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A NOTE ON FRAMES IN NON-LOCALLY CONVEX BANACH SPACES

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Abstract. Shrinking atomic decompositions in locally convex Banach spaces were studied by Carando and Lassalle [2]. In this paper, we define strongly shrinking atomic decompositions in p-Banach spaces and give necessary and sufficient condition for atomic decomposition to be shrinking and boundedly complete.

1. Introduction

Let X be a vector space over a field $\mathbb{F}$. A p-norm $\|.\|_p$ for $0 < p \leq 1$ on X is a mapping from $X \longrightarrow \mathbb{R}$ satisfying the following properties:

(1) $\|x\|_p \geq 0$, for all $x \in X$.
(2) $\|x\|_p = 0 \iff x = 0$.
(3) $\|\alpha x\|_p = |\alpha|^p \|x\|_p$, for all $x \in X$ and $\alpha \in \mathbb{F}$.
(4) $\|x+y\|_p \leq \|x\|_p + \|y\|_p$, for all $x, y \in X$.

The pair $(X, \|.\|_p)$ is called a p-normed linear space.
If $p = 1$, then the p-norm is equal to norm on X.
A p-normed linear space X over a field $\mathbb{F}$ is called a p-Banach space if it is complete.
A linear operator $T : (X, \|.\|_p) \longrightarrow (Y, \|.\|_q)$ is said to be bounded if there exists a real number $M > 0$ such that $\|T(x)\|_q^{\frac{1}{q}} \leq M \|x\|_p^{\frac{1}{p}}$, for all $x \in X$.

The collection of all bounded linear operators from the p-Banach space X to the q-Banach space Y is denoted by $B(X, Y)$ which is a Banach space with norm given by

$$\|T\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|T(x)\|_q^{\frac{1}{q}}}{\|x\|_p^{\frac{1}{p}}}.$$

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Let X be a p-Banach space. A linear functional $f : X \longrightarrow \mathbb{F}$ is said to be bounded on X if there exists a real number $M > 0$ such that

$$|f(x)| \leq M \|x\|^{\frac{1}{p}}, \text{ for all } x \in X.$$

The collection of all bounded linear functionals on the p-Banach space X is denoted by X^* which is also a Banach space with norm given by $\|f\| = \sup_{\|x\|_p \leq 1} |f(x)|$ and is called the conjugate space of X.
The concept of frame was first defined by Duffin and Schaeffer [7] in 1952. Frames were reintroduced in 1986 by Daubechies, Grossmann and Y. Meyer [6]. Coifman and Weiss [5], introduced the notion of atomic decomposition for certain Function spaces. Later, the notion of atomic decomposition to certain Banach spaces was extended by Feichtinger and Gröchenig [8]. Atomic decompositions were further studied by Kaushik and Sharma [10, 11, 12, 13]
In this paper, we define atomic decompositions in a non- locally convex Banach space $l^p(0 < p \leq 1)$. Also, the existence of atomic decomposition is exhibited through examples. Further, a sufficient condition for its existence is given and it is proved that if a p-Banach space has an atomic decomposition, then the space is isomorphic to its associated p-Banach sequence space. Furthermore, necessary and sufficient conditions for an atomic decomposition in a p-Banach space is given. Finally, shrinking atomic decomposition is defined and a necessary and sufficient condition for it is given.

2. Shrinking Atomic Decomposition in p-Banach Space

Shrinking Schauder frames were introduced and studied by Liu [15] while shrinking atomic decompositions in locally convex Banach spaces were studied by Carando and Lassalle [2]. In [16], Pahari, Kohli and Ghimire defined shrinking atomic decompositions in p-Banach spaces. In this paper, we define strongly shrinking atomic decompositions in p-Banach spaces and give necessary and sufficient condition for atomic decomposition to be shrinking and boundedly complete. We begin with the following definition:

Definition 2.1. Let $(X, \|.\|_p)$ be a p-Banach space and let X_d be a p-Banach sequence space associated with X. Let $\{x_n\} \subset X$ and $\{f_n\} \subset X^*$. Then the atomic decomposition $(\{x_n\}, \{f_n\})$ is said to be shrinking if

$$\lim_{n \to \infty} \|f \circ T_N\| = 0, \text{ for all } f \in X^*,$$

where $T_N : X \longrightarrow X$ is defined as

$$T_N(x) = \sum_{n=N}^{\infty} f_n(x) x_n, \text{ for all } x \in X.$$

Remark 2.2. Since $T_N = I - P_N$, it is uniformly bounded on X.

Theorem 2.3. [16] *Let $(\{x_n\}, \{f_n\})$ be an atomic decomposition for the p-Bancah space X with respect to the p-Banach sequence space X_d and let $\pi : X \longrightarrow X^{**}$ be the natural embedding. Then the following statements are equivalent:*

- *(i)* $(\{x_n\}, \{f_n\})$ *is shrinking.*
- *(ii)* $(\{f_n\}, \{\pi_{x_n}\})$ *is an atomic decomposition for X^* with respect to the p-Banach sequence space X_d which has canonical unit vectors as basis.*

Definition 2.4. A p-Banach space X is said to have the bounded approximation property iff there exists a sequence $\{A_n\}$ of finite rank operators on X such that $\lim\limits_{n\to\infty} \|A_n x - x\|_p = 0,\ \text{ for all } x \in X.$

Corollary 2.5. Let X be a p-Banach space, X_d be a p-Banach sequence space such that $(\{x_n\}, \{f_n\})$ is shrinking atomic decomposition for X with respect to X_d then X^* is separable and has bounded approximation property.

Proof. Since $(\{x_n\}, \{f_n\})$ is shrinking atomic decomposition therefore using Theorem 3.1 (of research paper Frame Systems in Non-Locally Convex Banach Spaces), $(\{f_n\}, \{\pi_{x_n}\})$ is an atomic decomposition for X^* and hence

$$f = \sum_{n=1}^{\infty} \pi_{x_n}(f) f_n = \sum_{n=1}^{\infty} f(x_n) f_n, \text{ for all } f \in X^*.$$

which clearly implies that X^* is separable and has bounded approximation property. $\square$

Now, we define the notion of strongly shrinking atomic decomposition for p-Banach spaces. We give the following definition.

Definition 2.6. Let $(X, \|.\|_p)$ be a p-Banach space and let X_d be a p-Banach sequence space associated with X. Let $\{x_n\} \subset X$ and $\{f_n\} \subset X^*$ such that $(\{x_n\}, \{f_n\})$ is an atomic decomposition of X with respect to X_d. Then $(\{x_n\}, \{f_n\})$ is strongly shrinking if

$$\lim_{n\to\infty} \|f \circ S_N\| = 0, \text{ for all } f \in X^*,$$

where $S_N : X_d \longrightarrow X$ is defined as

$$S_N(\{a_n\}) = \sum_{n=N}^{\infty} a_n x_n, \text{ for all } \{a_n\} \in X_d.$$

Lemma 2.7. *Every strongly shrinking atomic decomposition is shrinking.*

Proof. Let $(\{x_n\}, \{f_n\})$ be strongly shrinking atomic decomposition of X w.r.t. X_d. Hence

(i) $\{f_n(x)\}_n \in X_d$, for all $x \in X$.
(ii) There exists constants A,$B > 0$ such that

$$A \|x\|_p \le \|\{f_n(x)\}\|_{X_d} \le B \|x\|_p, \text{ for all } x \in X.$$

(iii) $x = \sum_{n=1}^{\infty} f_n(x) x_n$, for all $x \in X$.
(iv) $\lim\limits_{n\to\infty} \|f \circ S_N\| = 0$, for all $f \in X^*$.

Using (iv) for every $\epsilon > 0$ there exists K s.t.

$$\|f \circ S_N\| < \frac{\epsilon}{2B}, \text{ for all } N \ge K,$$

which implies for all $\{a_n\} \in X_d$ with $\|\{a_n\}\|_{X_d} \le 1$

$$\left| \sum_{n=N}^{\infty} a_n f(x_n) \right| < \frac{\epsilon}{2B}, \text{ for all } N \ge K.$$

Hence for $a_n = \dfrac{f_n(x)}{B}$ where $x \in X$ such that $\|x\|_p \leq 1$ we have

$$\|f \circ T_N\| < \epsilon, \text{ for all } N \geq K.$$

Hence $(\{x_n\}, \{f_n\})$ is shrinking. $\square$

Next example shows that converse of the above lemma is not true in general.

Example 2.8. Let $X = l_p, 0 < p < 1$ with p-norm $\|x\|_p = \sum_{n=1}^{\infty} |x_n|^p$ and $X_d = l_p \oplus l_p$ be a p-Banach sequence space with p-norm $\|x+y\|_{X_d} = \|x\|_p + \|y\|_p$. Define $\{f_n\}$ of functions on X as

$$f_{2n-1}(e_i) = 0 \text{ and } f_{2n}(e_i) = \delta_{i,n}, \ \forall\, i, n \in \mathbb{N}.$$

Consider $g_n = f_{2n-1} + f_{2n}$ $\forall\, n \in \mathbb{N}$. Then one can easily prove that $(\{e_n\}, \{g_n\})$ is an atomic decomposition for X with respect to X_d. Also using property (iii) of atomic decomposition one can prove that $(\{e_n\}, \{g_n\})$ is shrinking.

But we claim that $(\{e_n\}, \{g_n\})$ is not strongly shrinking.

Take $g \in X^*$ as $g(e_i) = \delta_{i,N}$ $\forall\, i \in \mathbb{N}$. Then for all $N \in \mathbb{N}$

$$\|g \circ S_N\| = \sup_{\|x+y\|_{X_d} \leq 1} |(g \circ S_N)(x+y)| = \sup_{\|x+y\|_{X_d} \leq 1} |x_N + y_N| \leq 1.$$

Also, if $x = \{e_N\}$ and $y = \{0\}$, then $\|g \circ S_N\| \geq 1$ and hence our claim.

Definition 2.9. Let $(X, \|.\|_p)$ be a p-Banach space and let X_d be a p-Banach sequence space associated with X. Let $\{x_n\} \subset X$ and $\{f_n\} \subset X^*$ such that $(\{x_n\}, \{f_n\})$ is an atomic decomposition of X with respect to X_d. Then $(\{x_n\}, \{f_n\})$ is said to be boundedly complete if the series $\sum_{n=1}^{\infty} T(f_n)x_n$ is convergent in X for each $T \in X^{**}$.

Lemma 2.10. *In a reflexive p-Banach space X, every atomic decomposition is boundedly complete.*

Proof. Let $(\{x_n\}, \{f_n\})$ be an atomic decomposition of a reflexive p-Banach space X w.r.t. X_d. Hence

(i) $\{f_n(x)\}_n \in X_d$, for all $x \in X$.

(ii) There exists constants $A, B > 0$ such that

$$A\|x\|_p \leq \|\{f_n(x)\}\|_{X_d} \leq B\|x\|_p, \text{ for all } x \in X.$$

(iii) $x = \sum_{n=1}^{\infty} f_n(x)x_n$, for all $x \in X$.

Since X is reflexive for each $T \in X^{**}$,

$$\sum_{n=1}^{\infty} T(f_n)x_n = \sum_{n=1}^{\infty} f_n(t)x_n = t, \text{ for all } t \in X,$$

which is clearly convergent from (iii). $\square$

Next theorem shows that the converse of above lemma holds if atomic decomposition is shrinking also alongwith boundedly complete.

Theorem 2.11. *Let X be a p-Banach space and X_d be a p-Banach sequence space with atomic decomposition $(\{x_n\}, \{f_n\})$. If $(\{x_n\}, \{f_n\})$ is shrinking and boundedly complete, then X is reflexive.*

Proof. Since $(\{x_n\},\{f_n\})$ is an atomic decomposition of X, therefore

$$x=\sum_{n=1}^{\infty} f_n(x)x_n,\ \text{for all } x\in X.$$

Clearly $X\cong\pi(X)\subseteq X^{**}$ where $\pi : X\longrightarrow X^{**}$ is the canonical map. In order to prove that X is reflexive it is enough to prove that π is onto.

Let $T\in X^{**}$, then since $(\{x_n\},\{f_n\})$ is boundedly complete therefore $\sum_{n=1}^{\infty} T(f_n)x_n$ is convergent in X to say x i.e.$\sum_{n=1}^{\infty} T(f_n)x_n = x$.

Hence for all $f\in X^*$, $\pi_x(f)=f(x)=\sum_{n=1}^{\infty} T(f_n)f(x_n)$.

Also since $(\{x_n\},\{f_n\})$ is shrinking therefore using Theorem 1.1

$$f=\sum_{n=1}^{\infty} f(x_n)f_n,\ \text{for all } f\in X^*,$$

which implies

$$T(f)=\sum_{n=1}^{\infty} f(x_n)T(f_n),\ \text{for all } f\in X^*,$$

which proves that $T=\pi_x$ and hence X is reflexive. □

Next result gives necessary and sufficient condition for a p-Banach space to be reflexive.

Theorem 2.12. *Let X be a p-Banach space and X_d be a p-Banach sequence space with unconditional atomic decomposition $(\{x_n\},\{f_n\})$, then $(\{x_n\},\{f_n\})$ is shrinking and boundedly complete iff X is reflexive.*

Proof. If atomic decomposition $(\{x_n\},\{f_n\})$ is shrinking and boundedly complete, then X is reflexive using the above theorem.

For sufficient condition, as X is reflexive so in view of Lemma 1.2 it is enough to prove that the atomic decomposition is shrinking.

Consider for $f\in X^*$,

$$\begin{aligned}\|f\circ T_N\| &= \sup_{\|x\|_p\le 1} |(f\circ T_N)(x)|\\ &= \sup_{\|x\|_p\le 1} \left| f\Big(\sum_{n=N}^{\infty} f_n(x)x_n\Big)\right|\\ &= \sup_{\|x\|_p\le 1} \left|\sum_{n=N}^{\infty} \pi_x(f_n)f(x_n)\right|\end{aligned}$$

which tends to 0 using Orlicz-Pettis Theorem for non-locally convex spaces, since the series in 1.1 is weakly convergent due to boundedly completeness of unconditional atomic decomposition $(\{x_n\},\{f_n\})$. □

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