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WEAKLY NANO SEMI-I-OPEN SETS AND WEAKLY NAMO SEMI-I-CONTINUOUS FUNCTIONS

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Abstract. In this paper, the notion of weakly nano semi-I-open sets is introduced and used to define the notions of weakly nano semi-I-continuous functions, weakly nano semi-I-open functions, and weakly nano semi-I-closed functions. Some characterizations and properties regarding these concepts are discussed.

1. Introduction

Thivagar and Richard [22] established the field of nano topological spaces. In 2016, Thivagar and Devi [20] introduced the notion of nano local functions and explore the field of nano topological spaces. In 2018, Parimala and Jafari [16] have introduced the notion of nano I-continuous functions in nano ideal topological spaces. Jamal M. Mustafa [14] studied the weakly b-I continuous functions in ideal topological spaces, also, in [11 - 13] he studied some covering properties and continuous functions using the semi-open sets. In this paper we introduce and study the new classes of continuous, irresolute and open functions namely *weakly nano semi-I-continuous*, *weakly nano semi-I-irresolute* and *weakly nano semi-I-open* functions in nano ideal topological spaces and we discuss some of their properties.

Let (X, τ) be a topological space and $A \subseteq X$. The complement of A in X, the closure of A, the interior of A and the power set of A will be denoted by $X - A = A^c$, $Cl\,(A)$, $Int\,(A)$ and $\mathcal{P}\,(A)$, respectively. The subject of ideals in topological spaces has been studied by Kuratowski [8] and Vaidyanathaswamy [24]. An ideal on a topological space (X, τ) is defined as a non-empty collection I of subsets of X satisfying the following two conditions: (1) If $A \in I$ and $B \subseteq A$, then $B \in I$; (2) If $A \in I$ and $B \in I$, then

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$A \cup B \in I$. An ideal topological space is a topological space (X, τ) with an ideal I on X and is denoted by (X, τ, I). For a subset $A \subseteq X$, $A^*(I) = \{x \in X : U \cap A \notin I$ for every $U \in \tau$ with $x \in U\}$ is called the local function of A with respect to I and τ [8]. We simply write A^* instead of $A^*(I)$ in case there is no chance of confusion. It is well known that $Cl^*(A) = A \cup A^*$ defines a Kuratowski closure operator for $\tau^*(I)$. Also [1, 18] have introduced some property related to nano semi-I-open.

First we shall recall the following definition used in the sequel.

Definition 1.1. A subset A of an ideal topological space (X, τ, I) is said to be

a) $I - open$ [7] if $A \subseteq Int(A^*)$.
b) $semi - I - open$ [5] if $A \subseteq Cl^*(Int(A))$.
c) $pre - I - open$ [3] if $A \subseteq Int(Cl^*(A))$.
d) $b - I - open$ [4] if $A \subseteq Cl^*(Int(A)) \cup Int(Cl^*(A))$.

2. Preliminaries

Definition 2.1. [21] Let U be a non-empty finite set of all objects called the universe and R be an equivalence relation on U named as indiscernibility relation. Then U is divided into disjoint equivalence classes. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$. Then,

(1) The lower approximation of X with respect to R is the set of all objects which can be for certain classified as X with respect to R and is denoted by $L_R(X)$. $L_R(X) = \cup\{R(x) : R(x) \subseteq X, x \in U\}$ where $R(x)$ denotes the equivalence class determined by $x \in U$.

(2) The upper approximation of X with respect to R is the set of all objects which can be possibly classified as X with respect to R and is denoted by $U_R(X) = \cup\{R(x) : R(x) \cap X \neq \phi, x \in U\}$.

(3) The boundary region of X with respect to R is the set of all objects which can be classified neither as X nor as not $-X$ with respect to R and is denoted by $B_R(X)$. $B_R(X) = U_R(X) - L_R(X)$.

Remark 2.2. [21] If (U, R) is an approximation space and $X, Y \subseteq U$, then

(1) $L_R(X) \subseteq X \subseteq U_R(X)$.
(2) $L_R(\phi) = U_R(\phi) = \phi$.
(3) $L_R(U) = U_R(U) = U$.
(4) $U_R(X \cup Y) = U_R(X) \cup U_R(Y)$.
(5) $U_R(X \cap Y) \subseteq U_R(X) \cap U_R(Y)$.
(6) $L_R(X \cup Y) \supseteq L_R(X) \cup L_R(Y)$.
(7) $L_R(X \cap Y) = L_R(X) \cup L_R(Y)$.
(8) $L_R(X) \subseteq L_R(Y)$ and $U_R(X) \subseteq U_R(Y)$ whenever $X \subseteq Y$.
(9) $U_R(X^c) = [L_R(X)]^c$ and $L_R(X^c) = [U_R(X)]^c$.

(10) $U_R(U_R(X)) = L_R(U_R(X)) = U_R(X)$.
(11) $L_R(L_R(X)) = U_R(L_R(X)) = L_R(X)$.

Definition 2.3. [21] Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. Then by the last remark, $\tau_R(X)$ satisfies the following axioms:

(1) U and $\phi \in \tau_R(X)$.
(2) The union of the elements of any subcollection of $\tau_R(X)$ is in $\tau_R(X)$.
(3) The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.
Then $\tau_R(X)$ is a topology on U called the nano topology on U with respect to X. $(U, \tau_R(X))$ is called the nano topological space. Elements of the nano topology are known as nano open sets in U and the complement of a nano open set is called nano closed.

Definition 2.4. [21] If $\tau_R(X)$ is the nano topology on U with respect to X, then the set $B = \{U, L_R(X), B_R(X)\}$ is the basis for $\tau_R(X)$.

Definition 2.5. [201 If $(U, \tau_R(X))$ is a nano topological space with respect to X where $X \subseteq U$ and if $A \subseteq U$, then

(1) The nano interior of the set A is defined as the union of all nano open subsets contained in A and is denoted by $nInt(A)$. $nInt(A)$ is the largest nano open subset of A.
(2) The nano closure of the set A is defined as the intersection of all nano closed sets containing A and is denoted by $nCl(A)$. $nCl(A)$ is the smallest nano closed set containing A.

Definition 2.6. [21] Let $(U, \tau_R(X))$ be a nano topological space and $A \subseteq U$. Then A is said to be:

(1) nano semi-open if $A \subseteq nCl(nInt(A))$.
(2) nano preopen if $A \subseteq nInt(nCl(A))$.

Definition 2.7. Let $(U, \tau_R(X))$ and $(V, \tau_{R'}(Y))$ be two nano topological spaces. A function $f : (U, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is called:

(1) nano continuous [22] if $f^{-1}(B)$ is nano open in U for every nano open set B in V .
(2) nano semi-continuous [19] if $f^{-1}(B)$ is nano semi-open in U for every nano open set B in V .
(3) nano precontinuous [20] if $f^{-1}(B)$ is nano preopen in U for every nano open set B in V .
(4) nano open if $f(A)$ is nano open in V for every nano open set A in U.
(5) nano closed if $f(C)$ is nano closed in V for every nano closed set C in U.

3. Nano ideal topological spaces

In 2016, Thivagar and Devi [20] have considered the nano local function in nano ideal topological space and they have obtained a new topology. A nano ideal topological space is a nano topological space $(U, \tau_R(X))$ with an ideal I on U and is denoted by $(U, \tau_R(X), I)$. For a subset $A \subseteq U$, $nA^*(I) = \{x \in U : W \cap A \notin I$ for every $W \in \tau_R(X)$ with $x \in W\}$ is called the nano local function of A with respect to I and $\tau_R(X)$ [20]. We simply write nA^* instead of $nA^*(I)$ in case there is no chance of confusion. It is well known that $nCl^*(A) = A \cup nA^*$ defines a nano closure operator for $\tau_R(X)^*(I)$.

Theorem 3.1. *[20] Let $(U, \tau_R(X))$ be a nano topological space with ideals I, J on U and A, B be subsets of U. Then the following statements are true:*

(i) if $A \subseteq B$, then $nA^* \subseteq nB^*$
(ii) if $I \subseteq J$, then $nA^*(I) \subseteq nA^*(J)$.
(iii) $nA^* = nCl(nA^*) \subseteq nCl(A)$.
(iv) $n(nA^*)^* = nA^*$.
(v) $nA^* \cup nB^* = n(A \cup B)^*$.
(vi) $nA^* - nB^* = n(A - B)^* - nB^* \subseteq n(A - B)^*$.
(vii) if $V \in \tau_R(X)$, then $V \cap nA^* = V \cap n(V \cap A)^* \subseteq n(V \cap A)^*$.
(viii) if $E \in I$, then $n(A \cup E)^* = nA^* = n(A - E)^*$.

Theorem 3.2. *[20] The nano closure operator nCl^* satisfies the following conditions :*

(i) $A \subseteq nCl^*(A)$.
(ii) $nCl^*(\phi) = \phi$ and $nCl^*(U) = U$.
(iii) if $A \subseteq B$ then $nCl^*(A) \subseteq nCl^*(B)$.
(iv) $nCl^*(A) \cup nCl^*(B) = nCl^*(A \cup B)$.
(v) $nCl^*(nCl^*(A)) = nCl^*(A)$.

4. Weakly nano semi-I-open sets

Definition 4.1. [20] A subset A of a nano ideal topological space $(U, \tau_R(X), I)$ is said to be *nano* $semi - I - open$ if $A \subseteq nCl^*(nInt(A))$.

Definition 4.2. A subset $A \subseteq U$ in a nano ideal topological space $(U, \tau_R(X), I)$ is said to be *weakly nano* $semi - I - open$ if $A \subseteq nCl^*(nInt(nCl(A)))$.

The family of all *weakly nano* $semi - I - open$ sets of the space $(U, \tau_R(X), I)$ will be denoted by $WNSIO(U, \tau_R(X))$.

A subset $A \subseteq U$ in a nano ideal topological space $(U, \tau_R(X), I)$ is said to be *weakly nano* $semi - I - closed$ if its complement is weakly nano semi-I-open.

Remark 4.3. Every *nano* $semi - I - open$ set is *weakly nano* $semi - I - open$.

The converse of the above remark need not be true as shown in the following example.

Example 4.4. Let $U = \{a, b, c, d\}$ be the universe, $X = \{b, d\} \subseteq U$, $U/R = \{\{a\}, \{b\}, \{c, d\}\}$, $\tau_R(X) = \{\phi, U, \{b\}, \{c, d\}, \{b, c, d\}\}$ and the ideal $I = \{\phi, \{a\}\}$. Then, the set $\{a, b, d\}$ is a *weakly nano* $semi-I-open$ *set* but it is not *nano* $semi-I-open$.

Lemma 4.5. *[7] Let A and B be subsets of U in a nano ideal topological space $(U, \tau_R(X), I)$.*

a) If $A \subseteq B$, then $A^* \subseteq B^*$.
b) If $V \in \tau_R(X)$, then $V \cap A^* \subseteq (V \cap A)^*$.
c) A^* is nano closed in $(U, \tau_R(X))$.

Theorem 4.6. *Let $(U, \tau_R(X), I)$ be an ideal topological space and A, B subsets of U.*

1) If $A_\alpha \in WNSIO(U, \tau_R(X))$ for each $\alpha \in \Delta$, then $\bigcup\{A_\alpha : \alpha \in \Delta\} \in WNSIO(U, \tau_R(X))$.
2) If $A \in WNSIO(U, \tau_R(X))$ and $B \in \tau_R(X)$, then $A \cap B \in WNSIO(U, \tau_R(X))$.

Proof. 1) Since $A_\alpha \in WNSIO(U, \tau_R(X))$, we have $\bigcup_{\alpha\in\Delta} A_\alpha \subseteq \bigcup_{\alpha\in\Delta} nCl^*(nInt(nCl(A_\alpha))) \subseteq \bigcup_{\alpha\in\Delta} [(nInt(nCl(A_\alpha))) \cup (nInt(nCl(A_\alpha)))^*]$
$\subseteq nInt(nCl(\bigcup_{\alpha\in\Delta} A_\alpha)) \cup (nInt(nCl(\bigcup_{\alpha\in\Delta} A_\alpha)))^* = nCl^*(nInt(nCl(\bigcup_{\alpha\in\Delta} A_\alpha)))$. Hence $\bigcup_{\alpha\in\Delta} A_\alpha \in WNSIO(U, \tau_R(X))$.

2) Let $A \in WNSIO(U, \tau_R(X))$ and $B \in \tau_R(X)$. Then $A \subseteq nCl^*(nInt(nCl(A)))$ and so $A \cap B \subseteq nCl^*(nInt(nCl(A))) \cap B = [nInt(nCl(A)) \cup (nInt(nCl(A)))^*] \cap B$
$\subseteq [(nInt(nCl(A)) \cap B) \cup ((nInt(nCl(A))) \cap B)^*] \subseteq [(nInt(nCl(A \cap B))) \cup (nInt(nCl(A \cap B)))^*] = nCl^*[nInt(nCl(A \cap B))]$. This shows that $A \cap B \in WNSIO(U, \tau_R(X))$. ☐

The following example shows that the finite intersection of weakly nano semi-I-open sets need not be weakly nano semi-I-open.

Example 4.7. Let $U = \{a, b, c, d, e\}$ be the universe, $X = \{b, c, d\} \subseteq U$ with $U/R = \{\{a\}, \{b, c\}, \{d, e\}\}$, $\tau_R(X) = \{\phi, U, \{b, c\}, \{b, c, d, e\}, \{d, e\}\}$ and the ideal $I = \{\phi\}$. The sets $A = \{a, b, c\}$ and $B = \{a, d, e\}$ are *weakly nano* $semi-I-open$ sets, but $A \cap B = \{a\}$ is not *weakly nano* $semi - I - open$ in U.

Theorem 4.8. *If a. subset $A \subseteq U$ in a nano ideal topological space $(U, \tau_R(X), I)$ is weakly nano $semi - I - closed$, then $nInt(nCl(nInt(A))) \subseteq A$.*

Proof. Since A is weakly nano semi-I-closed, $X - A \in WNSIO(U, \tau_R(X))$. Then $X - A \subseteq nCl^*(nInt(nCl(X - A))) \subseteq nCl(nInt(nCl(X - A))) = X - nInt(nCl(nInt(A)))$. Therefore, $nInt(nCl(nInt(A))) \subseteq A$. ☐

5. Weakly nano semi-I-continuous functions

Definition 5.1. [6] A function $f : (U, \tau_R(X), I) \rightarrow (V, \tau_{R'}(Y))$ is said to be *nano* $semi - I - continuous$ if $f^{-1}(B)$ is *nano* $semi - I - open$ set in $(U, \tau_R(X), I)$ for every nano open set B in $(V, \tau_{R'}(Y))$.

Definition 5.2. A function $f : (U, \tau_R(X), I) \to (V, \tau_{R'}(Y))$ is called *weakly nano semi* $- I -$ *continuous* if the inverse image of each nano open set in V is a *weakly nano semi* $- I -$ *open* set in U.

Remark 5.3. *Every nano semi* $- I -$ *continuous function is weakly nano semi* $- I -$ *continuous.*

The converse of the above remark need not be true as shown in the following example.

Example 5.4. Let $U = V = \{a, b, c, d\}$ be the universe, $X = Y = \{b, d\}$, $U/R = V/R' = \{\{a\}, \{b\}, \{c, d\}\}$, $\tau_R(X) = \tau_{R'}(Y) = \{\phi, U, \{b\}, \{c, d\}, \{b, c, d\}\}$ and the ideal $I = \{\phi, \{a\}\}$. Define $f : (U, \tau_R(X), I) \to (V, \tau_{R'}(Y))$ by $f(a) = c$, $f(b) = b$, $f(c) = a$, $f(d) = d$. We note that, the inverse image of each nano open set in V is a *weakly nano semi* $- I -$ *open* set in U but $f^{-1}(\{b, c, d\}) = \{a, b, d\}$ is a weakly nano semi-I-open set in U *but not* nano semi-I-open. Hence, f is *weakly nano semi* $- I -$ *continuous* but not nano semi-I-continuous.

Theorem 5.5. *For a function* $f : (U, \tau_R(X), I) \to (V, \tau_{R'}(Y))$ *the following are equivalent:*

1) f is weakly nano semi $- I -$ continuous.

2) For each $x \in X$ and each $B \in \tau_{R'}(Y)$ with $f(x) \in B$, there exists $A \in WNSIO(U, \tau_R(X))$ with $x \in A$ such that $f(A) \subseteq B$.

3) The inverse image of each nano closed set in $(V, \tau_{R'}(Y))$ is weakly nano semi $- I -$ closed in $(U, \tau_R(X), I)$.

Proof. Straightforward. □

Definition 5.6. Let $A \subseteq U$ in a nano ideal topological space $(U, \tau_R(X), I)$ and $x \in X$. Then A is called a *weakly nano semi* $- I -$ *neighborhood* of x, if there exists a *weakly nano semi* $- I -$ *open* set B containing x such that $B \subseteq A$.

Theorem 5.7. *For a function* $f : (U, \tau_R(X), I) \to (V, \tau_{R'}(Y))$, *the following statements are equivalent:*

1) f is weakly nano semi $- I -$ continuous.

2) For each $x \in X$ and each nano open set B in $(V, \tau_{R'}(Y))$ with $f(x) \in B$, $f^{-1}(B)$ is weakly nano semi $- I -$ neighborhood of x.

Proof. (1) $\Rightarrow$ (2). Let $x \in X$ and B be a nano open set in $(V, \tau_{R'}(Y))$ such that $f(x) \in B$. By Theorem 5.5, there exists a weakly nano semi-I-open set A in $(U, \tau_R(X), I)$ with $x \in A$ such that $f(A) \subseteq B$. So $x \in A \subseteq f^{-1}(B)$. Hence $f^{-1}(B)$ is a weakly nano semi-I-neighborhood of x.

(2) $\Rightarrow$ (1). Let B be a nano open set in $(V, \tau_{R'}(Y))$ and $f(x) \in B$. Then by assumption, $f^{-1}(B)$ is a *weakly nano semi* $- I -$ *neighborhood* of x. Thus for each $x \in f^{-1}(B)$ there exists a weakly nano semi-I-open set A_x containing x such that $x \in A_x \subseteq f^{-1}(B)$. Hence $f^{-1}(B) = \cup\{A_x : x \in f^{-1}(B)\}$ and so $f^{-1}(B) \in WNBIO(U, \tau_R(X))$. □

Definition 5.8. A function $f : (U, \tau_R(X), I) \to (V, \tau_{R'}(Y))$ is called *weakly nano* $semi - I - irresolute$ if $f^{-1}(B)$ is $weakly\ nano\ semi - I - open$ in $(U, \tau_R(X), I)$ for every $weakly\ nano\ semi - I - open$ set B in $(V, \tau_{R'}(Y))$.

Theorem 5.9. *Let* $f : (U, \tau_R(X), I) \to (V, \tau_{R'}(Y), J)$ *and* $g : (V, \tau_{R'}(Y), J) \to (W, \tau_{R''}(Z), K)$ *then*

1) gof *is weakly nano* $semi - I - continuous$ *if* f *is weakly nano* $semi - I - continuous$ *and* g *is nano continuous.*

2) gof *is weakly nano* $semi - I - continuous$ *if* f *is weakly nano* $semi - I - irresolute$ *and* g *is weakly nano* $semi - I - continuous$.

If $(U, \tau_R(X), I)$ is a nano ideal topological space and A is a subset of U, we denote by $\tau_R(X)_{|A}$ the relative nano topology on A and $I_{|A} = \{A \cap B : B \in I\}$ is obviously an ideal on A.

The proofs of the following two lemmas is similar to the proofs of Lemma 3.14 and Lemma 3.15 in [14].

Lemma 5.10. *Let* $(U, \tau_R(X), I)$ *be a nano ideal topological space and* A, B *be subsets of* U *such that* $B \subseteq A$. *Then* $nB^*(\tau_R(X)_{|A}, I_{|A}) = nB^*(\tau_R(X), I) \cap A$.

Lemma 5.11. *Let* $(U, \tau_R(X), I)$ *be a nano ideal topological space,* $A \subseteq U$ *and* $W \in \tau_R(X)$. *Then* $nCl^*(A) \cap W = nCl^*_W(A \cap W)$.

Theorem 5.12. *Let* $(U, \tau_R(X), I)$ *be a nano ideal topological space,* $A \subseteq W \in \tau_R(X)$. *If* $A \in WNSIO(U, \tau_R(X))$ *then* $A \in WNSIO(W, \tau_R(X)_{|W}, I_{|W})$.

Proof. Since $W \in \tau_R(X)$ and $A \in WNSIO(U, \tau_R(X))$, we have $A = W \cap A \subseteq W \cap nCl^*(nInt(nCl(A))) = [W \cap (nCl^*(nInt(nCl(A))))] \subseteq nCl^*(W \cap nInt(nCl(A)))$
$= nCl^*(Int(W \cap nCl(A))) = nCl^*(nInt_W(W \cap nCl(A)))$. Since $W \in \tau_R(X) \subseteq \tau_R(X)^*$, we obtain $A = W \cap A \subseteq W \cap nCl^*(nInt_W(W \cap nCl(A)))$
$= nCl^*_W(nInt_W(W \cap nCl(A))) = nCl^*_W(nInt_W(nCl_W(A)))$.

Then $A \in WNSIO(W, \tau_R(X)_{|W}, I_{|W})$. □

Corollary 5.13. Let $(U, \tau_R(X), I)$ be a nano ideal topological space, $W \in \tau_R(X)$ and $A \in WNSIO(U, \tau_R(X))$, then $W \cap A \in WNSIO(U, \tau_{|W}, I_{|W})$.

Proof. Since $W \in \tau_R(X)$ and $A \in WNSIO(U, \tau_R(X))$, by Theorem 4.6, $W \cap A \in WNBIO(U, \tau_R(X))$.

Since $W \in \tau_R(X)$, by Theorem 5.12, $W \cap A \in WNBIO(W, \tau_R(X)_{|W}, I_{|W})$. □

Theorem 5.14. *Let* $f : (U, \tau_R(X), I) \to (V, \tau_{R'}(Y))$ *be a weakly nano* $semi - I - continuous$ *function and* $W \in \tau_R(X)$. *Then the restriction* $f_{|W} : (W, \tau_R(X)_{|W}, I_{|W}) \to (V, \tau_{R'}(Y))$ *is weakly nano* $semi - I - continuous$.

Proof. Let G be any nano open set in $(V, \tau_{R'}(Y))$. Since f is $weakly\ nano\ semi - I - continuous$, we have $f^{-1}(G) \in WNSIO(U, \tau_R(X))$. Since $W \in \tau_R(X)$, by Theorem 5.12, we have $W \cap f^{-1}(G) \in WNSIO(W, \tau_R(X)_{|W}, I_{|W})$. On the other hand, $(f_{|W})^{-1}(G) = W \cap f^{-1}(G)$ and $(f_{|W})^{-1}(G) \in WNSIO(W, \tau_R(X)_{|W}, I_{|W})$. This shows that $f_{|W} : WNSIO(W, \tau_R(X)_{|W}, I_{|W}) \to (V, \tau_{R'}(Y))$ is *weakly nano* $semi - I - continuous$. □

Definition 5.15. A nano ideal topological space $(U, \tau_R(X), I)$ is said to be *weakly nano* $semi - I - normal$ if for each pair of non-empty disjoint nano closed subsets A and B of U, there exist two *weakly nano* $semi - I - open$ subsets G and W of U such that $A \subseteq G$, $B \subseteq W$ and $G \cap W = \phi$.

Theorem 5.16. *If* $f : (U, \tau_R(X), I) \to (V, \tau_{R'}(Y))$ *is weakly nano* $semi - I - continuous$*, nano closed injection and* V *is nano normal, then* $(U, \tau_R(X), I)$ *is weakly nano* $semi - I - normal$*.*

Proof. Let A and B be two disjoint nano closed subsets of U. Since f is nano closed and injective, $f(A)$ and $f(B)$ are disjoint nano closed subsets of $(V, \tau_{R'}(Y))$. Since V is nano normal, there exist two nano open subsets G and W of V such that $f(A) \subseteq G$, $f(B) \subseteq W$ and $G \cap W = \phi$. Now $f^{-1}(G)$ and $f^{-1}(W)$ are *weakly nano* $semi - I - open$ in U with $A \subseteq f^{-1}(G)$, $B \subseteq f^{-1}(W)$ and $f^{-1}(G) \cap f^{-1}(W) = \phi$. Thus $(U, \tau_R(X), I)$ is *weakly nano* $semi - I - normal$. □

Definition 5.17. A nano ideal topological space $(U, \tau_R(X), I)$ is said to be *weakly nano* $semi - I - connected$ if U can't be written as a union of two disjoint *weakly nano* $semi - I - open$ subsets of X.

Theorem 5.18. *The weakly nano* $semi - I - continuous$ *image of a weakly nano* $semi - I - connected$ *space is nano connected.*

Proof. Let $f : (U, \tau_R(X), I) \to (V, \tau_{R'}(Y))$ be a *weakly nano* $semi - I - continuous$ function of a *weakly nano* $semi - I - connected$ space $(U, \tau_R(X), I)$ onto a nano topological space $(V, \tau_{R'}(Y))$. Assume that V is nano disconnected, then $V = A \cup B$ where A and B are non-empty nano clopen with $A \cap B = \phi$. Since f is *weakly nano* $semi - I - continuous$, $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty *weakly nano* $semi - I - open$ in U. Also, $U = f^{-1}(V) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A) \cap f^{-1}(B) = \phi$. Hence U is not *weakly nano* $semi - I - connected$ which is a contradiction. Therefore, V is nano connected. □

6. Weakly nano semi-I-open functions

Recall that a subset F of a nano ideal topological space $(U, \tau_R(X), I)$ is said to be *weakly nano* $semi - I - closed$ if its complement is weakly nano semi-I-open.

Definition 6.1. A function $f : (U, \tau_R(X)) \to (V, \tau_{R'}(Y), I)$ is called *weakly nano* $semi - I - open$ (resp.,*weakly nano* $semi - I - closed$) if the image of every nano open (resp., nano closed) set in $(U, \tau_R(X))$ is *weakly nano* $semi - I - open$ (resp., *weakly nano* $semi - I - closed$) in $(V, \tau_{R'}(Y), I)$.

Theorem 6.2. *A function $f : (U, \tau_R(X)) \to (V, \tau_{R'}(Y), I)$ is weakly nano $semi - I - open$ if and only if for each $x \in U$ and each nano neighborhood W of x there exists $G \in WNSIO(V, \tau_{R'}(Y))$ containing $f(x)$ such that $G \subseteq f(W)$.*

Proof. $\Rightarrow$) Suppose that f is a *weakly nano* $semi - I - open$ function. For each $x \in U$ and each nano neighborhood W of x, there exists $W_x \in \tau_R(X)$ such that $x \in W_x \subseteq W$. Let $G = f(W_x)$. Since f is *weakly nano* $semi - I - open$, $G \in WNSIO(V, \tau_{R'}(Y))$ and $f(x) \in G \subseteq f(W)$.

$\Leftarrow$) Let W be a nano open set in $(U, \tau_R(X))$. For each $x \in W$, there exists $G_x \in WNSIO(V, \tau_{R'}(Y))$ such that $f(x) \in G_x \subseteq f(W)$. Now $f(W) = \cup\{G_x : x \in W\}$ and so $f(W) \in WNSIO(V, \tau_{R'}(Y))$. This shows that f is *weakly nano* $semi - I - open$. □

Theorem 6.3. *Let $f : (U, \tau_R(X)) \to (V, \tau_{R'}(Y), I)$ be a weakly nano $semi - I - open$ function. If G is any subset of V and C is a nano closed subset of U with $f^{-1}(G) \subseteq C$, then there exists a weakly nano $semi - I - closed$ subset H of V with $G \subseteq H$ such that $f^{-1}(H) \subseteq C$.*

Proof. Suppose that f is a *weakly nano* $semi - I - open$ function. Let G be any subset of V and C a nano closed subset of U with $f^{-1}(G) \subseteq C$. Then $U - C$ is nano open. Since f is *weakly nano* $semi - I - open$, $f(U - C)$ is *weakly nano* $semi - I - open$ in V. Let $H = V - f(U - C)$. Then H is *weakly nano* $semi - I - closed$ in V. Since $f^{-1}(G) \subseteq C$, $G \subseteq H$. Also, we obtain $f^{-1}(H) \subseteq C$. □

Theorem 6.4. *Let $f : (U, \tau_R(X)) \to (V, \tau_{R'}(Y), I)$ be weakly nano $semi - I - closed$. If G is any subset of V and W is a nano open subset of U with $f^{-1}(G) \subseteq W$, then there exists a weakly nano $semi - I - open$ subset H of V with $G \subseteq H$ such that $f^{-1}(H) \subseteq W$.*

Proof. Similar to that used in Theorem 6.3. □

Theorem 6.5. *For any bijective function $f : (U, \tau_R(X)) \to (V, \tau_{R'}(Y), I)$, the following are equivalent:*

1) *$f^{-1} : (V, \tau_{R'}(Y), I) \to (U, \tau_R(X))$ is weakly nano $semi - I - continuous$.*
2) *f is weakly nano $semi - I - open$.*
3) *f is weakly nano $semi - I - closed$.*

Proof. It is straightforward. □

Theorem 6.6. *Let $f : (U, \tau_R(X)) \to (V, \tau_{R'}(Y), I)$ and $g : (V, \tau_{R'}(Y), I) \to (W, \tau_{R''}(Z), K)$.*

1) gof is weakly nano $semi - I - open$ if f is nano open and g is a weakly nano $semi - I - open$.

2) f is weakly nano $semi - I - open$ if gof is nano open and g is a weakly nano $semi - I - continuous$ injection.

7. Conclusion

In this paper, we define and study the notions of weakly nano semi-I-continuous functions, weakly nano semi-I-open functions, and weakly nano semi-I-closed functions. Also some characterizations and properties regarding these concepts are discussed. In the future, we will define and study the notions of weakly nano b-I-continuous functions, weakly nano b-I-open functions, and weakly nano b-I-closed functions.

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