Following the proof of Theorem 5.1,

$$\begin{aligned} |< f, \psi_{j,k}^{(\lambda)} >| &\leq \frac{2^{\frac{j}{2}}}{\sqrt{m}} \int_{\frac{k\lambda}{2^j}}^{\frac{(k+\frac{1}{2})\lambda}{2^j}} \left| f(t) - f\left(t + \frac{\lambda}{2^{j+1}}\right)\right| dt \\ &\leq \frac{2^{\frac{j}{2}}}{\sqrt{m}} \left(\frac{\lambda}{2^{j+1}}\right)^{\alpha} \int_{\frac{k\lambda}{2^j}}^{\frac{(k+\frac{1}{2})\lambda}{2^j}} \omega\left(\frac{\lambda}{2^{j+1}}\right) dt, \ f \in H^{\alpha,\omega}[0,\lambda) \\ &= \frac{2^{\frac{j}{2}}}{\sqrt{m}} \left(\frac{\lambda}{2^{j+1}}\right)^{\alpha} \omega\left(\frac{\lambda}{2^{j+1}}\right)\left(\frac{\lambda}{2^{j+1}}\right) \end{aligned}$$

$$= \frac{2^{\frac{j}{2}}}{\sqrt{m}} \left(\frac{\lambda}{2^{j+1}}\right)^{\alpha+1} \omega\left(\frac{\lambda}{2^{j+1}}\right). \tag{5.12}$$

By Eqn. (5.12),

$$\begin{aligned} ||f - (S_N f)||_2^2 &\leq \frac{m}{\lambda}\sum_{j=N}^{\infty}\sum_{k=0}^{2^j-1} \frac{2^j}{m}\left(\frac{\lambda}{2^{j+1}}\right)^{2(\alpha+1)} \omega^2\left(\frac{\lambda}{2^{j+1}}\right) \\ &= \frac{\lambda^{2\alpha+1}}{2^{2\alpha+2}}\omega^2\left(\frac{\lambda}{2^{N+1}}\right)\sum_{j=N}^{\infty}\frac{1}{2^{2j\alpha}}, \text{ by the hypothesis of Theorem 5.1,} \\ &= \frac{\lambda^{2\alpha+1}}{2^{2\alpha+2}}\omega^2\left(\frac{\lambda}{2^{N+1}}\right)\frac{\left(\frac{1}{2^{2N\alpha}}\right)}{\left(1 - \frac{1}{2^{2\alpha}}\right)} \\ ||f - (S_N f)||_2 &\leq \frac{\lambda^{\alpha+\frac{1}{2}}}{2^{\alpha+1}2^{N\alpha}}\omega\left(\frac{\lambda}{2^{N+1}}\right)\sqrt{\frac{1}{1-\frac{1}{2^{2\alpha}}}} \\ \text{Therefore } E_N^{(3)}(f) &= O\left(\frac{\lambda^{\alpha+\frac{1}{2}}}{2^{(N+1)\alpha}}\omega\left(\frac{\lambda}{2^{N+1}}\right)\right). \end{aligned} \tag{5.13}$$

This completes the proof of Theorem 5.3.

Note. If $\omega(|t|) = 1 \ \forall \ t \in [0,\lambda)$, then class $H^{\alpha,w}[0,\lambda)$ coincides with $H^{\alpha}[0,\lambda)$. $\square$

6. Extended Haar Wavelet Operational Matrix of Integration

The operational matrix of integration P_4 is determined by the four basis functions as

$$\phi_{0,0}^{(\lambda)}(t) = \frac{1}{\sqrt{m}}\begin{cases}1, & if \;\; 0 \le t < \lambda,\\ 0, & \text{otherwise,}\end{cases}$$

$$\psi_{0,0}^{(\lambda)}(t) = \frac{1}{\sqrt{m}}\begin{cases}1, & if \;\; 0 \le t < \frac{\lambda}{2},\\ -1, & if \;\; \frac{\lambda}{2} \le t < \lambda,\\ 0, & \text{otherwise,}\end{cases}$$

$$\psi_{1,0}^{(\lambda)}(t) = \frac{1}{\sqrt{m}}\begin{cases}\sqrt{2}, & if \;\; 0 \le t < \frac{\lambda}{4},\\ -\sqrt{2}, & if \;\; \frac{\lambda}{4} \le t < \frac{\lambda}{2},\\ 0, & \text{otherwise,}\end{cases}$$

$$\psi_{1,1}^{(\lambda)}(t) = \frac{1}{\sqrt{m}}\begin{cases}\sqrt{2}, & if \;\; \frac{\lambda}{2} \le t < \frac{3\lambda}{4},\\ -\sqrt{2}, & if \;\; \frac{3\lambda}{4} \le t < \lambda,\\ 0, & \text{otherwise.}\end{cases}$$

$$\left.\begin{aligned}\phi_{0,0}^{(\lambda)}(t) &= \frac{1}{\sqrt{m}}[1\ 1\ 1\ 1]\\ \psi_{0,0}^{(\lambda)}(t) &= \frac{1}{\sqrt{m}}[1\ 1\ -1\ -1]\\ \psi_{1,0}^{(\lambda)}(t) &= \frac{1}{\sqrt{m}}[\sqrt{2}\ -\sqrt{2}\ 0\ 0]\\ \psi_{1,1}^{(\lambda)}(t) &= \frac{1}{\sqrt{m}}[0\ 0\ \sqrt{2}\ -\sqrt{2}]\end{aligned}\right\} \tag{6.1}$$

From Eqn. (6.1),

$$H_4 = \frac{1}{\sqrt{m}}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ \sqrt{2} & -\sqrt{2} & 0 & 0\\ 0 & 0 & \sqrt{2} & -\sqrt{2}\end{bmatrix}$$

The integrals of the first four Haar wavelets $\phi_{0,0}^{(\lambda)}, \psi_{0,0}^{(\lambda)}, \psi_{1,0}^{(\lambda)}, \psi_{1,1}^{(\lambda)}$ can be expressed as:

$$\int_0^t \phi_{0,0}^{(\lambda)}(x)dx \approx \frac{\lambda}{8\sqrt{m}}[1\ \ 3\ \ 5\ \ 7] \tag{6.2}$$

$$\int_0^t \psi_{0,0}^{(\lambda)}(x)dx \approx \frac{\lambda}{8\sqrt{m}}[1\ \ 3\ \ 3\ \ 1] \tag{6.3}$$

$$\int_0^t \psi_{1,0}^{(\lambda)}(x)dx \approx \frac{\lambda}{8\sqrt{m}}[1\ \ 1\ \ 0\ \ 0] \tag{6.4}$$

$$\int_0^t \psi_{1,1}^{(\lambda)}(x)dx \approx \frac{\lambda}{8\sqrt{m}}[0\ \ 0\ \ 1\ \ 1] \tag{6.5}$$

From Eqns. (6.2) to (6.5),

$$\int_0^t H_4(x)dx \approx \frac{\lambda}{8\sqrt{m}} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 1 & 3 & 3 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

$$P_4 = \left(\int_0^t H_4(x)dx \right) H_4^{-1}(t) = \frac{\lambda}{2\times 4} \begin{bmatrix} 4 & -2 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 2 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}.$$

Following the same procedures, the integration matrix P_n can be obtained as follows:

$$P_n = \frac{\lambda}{2\times n} \begin{bmatrix} \frac{2n}{\lambda} P_{\frac{n}{2}} & -\frac{1}{\sqrt{2}} H_{\frac{n}{2}} \\ \sqrt{2} H_{\frac{n}{2}}^{-1} & 0 \end{bmatrix}.$$

Example 6.1. Consider the Bessel's differential equation of order zero

$$ty'' + y' + ty = 0,\ y(0) = 1,\ y'(0) = 0. \tag{6.6}$$

The solution of Bessel's differential equation of order zero is denoted by $J_0(t)$, and defined by

$$J_0(t) = \sum_{v=0}^{\infty} \frac{(-1)^v}{(v!)^2} \left(\frac{t}{2}\right)^{2v}.$$

Here, using extended Haar wavelet for the solution of this example, consider $\lambda = 1,\ 2,$ and $m = 2$ then values of scale j as $j = 0, 1$.
Let $y''(t) = C^T \psi^{(\lambda)}(t)$, then by using extended Haar wavelets and boundary conditions of this example, we obtain (Aziz[3], Aziz and Amin[2]),

$$y'(t) = C^T P_n \psi^{(\lambda)}(t),\ y(t) = C^T P_n^2 \psi^{(\lambda)}(t) + d^T \psi^{(\lambda)}(t),$$

where $d^T = [\sqrt{2}\ \ 0\ \ 0\ \ 0],\ x = e^T \psi^{(\lambda)}(t)$.
Therefore,

$$e^T \psi^{(\lambda)} (\psi^{(\lambda)})^T C + C^T P_n \psi^{(\lambda)} + e^T \psi^{(\lambda)} (\psi^{(\lambda)})^T (P_n^2)^T C + e^T \psi^{(\lambda)} (\psi^{(\lambda)})^T d = 0 \tag{6.7}$$

$$\text{since } e^T \psi^{(\lambda)} (\psi^{(\lambda)})^T \approx (\psi^{(\lambda)})^T \tilde{E} \tag{6.8}$$

Now from Eqns. (6.7) and (6.8),

$$\tilde{E}C + P_n^T C + \tilde{E}(P_n^2)^T C + \tilde{E}d = 0 \tag{6.9}$$

$$\tilde{E} = \int_0^t x\psi^{(\lambda)}(x) (\psi^{(\lambda)}(x))^T dx \tag{6.10}$$

$$\tilde{E} = \lambda \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{8\sqrt{2}} & -\frac{1}{8\sqrt{2}} \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{8\sqrt{2}} & \frac{1}{8\sqrt{2}} \\ -\frac{1}{8\sqrt{2}} & -\frac{1}{8\sqrt{2}} & \frac{1}{4} & 0 \\ -\frac{1}{8\sqrt{2}} & \frac{1}{8\sqrt{2}} & 0 & \frac{3}{4} \end{bmatrix}.$$

$$y(t) = (C^T P_n^2 + d^T)\psi^{(\lambda)}(t). \tag{6.11}$$

The exact solution and approximate solution of the Bessel's differential equation of order zero obtained by the extended Haar wavelet method for different values of t in the interval [0,1) and [0,2) are given in Table 1 .

t	HWS for λ=1	Exact sol.	t	HWS for λ=2	Exact sol.
0	0.992248	1.000000	0	0.969697	1.000000
0.25	0.961599	0.984436	0.50	0.853912	0.938470
0.50	0.901721	0.938470	1.00	0.642978	0.765197
0.75	0.815375	0.864242	1.50	0.374087	0.511828

Table 1

The graphs of the exact solution and approximate solution of the Bessel's differential equation of order zero obtained by the extended Haar wavelet method for λ =1, 2, and $m = 2$ are shown in Figure 1 and Figure 2, respectively.

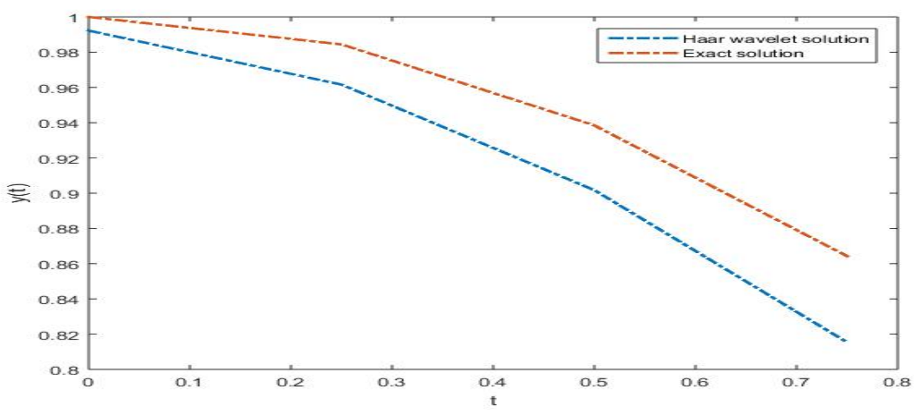

Figure 1

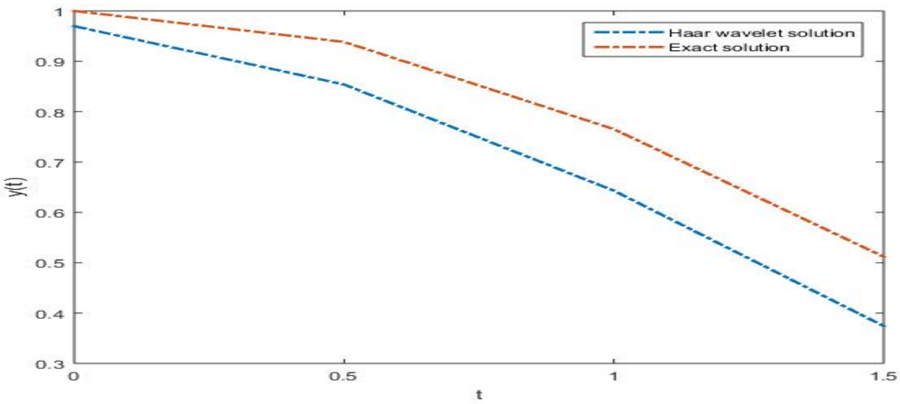

Figure 2

Again, using extended Haar wavelet for the solution of this example, consider $\lambda = 1,\ 2$, and $m = 4$ then values of scale j as $j = 0, 1, 2$.
Following the above process, the exact solution and approximate solution of the Bessel's differential equation of order zero obtained by the extended Haar wavelet method for different values of t in the interval [0,1) and [0,2) are given in Table 2.

t	HWS for λ=1	Exact sol.	t	HWS for λ=2	Exact sol.
0	0.998051	1.000000	0	0.992249	1.000000
0.125	0.990227	0.996098	0.25	0.961603	0.984436
0.250	0.974820	0.984436	0.50	0.901728	0.938470
0.375	0.951859	0.965152	0.75	0.815382	0.864242
0.500	0.921662	0.938470	1.00	0.706537	0.765197
0.625	0.884581	0.904702	1.25	0.580158	0.645906
0.750	0.841049	0.864242	1.50	0.441962	0.511828
0.875	0.791565	0.817560	1.75	0.298118	0.369032

Table 2

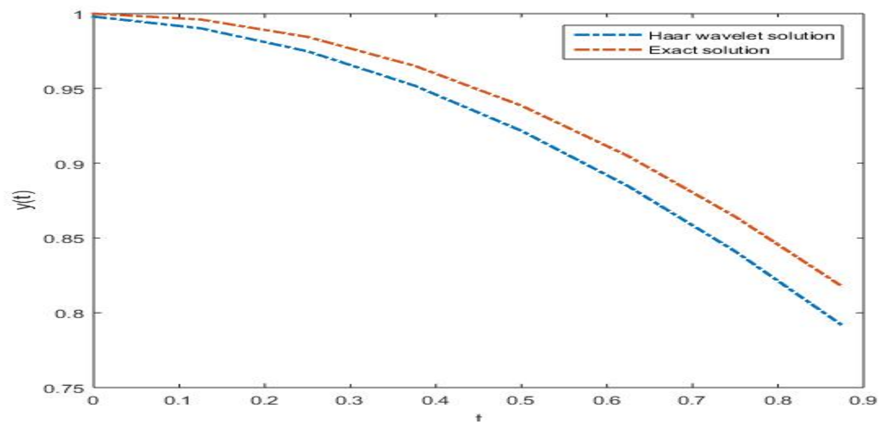

Figure 3

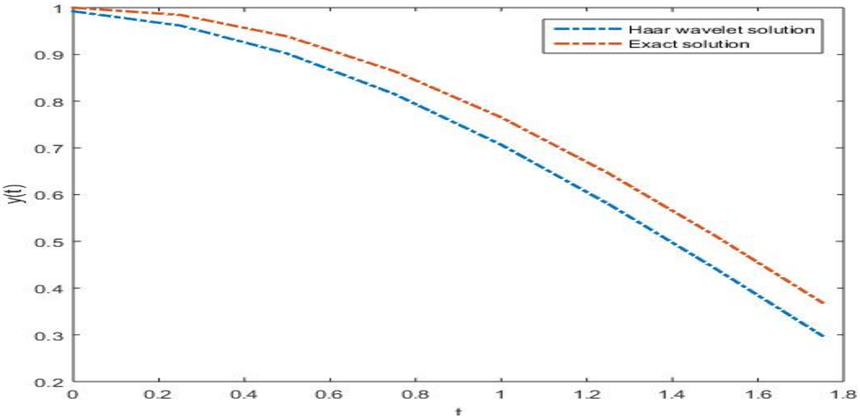

Figure 4

The graphs of the exact solution and approximate solution of the Bessel's differential equation of order zero obtained by the extended Haar wavelet method for λ =1, 2 and $m = 4$ are shown in Figure 3 and Figure 4, respectively.

Again, using extended Haar wavelet for the solution of this example, consider $\lambda = 1,\ 2$, and $m = 8$ then values of scale j as $j = 0, 1, 2, 3$.
Following the above process, the exact solution and approximate solution of the Bessel's differential equation of order zero obtained by the extended Haar wavelet method for different values of t in the interval [0,1) and [0,2) are given in Table 3.

t	HWS for λ=1	Exact sol.	t	HWS for λ=2	Exact sol.
0	0.999512	1.000000	0	0.998051	1.000000
0.0625	0.997562	0.999024	0.125	0.990278	0.996098
0.1250	0.993666	0.996098	0.250	0.974822	0.984436
0.1875	0.987838	0.991230	0.375	0.951863	0.965152
0.2500	0.980092	0.984436	0.500	0.921667	0.938470
0.3125	0.970452	0.975735	0.625	0.884588	0.904702
0.3750	0.958947	0.965152	0.750	0.841054	0.864242
0.4375	0.945609	0.952718	0.875	0.791572	0.817560
0.5000	0.930478	0.938470	1.000	0.736712	0.765198
0.5625	0.913598	0.922449	1.125	0.677106	0.707759
0.6250	0.895019	0.904702	1.250	0.613439	0.645906
0.6875	0.874793	0.885281	1.375	0.546435	0.580347
0.7500	0.852981	0.864242	1.500	0.476855	0.511828
0.8125	0.829645	0.841647	1.625	0.405481	0.441125
0.8750	0.804853	0.817560	1.750	0.333111	0.369033
0.9375	0.778676	0.792053	1.875	0.260546	0.296354

Table 3

The graphs of the exact solution and approximate solution of the Bessel's differential equation of order zero obtained by the extended Haar wavelet method for λ=1, 2 and $m = 8$ are shown in Figure 5 and Figure 6, respectively.

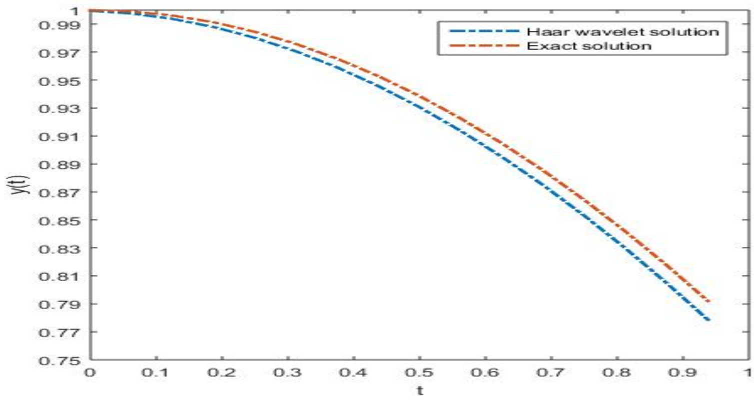

Figure 5

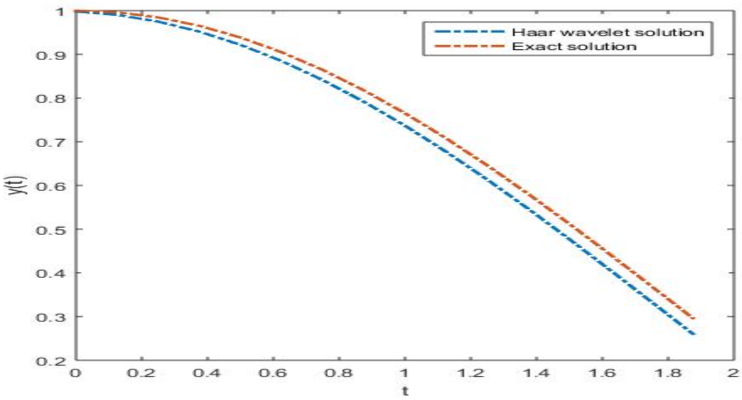

Figure 6

Note. If scales are increasing from $j = 0, 1$; $j = 0, 1, 2$; $j = 0, 1, 2, 3$, and $\lambda = 1, 2$, then it is observed from Figures 1, 2, 3, 4, 5, and 6 that the solutions of Bessel's differential equation of order zero appear to shift closer to its exact solution.

7. Conclusions

Following conclusions are derived from our theorems and example.

1. The extended Haar wavelet approximations of Theorems 5.1, 5.2 and 5.3 are given by

$E_N^{(1)}(f) = O\left(\frac{\lambda^{\alpha+\frac{1}{2}}}{2^{(N+1)\alpha}}\right)$, $E_N^{(1)}(f) \to 0$ as $N \to \infty$,

$E_{2^N}^{(2)}(f) = O\left(\frac{\lambda^{\alpha+\frac{1}{2}}}{2^{(N+1)\alpha+\frac{1}{2}}}\right)$, $E_{2^N}^{(2)}(f) \to 0$ as $N \to \infty$,

$$E_N^{(3)}(f) = O\left(\frac{\lambda^{\alpha+\frac{1}{2}}}{2^{(N+1)\alpha}}\omega\left(\frac{\lambda}{2^{N+1}}\right)\right),\ E_N^{(3)}(f) \to 0 \text{ as } N \to \infty\ .$$

Therefore, estimators $E_N^{(1)}(f), E_{2^N}^{(2)}(f)$ and $E_N^{(3)}(f)$ are best possible in wavelet analysis.
2. Since $E_{2^N}^{(2)}(f) \leq E_N^{(1)}(f)$, therefore estimator $E_{2^N}^{(2)}(f)$ is sharper than estimator $E_N^{(1)}(f)$. This shows that the estimator of $f \in H_2^\alpha[0, \lambda)$ is sharper than the estimator of $f \in H^\alpha[0, \lambda)$.
3. The solution of Bessel's differential equation of order zero by extended Haar operational matrix and its comparison by exact solution are significant achievements of this research paper.

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References

[1] A. A. Ahmedov, M. H. A. Sathar, A. F. N. Rasedee, and N. F. B. Mokhtar, Approximating of functions from Holder classes $H^\alpha[0, 1)$ by Haar wavelets, J. Phys. Conf. Ser., 890(2017), no. 1, 012073.

[2] I. Aziz, R. Amin, Numerical solution of a class of delay differential and delay partial differential equations via Haar wavelet, Appl. Math. Model., 40(2016), no. 1, 10286-10299.

[3] I. Aziz, B. Sarler, P. Siraj-ul-Islam, The numerical solution of second-order boundary-value problems by collocation method with the Haar wavelet, Math. Comput. Model, 52(2010), no. 9–10, 1577-1590.

[4] S. M. Aznam and M. S. Chowdhury, Generalized Haar wavelet operational matrix method for solving hyperbolic heat conduction in thin surface layers, Res. Phys., 11(2018), 243-252.

[5] D. Baleanu, J. Alzabut, M. Ismail and U. Saeed, Green–Haar wavelets method for generalized fractional differential equations, Adv. Difference Equ., (2020), Paper No. 515, 25 pp.

[6] C. K. Chui, *An introduction to wavelets.* Wavelet Anal. Appl., 1 Academic Press, Inc., Boston, MA, 1992.

[7] L. Debnath, *Wavelet transforms and their applications*, Birkhäuser Boston, Inc., Boston, MA, 2002.

[8] S. Lal and S. Kumar, CAS wavelet approximation of functions of Hölder's class $H^\alpha[0, 1)$ and Solution of Fredholm Integral Equations, Ratio Mathematica, 39(2020), 187-212.

[9] C. Y. Lee, H. M. Srivastava and W. C. Yueh, Explicit of some linear ordinary and partial fractional differintegral equations, Appl. Math. Comput., 144(2003), 11-25.

[10] Ü. Lepik, Application of Haar wavelet transform to solving integral and differential equations, Sci. Phys. Math., 56(2007), 28-46.

[11] Y. Meyer, *Wavelets: their past and their future.* Progress in wavelet analysis and applications (Toulouse, 1992), 9-18. Editions Frontières, Gif-sur-Yvette, 1993.

[12] E.C. Titchmarsh, *The theory of functions*. Second edition, Oxford University Press, Oxford, 1939.

[13] U. Saeed and M. ur Rehman, Haar wavelet-quasilinearization technique for fractional nonlinear differential equations, Appl. Math. Comput., 220(2013), 630-648.

[14] A. Zygmund, Trigonometric Series, Cambridge University Press, Cambridge, 1959.

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