Poincare Journal of Analysis & Applications
Vol. 10, No. 1 (2023), 61-73

DOI: 10.46753/pjaa.2023.v010i01.005

NUMERICAL SOLUTIONS OF SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS USING HAAR WAVELET APPROACH

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Date of Receiving	:	28. 02. 2021
Date of Revision	:	06. 08. 2022
Date of Acceptance	:	30. 11. 2022

Abstract. In this article, a general procedure of forming the Haar wavelets operational matrix is discussed. The Haar wavelet system which has localization property is applied to find approximate solution to a given system which is very near to the exact solution of the system. We demonstrate this procedure through numerical examples. A comparison of approximate solutions and the exact solutions is done along with the error analysis in order to establish that the Haar wavelet system gives better approximate solutions.

1. Introduction

The concept of physical systems in engineering and applied sciences is modeled through differential equations. Since then, many numerical methods for finding the solutions to these differential equations have been proposed. Some of them are variational iteration method [9], perturbation method [3], finite difference method [10, 11], Bernoulli matrix method [4, 6, 7, 13, 19, 20, 21, 22], and Galerkin method [2, 8, 12, 17], etc. These techniques aim to approach a better approximation of the solution with restricted computational inceptions.
Wavelet theory is a comparatively new emanating discipline in applied sciences and soon wavelet techniques have been used in almost every branch of applied sciences. One advantage is it reduces most complex problems to a system of algebraic equations which felicitates these problems remarkably. This procedure for a differential equation is based on the reformation of the given differential equations into a system of algebraic equations and then using an operational matrix to tackle various integral operators. For more of these procedures, one may refer to [1, 14, 15, 16, 18, 23].
In this paper, we discuss a procedure for forming the Haar wavelets basis and operational

2010 *Mathematics Subject Classification.* 34A35, 65T60.
Key words and phrases. Haar Wavelet, Operational Matrix, Function Approximation.
Communicated by. Shah Jahan
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matrix, which can be useful to resolve many practical problems. Further, as the Haar wavelet basis possesses the localization property, so with the help of the Haar wavelets basis and operational matrix, a system of differential equations is transformed into a system of algebraic equations which helps to achieve an approximate solution of the system nearer to the exact solution. Furthermore, examples have been given in support of this process. Finally, the approximate solutions obtained are compared with the exact solutions and their error analysis.

2. Haar Wavelet

The easiest of wavelets is Haar Wavelet. It was proposed by Alfred Haar in 1910. For $t \in [0,1]$, the Haar wavelet functions are defined by

$$\phi_0(t) = 1$$

$$\phi_i(t) = \begin{cases} 1, & \frac{k-1}{2^j} \leq t < \frac{k-\frac{1}{2}}{2^j} \\ -1, & \frac{k-\frac{1}{2}}{2^j} \leq t < \frac{k}{2^j} \\ 0, & otherwise \end{cases}$$

where $i = 1, 2, ..., n-1, n = 2^N (n \in \mathbb{N}), j$ and k constitute the integer decomposition of i, i.e., $i = 2^j + k - 1, 0 \leq j < i$ and $1 \leq k < 2^j + 1$.
Any arbitrary function f$\in L^2([0,1])$ can be represented using the Haar series by

$$f(t) = \sum_{i=0}^{\infty} \alpha_i \phi_i(t) \tag{2.1}$$

where each $\alpha_i (i = 0, 1, 2, ...)$, called the Haar coeffiient, is given by

$$c_i = 2^j \int_0^1 f(t)\, \phi_i(t)\, dt$$

The series representation of f(see (2.1)) contains infinitely many terms. In case, f is piecewise constant or can be approximated by a piecewise constant throughout each subinterval, then f will termiate at finite terms.
In other words, we may write

$$f(t) \approx \sum_{i=0}^{m-1} \alpha_i \phi_i(t) = A_m^T H_m(t) = \hat{f}(t),$$

where $m = 2^j$, the superscript T designate transposition and $\hat{f}(t)$ indicate the truncated sum.
The Haar coefficient vector A_m and the Haar function vector $H_m(t)$ are given by

$$A_m \triangleq [\alpha_0, \alpha_1, ..., \alpha_{m-1}]^T$$
$$H_m \triangleq [\phi_0, \phi_1, ..., \phi_{m-1}]^T$$

For more details related to Haar wavelet, one may refer to [1,18].

3. Operational Matrix

Chen and Hsiao [5] approximated the integration of $\mathcal{H}_m(t)$ as

$$\int_0^t \mathcal{H}_m(w)\,dw \cong \mathcal{P}^T_{m\times m}\mathcal{H}_m(t),$$

where $\mathcal{P}^T_{m\times m}$ is the Haar wavelet operational matrix of integration and is given by

$$\int_0^t \mathcal{P}^{k-1}_{m\times m}\mathcal{H}_m(w)\,dw \cong \mathcal{P}^k_{m\times m}\mathcal{H}_m(t), \quad k = 2, 3, ...$$

Thus, we may write

$$\begin{aligned}\mathcal{P}_m &= \left[\int_0^t \mathcal{H}_m(w)\,dw\right]\mathcal{H}_m^{-1}(t), \quad m = 1, 2, ...\\ &= \frac{1}{2m}\begin{bmatrix} 2m\,\mathcal{P}_{m/2} & -\mathcal{H}_{m/2} \\ \mathcal{H}^{-1}_{m/2} & 0 \end{bmatrix},\end{aligned}$$

where $\mathcal{H}_m(t) = \begin{bmatrix} h_0(t) \\ h_1(t) \\ \vdots \\ h_{m-1}(t) \end{bmatrix}$

4. Application of Haar Wavelet Method

In this section, we shall find the numerical solution of a linear system of equations given by

$$x'(t) + a_1\,x(t) + a_2\,y(t) = 0 \tag{4.1}$$

$$y'(t) + b_1\,x(t) + b_2\,y(t) = 0 \tag{4.2}$$

with initial conditions

$$x(0) = a_3, \quad y(0) = b_3 \tag{4.3}$$

where $a_i's$ and $b_i's$ are constants.

The Haar wavelet expansion of the given quantities can be expressed as

$$a_3 = A_3^T\mathcal{H}(t),\, b_3 = B_3^T\mathcal{H}(t) \tag{4.4}$$

We generally approximate the variable $x'(t)$ and $y'(t)$ instead of $x(t)$ and $y(t)$ using Haar wavelet.

$$x'(t) \approx \tilde{x}'(t) = X^T\mathcal{H}(t) \tag{4.5}$$

Which on integrating yields

$$x(t) = \int_0^t x'(t)dt + x(0)$$

$$\tilde{x}(t) = X^T\mathcal{P}\mathcal{H}(t) + A_3^T\mathcal{H}(t), \tag{4.6}$$

Where $\mathcal{P}$ is an operational matrix for Haar wavelet.

Similarly

$$y'(t) \approx \tilde{y}'(t) = Y^T\mathcal{H}(t) \tag{4.7}$$

$$\tilde{y}(t) = Y^T \mathcal{P}\mathcal{H}(t) + B_3^T \mathcal{H}(t) \tag{4.8}$$

Substituting equations (4.4)-(4.8) into equations (4.1) and (4.2), we obtain the approximation of equation (4.1)

$$X^T \mathcal{H}(t) + a_1[X^T \mathcal{P}\mathcal{H}(t) + A_3^T \mathcal{H}(t)] + a_2[Y^T \mathcal{P}\mathcal{H}(t) + B_3^T \mathcal{H}(t)] = 0$$

$$X^T(I + a_1\mathcal{P})\mathcal{H}(t) + Y^T(a_2\mathcal{P})\mathcal{H}(t) = -(a_1 A_3^T \mathcal{H}(t) + a_2 B_3^T \mathcal{H}(t))$$

$$X^T(I + a_1\mathcal{P}) + Y^T(a_2\mathcal{P}) = -(a_1 A_3^T + a_2 B_3^T) \tag{4.9}$$

Similarly, we can produce the approximation of equation (4.2)

$$Y^T(I + b_2\mathcal{P}) + X^T(b_1 P) = -(b_1 A_3^T + b_2 B_3^T) \tag{4.10}$$

$$\begin{aligned} \text{Let} \qquad (I + a_1\mathcal{P}) = K_1, &\quad (I + b_2\mathcal{P}) = K_2 \\ (a_2\mathcal{P}) = N_1, &\quad (b_1\mathcal{P}) = N_2 \\ -(a_1 A_3^T + a_2 B_3^T) &= M_1 \\ -(b_1 A_3^T + b_2 B_3^T) &= M_2 \end{aligned}$$

So, equations (4.9) and (4.10) reduced into the system of algebraic equations

$$X^T K_1 + Y^T N_1 = M_1 \tag{4.11}$$

$$X^T N_2 + Y^T K_2 = M_2 \tag{4.12}$$

By solving the above system of algebraic equation, the vectors X and Y can be obtained and putting these values in equations (4.6) and (4.8), we can find approximate solution $\tilde{x}(t)$ andy $\tilde{y}(t)$ of the given system of differential equations.

Example 1. Consider the system of homogeneous linear differential equations

$$x'(t) + y(t) = 0 \tag{4.13}$$

$$y'(t) + x(t) = 0 \tag{4.14}$$

with initial conditions

$$x(0) = -1, \quad y(0) = 1$$

Solution. The approximation form of the unknown functions are

$$x(t) \approx \tilde{x}(t) = X^T \mathcal{P}\mathcal{H}(t) + E^T \mathcal{H}(t) \tag{4.15}$$

$$y(t) \approx \tilde{y}(t) = Y^T \mathcal{P}\mathcal{H}(t) + D^T \mathcal{H}(t) \tag{4.16}$$

where $x(0) = -1 \approx E^T \mathcal{H}(t)$ and $y(0) = 1 \approx D^T \mathcal{H}(t)$
By using the above approximating method, we obtain the vectors X and Y,

$$Y = -[\mathcal{P}^T X + E]$$

$$(I - \mathcal{P}^{2T})X = \mathcal{P}^T E - D$$

Four sets of vectors (X, Y) are obtained via Haar approach:

For $m = 2, X^T = \begin{bmatrix} -\frac{16}{9}, & \frac{4}{9} \end{bmatrix}, \quad Y^T = \begin{bmatrix} \frac{16}{9}, & -\frac{4}{9} \end{bmatrix}$

For $m = 4, X^T = \left[-\frac{4160}{2401}, \quad \frac{1024}{2401}, \quad \frac{8}{49}, \quad \frac{648}{2401}\right]$,

$$Y^T = \left[\frac{4160}{2401}, \quad -\frac{1024}{2401}, \quad -\frac{8}{49}, \quad -\frac{648}{2401}\right]$$

For $m = 8$

$$X^T = \left[-\frac{4412866816}{2562890625}, \frac{1082146816}{2562890625}, \frac{8192}{50625}, \frac{684204032}{2562890625}, \frac{16}{225}, \frac{4624}{50625}, \frac{1336336}{11390625}, \quad \frac{386201104}{2562890625}\right],$$

$$Y^T = \left[\frac{4412866816}{2562890625}, \quad -\frac{1082146816}{2562890625}, \quad -\frac{8192}{50625}, \quad -\frac{684204032}{2562890625}, \quad -\frac{16}{225}, -\frac{4624}{50625}, \quad -\frac{1336336}{11390625}, \quad -\frac{386201104}{2562890625}\right]$$

For $m = 16$

$$X^T = \left[-\frac{1250562079715373614105600}{7274231217471852638281}, \frac{3063817122546845286400000}{7274231217471852638281}, \frac{137707520000}{852891037441}, \frac{19367304292459487232 0000}{7274231217471852638281}, \frac{65536}{923521}, \frac{77720518656}{852891037441}, \frac{92170395205042176}{787662783788549761}, \frac{1093068072519588224040 96}{7274231217471852638281}, \frac{32}{961}, \frac{34848}{923521}, \frac{37949472}{887503681}, \frac{41326975008}{852891037441}, \frac{45005075783712}{819628286980801}, \frac{49010527528462368}{787662783788549761}, \frac{53372464478495518752}{756943935220796320321}, \frac{58122613817081619920928}{7274231217471852638281}\right]$$

$$Y^T = \left[\frac{1250562079715373614105600}{7274231217471852638281}, -\frac{3063817122546845286400000}{7274231217471852638281}, -\frac{137707520000}{852891037441}, -\frac{193673042924594872320000}{7274231217471852638281}, -\frac{65536}{923521}, -\frac{77720518656}{852891037441}, -\frac{92170395205042176}{787662783788549761}, -\frac{109306807251958822404096}{7274231217471852638281}, -\frac{32}{961}, -\frac{34848}{923521}, -\frac{37949472}{887503681}, -\frac{41326975008}{852891037441}, -\frac{45005075783712}{819628286980801}, -\frac{49010527528462368}{787662783788549761}, -\frac{53372464478495518752}{756943935220796320321}, -\frac{58122613817081619920928}{7274231217471852638281}\right]$$

By using equation (4.15) and (4.16), the approximate solution of the given problem can be obtained

For $m = 2, \tilde{x(t)} = \begin{cases} -\frac{20}{9} & \frac{1}{2} \leq t < 1 \\ -\frac{4}{3} & 0 \leq t < \frac{1}{2} \end{cases}$,

$$\tilde{y(t)} = \begin{cases} \frac{4}{3} & 0 \leq t < \frac{1}{2} \\ \frac{20}{9} & \frac{1}{2} \leq t < 1 \end{cases}$$

For $m = 4, \tilde{x(t)} = \begin{cases} -\frac{5832}{2401} & \frac{3}{4} \leq t < 1 \\ -\frac{648}{343} & \frac{1}{2} \leq t < \frac{3}{4} \\ -\frac{72}{49} & \frac{1}{4} \leq t < \frac{1}{2} \\ -\frac{8}{7} & 0 \leq t < \frac{1}{4} \\ 0 & True \end{cases}, \quad \tilde{y(t)} = \begin{cases} \frac{8}{7} & 0 \leq t < \frac{1}{4} \\ \frac{72}{49} & \frac{1}{4} \leq t < \frac{1}{2} \\ \frac{648}{343} & \frac{1}{2} \leq t < \frac{3}{4} \\ \frac{5832}{2401} & \frac{3}{4} \leq t < 1 \\ 0 & True \end{cases}$

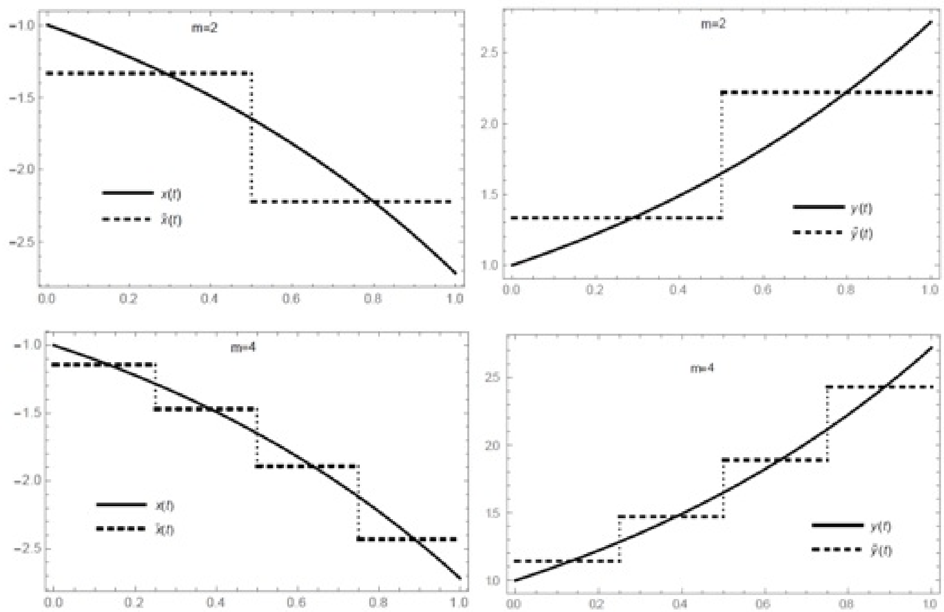
For $m = 8, \tilde{x(t)} = \begin{cases} -\frac{6565418768}{2562890625} & \frac{7}{8} \leq t < 1 \\ -\frac{386201104}{170859375} & \frac{3}{4} \leq t < \frac{7}{8} \\ -\frac{22717712}{11390625} & \frac{5}{8} \leq t < \frac{3}{4} \\ -\frac{1336336}{759375} & \frac{1}{2} \leq t < \frac{5}{8} \\ -\frac{78608}{50625} & \frac{3}{8} \leq t < \frac{1}{2} \\ -\frac{4624}{3375} & \frac{1}{4} \leq t < \frac{3}{8} \\ -\frac{272}{225} & \frac{1}{8} \leq t < \frac{1}{4} \\ -\frac{16}{25} & 0 \leq t < \frac{1}{8} \\ 0 & True \end{cases}, \quad \tilde{y(t)} = \begin{cases} \frac{16}{25} & 0 \leq t < \frac{1}{8} \\ \frac{272}{225} & \frac{1}{8} \leq t < \frac{1}{4} \\ \frac{4624}{3375} & \frac{1}{4} \leq t < \frac{3}{8} \\ \frac{78608}{50625} & \frac{3}{8} \leq t < \frac{1}{2} \\ \frac{1336336}{759375} & \frac{1}{2} \leq t < \frac{5}{8} \\ \frac{22717712}{11390625} & \frac{5}{8} \leq t < \frac{3}{4} \\ \frac{386201104}{170859375} & \frac{3}{4} \leq t < \frac{7}{8} \\ \frac{6565418768}{2562890625} & \frac{7}{8} \leq t < 1 \\ 0 & True \end{cases}$

For m=16,

$$\tilde{x(t)} = \begin{cases} -\frac{19180462559636934}{7274231217471852} & \frac{15}{16} \leq t < 1 \\ -\frac{581226138170816}{234652619918446} & \frac{7}{8} \leq t < \frac{15}{16} \\ -\frac{17612913277903}{7569439352207} & \frac{13}{16} \leq t < \frac{7}{8} \\ -\frac{533724644784}{244175462974} & \frac{3}{4} \leq t < \frac{13}{16} \\ -\frac{16173474084392581}{7876627837885497} & \frac{11}{16} \leq t < \frac{3}{4} \\ -\frac{490105275284623}{254084768964048} & \frac{5}{8} \leq t < \frac{11}{16} \\ -\frac{14851675008624}{8196282869808} & \frac{9}{16} \leq t < \frac{5}{8} \\ -\frac{450050757837}{264396221606} & \frac{1}{2} \leq t < \frac{9}{16} \\ -\frac{1363790175264}{852891037441} & \frac{7}{16} \leq t < \frac{1}{2} \\ -\frac{41326975008}{27512614111} & \frac{3}{8} \leq t < \frac{7}{16} \\ -\frac{1252332576}{887503681} & \frac{5}{16} \leq t < \frac{3}{8} \\ -\frac{37949472}{28629151} & \frac{1}{4} \leq t < \frac{5}{16} \\ -\frac{1149984}{923521} & \frac{3}{16} \leq t < \frac{1}{4} \\ -\frac{34848}{29791} & \frac{1}{8} \leq t < \frac{3}{16} \\ -\frac{1056}{961} & \frac{1}{16} \leq t < \frac{1}{8} \\ -\frac{32}{31} & 0 \leq t < \frac{1}{8} \\ 0 & True \end{cases},$$

$$\tilde{y(t)} = \begin{cases} \frac{32}{31} & 0 \leq t < \frac{1}{8} \\ \frac{1056}{961} & \frac{1}{16} \leq t < \frac{1}{8} \\ \frac{34848}{29791} & \frac{1}{8} \leq t < \frac{3}{16} \\ \frac{1149984}{923521} & \frac{3}{16} \leq t < \frac{1}{4} \\ \frac{37949472}{28629151} & \frac{1}{4} \leq t < \frac{5}{16} \\ \frac{1252332576}{887503681} & \frac{5}{16} \leq t < \frac{3}{8} \\ \frac{41326975008}{27512614111} & \frac{3}{8} \leq t < \frac{7}{16} \\ \frac{1363790175264}{852891037441} & \frac{7}{16} \leq t < \frac{1}{2} \\ \frac{450050757837}{264396221606} & \frac{1}{2} \leq t < \frac{9}{16} \\ \frac{14851675008624}{8196282869808} & \frac{9}{16} \leq t < \frac{5}{8} \\ \frac{490105275284623}{254084768964048} & \frac{5}{8} \leq t < \frac{11}{16} \\ \frac{16173474084392581}{7876627837885497} & \frac{11}{16} \leq t < \frac{3}{4} \\ \frac{533724644784}{244175462974} & \frac{3}{4} \leq t < \frac{13}{16} \\ \frac{17612913277903}{7569439352207} & \frac{13}{16} \leq t < \frac{7}{8} \\ \frac{581226138170816}{234652619918446} & \frac{7}{8} \leq t < \frac{15}{16} \\ \frac{19180462559636934}{7274231217471852} & \frac{15}{16} \leq t < 1 \\ 0 & True \end{cases}$$

The graphs of approximate $(\tilde{x}(t), \tilde{y}(t))$ and analytic solutions $(x(t), y(t))$ of the system of differential equations are shown in figure 1.



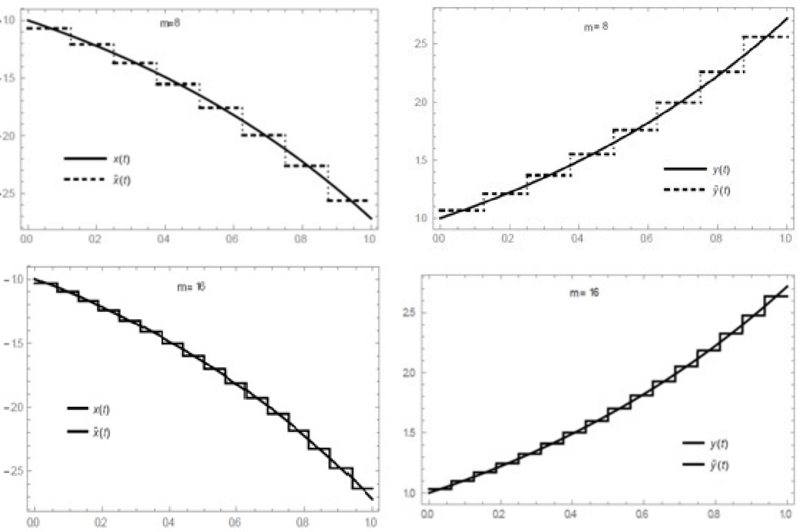


Figure 1. Haar vs Analytic solution

Though there are many ways to calculate error using different norms. We are using

$$\|e\|_{\infty} = \max \ \{e\}$$

to calculate the error e.

m	2	4	8	16
Error x(t)	0.5735009	0.2404915	0.111062762	0.053461745
Error y(t)	0.5735009	0.2404915	0.111062762	0.053461745

Table 1. Error Haar vs Analytic Solution

Example 2. Consider the system of homogeneous linear differential equations

$$x'(t) = x(t) + y(t) \tag{4.17}$$

$$y'(t) = y(t) \tag{4.18}$$

with initial conditions

$$x(0) = 0, \quad y(0) = 1$$

Solution. By using the approximation method mentioned in section 4 , we obtain the vectors X and Y,

$$(I - \mathcal{P}^T)X = \mathcal{P}^T Y + D$$
$$(I - \mathcal{P}^T)Y = D$$

the approximate solution of the given problem can be obtained

$$\text{For } m=2, \tilde{x} = \begin{cases} \frac{4}{9} & 0 \le t < \frac{1}{2} \\ \frac{52}{27} & \frac{1}{2} \le t < 1 \\ 0 & True \end{cases}$$

$$\tilde{y} = \begin{cases} \frac{4}{3} & 0 \le t < \frac{1}{2} \\ \frac{20}{9} & \frac{1}{2} \le t < 1 \\ 0 & True \end{cases}$$

$$\text{For } m=4, \tilde{x} = \begin{cases} \frac{8}{49} & 0 \le t < \frac{1}{4} \\ \frac{200}{343} & \frac{1}{4} \le t < \frac{1}{2} \\ \frac{2952}{2401} & \frac{1}{2} \le t < \frac{3}{4} \\ \frac{36936}{16807} & \frac{3}{4} \le t < 1 \\ 0 & True \end{cases}$$

$$\tilde{y} = \begin{cases} \frac{8}{7} & 0 \le t < \frac{1}{4} \\ \frac{72}{49} & \frac{1}{4} \le t < \frac{1}{2} \\ \frac{648}{343} & \frac{1}{2} \le t < \frac{3}{4} \\ \frac{5832}{2401} & \frac{3}{4} \le t < 1 \\ 0 & True \end{cases}$$

$$\text{For } m=8, \tilde{x} = \begin{cases} \frac{16}{225} & 0 \le t < \frac{1}{8} \\ \frac{784}{3375} & \frac{1}{8} \le t < \frac{1}{4} \\ \frac{272}{625} & \frac{1}{4} \le t < \frac{3}{8} \\ \frac{522512}{759375} & \frac{3}{8} \le t < \frac{1}{2} \\ \frac{2279632}{2278125} & \frac{1}{2} \le t < \frac{5}{8} \\ \frac{78843824}{56953125} & \frac{5}{8} \le t < \frac{3}{4} \\ \frac{4748001808}{2562890625} & \frac{3}{4} \le t < \frac{7}{8} \\ \frac{93074466064}{38443359375} & \frac{7}{8} \le t < 1 \\ 0 & True \end{cases}$$

$$\tilde{y} = \begin{cases} \frac{16}{25} & 0 \le t < \frac{1}{8} \\ \frac{272}{225} & \frac{1}{8} \le t < \frac{1}{4} \\ \frac{4624}{3375} & \frac{1}{4} \le t < \frac{3}{8} \\ \frac{78608}{50625} & \frac{3}{8} \le t < \frac{1}{2} \\ \frac{1336336}{759375} & \frac{1}{2} \le t < \frac{5}{8} \\ \frac{22717712}{11390625} & \frac{5}{8} \le t < \frac{3}{4} \\ \frac{386201104}{170859375} & \frac{3}{4} \le t < \frac{7}{8} \\ \frac{6565418768}{2562890625} & \frac{7}{8} \le t < 1 \\ 0 & True \end{cases}$$

For $m = 16, \tilde{x} = \begin{cases} \frac{32}{961} & 0 \le t < \frac{1}{16} \\ \frac{3104}{29791} & \frac{1}{16} \le t < \frac{1}{8} \\ \frac{170016}{923521} & \frac{1}{8} \le t < \frac{3}{16} \\ \frac{7840800}{28629151} & \frac{3}{16} \le t < \frac{1}{4} \\ \frac{332345376}{887503681} & \frac{1}{4} \le t < \frac{5}{16} \\ \frac{13396163616}{27512614111} & \frac{5}{16} \le t < \frac{3}{8} \\ \frac{522222684192}{852891037441} & \frac{3}{8} \le t < \frac{7}{16} \\ \frac{19878274978848}{26439622160671} & \frac{7}{16} \le t < \frac{1}{2} \\ \frac{7432656455188}{8196282869808} & \frac{1}{2} \le t < \frac{9}{16} \\ \frac{2740809115228}{2540847689640} & \frac{9}{16} \le t < \frac{5}{8} \\ \frac{9995177280804598}{7876627837885497} & \frac{5}{8} \le t < \frac{11}{16} \\ \frac{361207587884767652}{244175462974450425} & \frac{11}{16} \le t < \frac{3}{4} \\ \frac{12954952741598457}{7569439352207963} & \frac{3}{4} \le t < \frac{13}{16} \\ \frac{461671817738986237}{234652619918446859} & \frac{13}{16} \le t < \frac{7}{8} \\ \frac{16362396435172371183}{7274231217471852638} & \frac{7}{8} \le t < \frac{15}{16} \\ \frac{57715755520362048 5814}{22550116774162743 1786} & \frac{15}{16} \le t < 1 \\ 0 & True \end{cases}$

$$\tilde{y} = \begin{cases} \frac{32}{31} & 0 \le t < \frac{1}{8} \\ \frac{1056}{961} & \frac{1}{16} \le t < \frac{1}{8} \\ \frac{34848}{29791} & \frac{1}{8} \le t < \frac{3}{16} \\ \frac{1149984}{923521} & \frac{3}{16} \le t < \frac{1}{4} \\ \frac{37949472}{28629151} & \frac{1}{4} \le t < \frac{5}{16} \\ \frac{1252332576}{887503681} & \frac{5}{16} \le t < \frac{3}{8} \\ -\frac{41326975008}{27512614111} & \frac{3}{8} \le t < \frac{7}{16} \\ \frac{1363790175264}{852891037441} & \frac{7}{16} \le t < \frac{1}{2} \\ \frac{4500507578}{2643962216} & \frac{1}{2} \le t < \frac{9}{16} \\ \frac{148516750086}{81962828698} & \frac{9}{16} \le t < \frac{5}{8} \\ \frac{490105275284623}{254084768964048} & \frac{5}{8} \le t < \frac{11}{16} \\ \frac{16173474084392581}{7876627837885497} & \frac{11}{16} \le t < \frac{3}{4} \\ \frac{533724644784955}{244175462974450} & \frac{3}{4} \le t < \frac{13}{16} \\ \frac{17612913277903521}{7569439352207963} & \frac{13}{16} \le t < \frac{7}{8} \\ \frac{581226138170816199}{234652619918446859} & \frac{7}{8} \le t < \frac{15}{16} \\ \frac{19180462559636934573}{7274231217471852638} & \frac{15}{16} \le t < 1 \\ 0 & True \end{cases}$$

The graphs of approximate $(\tilde{x}(t), \tilde{y}(t))$ and analytic solutions $(x(t), y(t))$ of the system of differential equations are shown in figure 2.

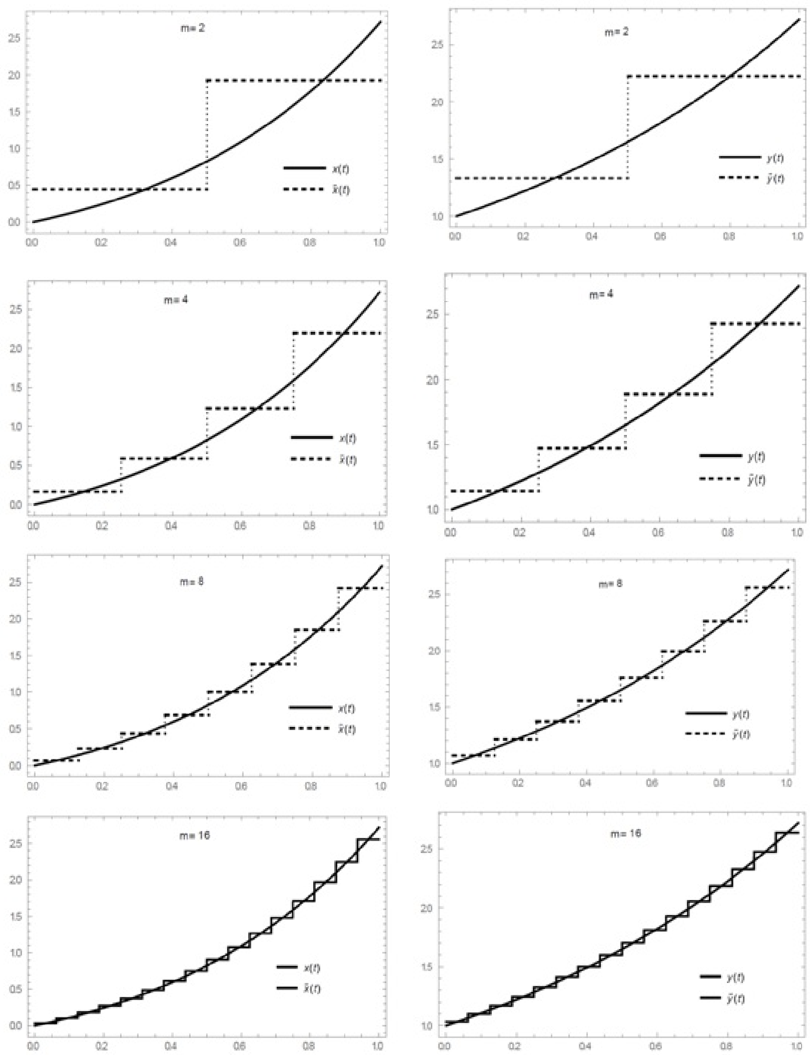


Figure 2. Haar vs Analytic solution

We use the following method

$$\|e\|_{\infty} = \max \ \{e\}$$

to calculate the error e.

m	2	4	8	16
Error x(t)	1.101565290	0.417222995	0.2074376564	0.082471958
Error y(t)	0.573500951	0.240491557	0.111062762	0.053461745

TABLE 2. Error Haar vs Analytic Solution

5. Conclusion

In this paper, the Haar wavelet has been applied for solving system of linear differential equations by reducing them into system of algebraic equations.The examples expressed that the accuracy of this method gives good approximate results.

Acknowledgements

The authors pay their sincere thanks to Professor Shiv Kumar Kaushik and the reviewer for his/her critical remarks and suggestions for the improvement of this paper.

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