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NONLINEAR DEGENERATE p-LAPLACIAN ELLIPTIC EQUATIONS WITH SINGULAR GRADIENT LOWER ORDER TERM

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Abstract. The present paper aims to study of the Dirichlet problem for a nonlinear degenerate elliptic equation with singular gradient lower order term, we establish existence and regularity estimates for weak solutions of p-Laplacian type elliptic equations of the form

$$\begin{cases} -\operatorname{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\gamma}}\right) + \frac{|\nabla u|^p}{|u|^{\theta}} = f + u^r & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset in $\mathbb{R}^N$, $0 < \theta < 1$, $\gamma > 0$, $2 < p < N$, $0 < r < p - \theta$ and f is a nonnegative function on whose summability we will make different assumptions.

1. Introduction and main results

In this article, the problems to be studied are the following

$$\begin{cases} -\operatorname{div}\left(a(x,u)\widehat{a}(x,u,\nabla u)\right) + b(x)\frac{|\nabla u|^p}{|u|^{\theta}} = \lambda u^r + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded set of $\mathbb{R}^N$ $(N \geq 3)$, $B > 0$, f is a positive function belonging to $L^m(\Omega)$ with $m \geq 1$, $0 < \theta < 1$, $\lambda \geq 0$, $0 < r < p - \theta$ and $2 \leq p < N$, moreover, assume that b is a measurable function that satisfies a certain condition, where ν_1 and ν_2 are positive numbers such that

$$0 < \nu_1 \leq b(x) \leq \nu_2. \tag{1.2}$$

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In this context, we assume that $a : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carath'eodory function such that, for almost every x in Ω, for any $s \in \mathbb{R}$, and a certain positive real number γ, we have the following condition:

$$\frac{\alpha}{(d(x)+|s|)^{\gamma}} \leq a(x,s) \leq \beta, \tag{1.3}$$

where α and β are positive numbers, and $d(x)$ is a measurable function satisfying the condition for some positive numbers ρ_1 and ρ_2 as follows:

$$0 < \rho_1 \leq d(x) \leq \rho_2. \tag{1.4}$$

We assume that $\widehat{a} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, is a Carathéodory function and satisfies for a.e. $x \in \Omega$, $\forall s \in \mathbb{R}$ and $\xi, \xi' \in \mathbb{R}^N$, the following conditions :

$$\widehat{a}(x,s,\xi).\xi \geq |\xi|^p, \tag{1.5}$$

$$|\widehat{a}(x,s,\xi)| \leq \widehat{c}_1 |s|^{\frac{\theta(p-1)}{p}} + \widehat{c}_2 |\xi|^{p-1}, \tag{1.6}$$

$$(\widehat{a}(x,s,\xi) - \widehat{a}(x,s,\xi'))(\xi - \xi') \geq \widehat{c}_3 |\xi - \xi'|^p, \quad \forall \xi \neq \xi', \tag{1.7}$$

where $\widehat{c}_1, \widehat{c}_2, \widehat{c}_3$ are strictly positive real numbers.

Recently, there has been a significant increase in interest among the mathematical community regarding problems with a singular lower order term. This research area has gained popularity since the pioneering work by [1]. Many authors have since studied singular semilinear elliptic equations, as seen in [2, 3, 4, 12, 19] and related references. Notably, contributions made in [7, 8, 9] have also addressed similar problems involving singular gradient lower order terms.

In reference to problem (1.1), [18] has established existence and regularity results under the condition that $\lambda = 0$ and $a(x,u)\widehat{a}(x,u,\nabla u) = \frac{b(x)}{(1+|u|)^p}\nabla u$, where $b : \Omega \to \mathbb{R}$ is a measurable function that satisfies $\alpha \leq b(x) \leq \beta$ almost everywhere in Ω, where α and β are positive constants.

In the classical case where $p = 2$ and $f \in L^m(\Omega)$ with $m \geq 1$, existence and regularity solutions have been studied in [19]. Here, the authors assume $0 < \theta < 1$, $\gamma > 0$, and $0 < r < 2 - \theta$ to address the regularity of u varying with m:

- (R1) If $\frac{2N}{2N-\theta(N-2)} \leq m < \frac{N}{2}$, then $u \in H_0^1(\Omega) \cap L^{(2-\theta)m^{**}}(\Omega)$.
- (R2) If $m \geq \frac{N}{2}$, then $u \in H_0^1(\Omega) \cap L^{\infty}(\Omega)$.
- (R3) If $1 < m < \frac{2N}{2N-\theta(N-2)}$, then $u \in W_0^{1,q}(\Omega) \cap L^{(2-\theta)m^{**}}(\Omega)$ with $q = \frac{Nm(2-\theta)}{N-m\theta}$.
- (R4) If $m = 1$, then $u \in W_0^{1,\delta}$ with $\delta = \frac{N(2-\theta)}{N-\theta}$.

In [10] under the hypotheses $\lambda = 0$, $a(x,u) = \frac{1}{(1+|u|)^p}$ and $p = 2$, the author proved that

- (R4) If $m \geq \frac{2N}{2N-\theta(N-2)}$, then $u \in H_0^1$.
- (R5) If $\frac{N}{2N-\theta(N-1)} < m < \frac{2N}{2N-\theta(N-2)}$, then $u \in W_0^{1,\sigma}(\Omega)$ with $\sigma = \frac{mN(2-\theta)}{N-\theta m}$.

In this subsection, we will discuss the main characteristics and difficulties encountered in problem (1.1). Additionally, we will provide comments on the equation at hand.

Firstly, from hypothesis (1.6), the operator $Au = -\mathrm{div}\,(a(x,u)\widehat{a}(x,u,\nabla u))$ is well defined between $W_0^{1,p}(\Omega)$ and its dual space $W^{-1,p'}(\Omega)$. However, by assumption (1.3), (1.5), the operator A is not coercive. Because, if $\|u_n\|_{W^{1,p}(\Omega)}$ tends to infinity then $\frac{\langle Au_n, u_n\rangle}{\|u_n\|_{W^{1,p}(\Omega)}} \to 0$. The absence of coercivity in problem (1.1) poses a challenge, rendering the classical theory for elliptic operators between spaces in duality (as discussed in [17]) inapplicable, even when the data f are sufficiently regular (as noted in [16]). Another difficulty arises when we impose a variable exponential growth condition (1.6) for $\widehat{a}$. A problematic consequence is that $a(x,u)\widehat{a}(x,u,\nabla u)$, which appears in the principal term, may not belong to $L^1(\Omega)$, meaning that the first term in (1.1) may not be well-defined even as a distribution.

Our approach relies on approximating problems in such a way that have finite energy solutions and establishing several a priori estimates for the behavior of the approximate solution. We collect all these results in the section 2 and section 3. Section 4 contain the proofs of our main results.

In order to state our main results, we give the definition of weak solutions to problem (1.1).

Definition 1.1. Let $f \in L^m(\Omega)$, $m \geq 1$. A measurable function u is said to be solution in the sense of distributions to the problem (1.1), if $u \in W_0^{1,1}(\Omega)$, $\widehat{a}(x,u,\nabla u) \in (L^1(\Omega))^N$, $\frac{|\nabla u|^p}{u^\theta} \in L^1(\Omega)$, $u > 0$ in Ω, and

$$\int_\Omega a(x,u)\widehat{a}(x,u,\nabla u)\nabla\varphi dx + \int_\Omega b(x)\frac{|\nabla u|^p}{u^\theta}\varphi dx = \int_\Omega (\lambda u^r + f)\varphi dx, \quad \forall \varphi \in C_0^1(\Omega) \tag{1.8}$$

The initial outcome pertains to a specific function f that produces solutions in the energy space $W_0^{1,p}(\Omega)$ that are unbounded.

Theorem 1.2. *Let $f \in L^m(\Omega)$ be a positive function, such that*

$$\frac{pN}{pN - \theta(N-p)} \leq m < \frac{N}{p}. \tag{1.9}$$

Let a and $\widehat{a}$ be Carathéodory functions satisfying (1.3)-(1.7). *Then, the problem* (1.1) *has at least one distributional solution $u \in W_0^{1,p}(\Omega) \cap L^{(p-\theta)\frac{Nm}{N-pm}}(\Omega)$.*

The next result considers the case where f has a high summability.

Theorem 1.3. *Let $f \in L^m(\Omega)$, be a positive function, such that*

$$m \geq \frac{N}{p} \tag{1.10}$$

and let a and $\widehat{a}$ be Carathéodory functions satisfying (1.3)-(1.7). *Then, the problem* (1.1) *has at least one distributional solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.*

The following outcome concerns the situation where the summability of f leads to the existence of a solution with infinite energy, represented by $u \in W_0^{1,\sigma}(\Omega)$, where $1 < \sigma < p$.

Theorem 1.4. *Let $f \in L^m(\Omega)$ be a positive function, such that*

$$1 < m < \frac{pN}{pN - \theta(N-p)}. \tag{1.11}$$

Let a and $\widehat{a}$ be Carathéodory functions satisfying (1.3)-(1.7). *Then, the problem* (1.1) *has at least one distributional solution $u \in W_0^{1,\sigma}(\Omega) \cap L^{(p-\theta)\frac{Nm}{N-pm}}(\Omega)$, with*

$$\sigma = \frac{mN(p-\theta)}{N - \theta m}. \tag{1.12}$$

The first result deals with the case where the source function f is an element of the $L^1(\Omega)$ space.

Theorem 1.5. *Let $f \in L^1(\Omega)$ be a positive function. Let a and $\widehat{a}$ be Carathéodory functions satisfying* (1.3)-(1.7). *Then, the problem* (1.1) *possesses at least one distributional solution $u \in W_0^{1,\eta}(\Omega)$, where*

$$\eta = \frac{N(p-\theta)}{N-\theta}, \tag{1.13}$$

The following truncation functions will be utilized, where k is a fixed value,

$$T_k(s) = \max\{-k, \min\{s,k\}\} \text{ and } G_k(s) = s - T_k(s) \text{ for every } s \in \mathbb{R}.$$

2. Approximating problems

Within this section, we demonstrate the existence of a resolution to the subsequent approximating problem, which is both non-degenerate and non-singular.

Let us consider, for $0 < \varepsilon < 1$, the sequence of approximating problems

$$\begin{cases} -\operatorname{div}\Big(a(x,T_{\frac{1}{\varepsilon}}(u_\varepsilon))\widehat{a}(x,u_\varepsilon,\nabla u_\varepsilon)\Big) + b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(|u_\varepsilon|+\varepsilon)^{\theta+1}} = \frac{\lambda|u_\varepsilon|^r}{1+\varepsilon|u_\varepsilon|^r} + f_\varepsilon, & \text{in } \Omega, \\ u_\varepsilon = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

where $f_\varepsilon = T_{\frac{1}{\varepsilon}}(f)$ ($f_\varepsilon > 0$) be a sequence of bounded functions defined in Ω which converges to $f > 0$ in $L^1(\Omega)$. From the results of [5], the problem (2.1) admits at least one solution $u_\varepsilon \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, moreover, $u_\varepsilon \geq 0$ since the right-hand side is nonnegative (by the assumptions on f), and since the gradient lower order term has the same sign of the solution. Therefore, u_ε solves

$$\begin{cases} -\operatorname{div}\Big(a(x,T_{\frac{1}{\varepsilon}}(u_\varepsilon))\widehat{a}(x,u_\varepsilon,\nabla u_\varepsilon)\Big) + b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\theta+1}} = \frac{\lambda u_\varepsilon^r}{1+\varepsilon u_\varepsilon^r} + f_\varepsilon, & \text{in } \Omega, \\ u_\varepsilon = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

in the sense that u_ϵ satisfies the weak formulation of (2.2):

$$\begin{aligned} &\int_\Omega a(x,T_{\frac{1}{\varepsilon}}(u_\varepsilon))\widehat{a}(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla\phi\,dx + \int_\Omega b(x)\frac{u_\varepsilon\,|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\theta+1}}\phi\,dx \\ &= \int_\Omega \frac{\lambda u_\varepsilon^r}{1+\varepsilon u_\varepsilon^r}\phi\,dx + \int_\Omega f_\varepsilon\phi\,dx, \end{aligned} \tag{2.3}$$

for every ϕ in $H_0^1(\Omega)\cap L^\infty(\Omega)$. Is usually easy, we just have by taking u_ϵ^- as test function in the weak formulation of problem (2.1).

In the remainder of this paper, we denote by C, various positive constants depending only on the data of the problem, but not on n. Also, all integrals are computed in the Lebesgue sense even if we omit the dx symbol.

3. A priori estimates

The focus of this section is to establish certain a priori estimates for the partial derivatives and approximate solutions u_ε.

Lemma 3.1. *Let u_ε be the solutions to problems* (2.2). *Then it results*

$$\int_\Omega b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\theta+1}} \leq \lambda\int_\Omega u_\varepsilon^r + \int_\Omega f. \tag{3.1}$$

Proof. The use of $\frac{T_h(u_\varepsilon)}{h}$ as test function in (2.2) and dropping the nonnegative first term, implies

$$\int_\Omega b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\theta+1}}\frac{T_h(u_\varepsilon)}{h} \le \lambda\int_\Omega u_\varepsilon^r\frac{T_h(u_\varepsilon)}{h} + \int_\Omega f_\varepsilon\frac{T_h(u_\varepsilon)}{h}.$$

Since $f_\varepsilon \le f$ and $\frac{T_h(u_\varepsilon)}{h} \le 1$, we have

$$\int_\Omega b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\theta+1}}\frac{T_h(u_\varepsilon)}{h} \le \lambda\int_\Omega u_\varepsilon^r + \int_\Omega f. \tag{3.2}$$

Letting h tend to 0 in (3.2), we deduce (3.1) by Fatou's Lemma. □

Subsequently, we will need the following lemma

Lemma 3.2. [19] *Let $\varsigma > 0$ and let $0 < \varepsilon < 1$, then there exists $C_0 > 0$ such that*

$$\frac{\alpha\varsigma(t+\varepsilon)^{\theta-1}}{(\rho_2+t)^\gamma} + \frac{\nu_1 t}{t+\varepsilon} \ge C_0, \tag{3.3}$$

for every $t \ge 0$.

Lemma 3.3. *Assume that f belongs to $L^m(\Omega)$ with m satisfies* (1.9) *and let u_ε be a solution of* (2.2)*. Then there exists a positive constant C, depending on N, $\|f\|_{L^m(\Omega)}$, γ, and r such that*

$$\|u_\varepsilon\|_{L^{(p-\theta)\frac{Nm}{N-pm}}(\Omega)} \le C, \tag{3.4}$$

and

$$\|u_\varepsilon\|_{W_0^{1,p}(\Omega)} \le C, \tag{3.5}$$

Proof. Take $\phi = (u_\varepsilon+\varepsilon)^{\theta+p\kappa} - \varepsilon^{\theta+p\kappa}$, with

$$\kappa = \frac{p^* - \theta m'}{pm' - p^*}, \tag{3.6}$$

as a test function in problems (2.2), (note that $\kappa \ge 0$ and $\theta + p\kappa > 0$), we get

$$\begin{aligned}
&(\theta+p\kappa)\int_\Omega a(x,T_{\frac{1}{\varepsilon}}(u_\varepsilon))\widehat{a}(x,u_\varepsilon,\nabla u_\varepsilon).\nabla u_\varepsilon(u_\varepsilon+\varepsilon)^{\theta+p\kappa-1}\\
&\quad+\int_\Omega b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\theta+1}}(u_\varepsilon+\varepsilon)^{\theta+p\kappa}\\
&\quad=\varepsilon^{\theta+p\kappa}\int_\Omega b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\theta+1}} + \int_\Omega\left[\frac{\lambda u_\varepsilon^r}{1+\varepsilon u_\varepsilon}+f_\varepsilon\right]\left[(u_\varepsilon+\varepsilon)^{\theta+p\kappa}-\varepsilon^{\theta+p\kappa}\right].
\end{aligned}$$

Through the utilization of hypothesis (1.2)-(1.5) and by eliminating the nonpositive term on the right-hand side, we obtain:

$$\begin{aligned}
&\int_\Omega|\nabla u_\varepsilon|^p(u_\varepsilon+\varepsilon)^{p\kappa}\left[\frac{(\theta+p\kappa)\alpha(u_\varepsilon+\varepsilon)^{\theta-1}}{(\rho_2+u_\varepsilon)^\gamma}+\frac{\nu_1 u_\varepsilon}{u_\varepsilon+\varepsilon}\right]\\
&\quad\le\varepsilon^{\theta+p\kappa}\int_\Omega b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\theta+1}}+\int_\Omega(\lambda u_\varepsilon^r+f_\varepsilon)(u_\varepsilon+\varepsilon)^{\theta+p\kappa}.
\end{aligned}$$

Recalling Lemma 3.2 (here $t = u_\varepsilon$, $\varsigma = \theta + p\kappa$), we have

$$C_0 \int_\Omega |\nabla u_\varepsilon|^p (u_\varepsilon+\varepsilon)^{p\kappa} \le \varepsilon^{\theta+p\kappa} \int_\Omega b(x) \frac{u_\varepsilon |\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\theta+1}} + \int_\Omega (\lambda u_\varepsilon^r + f_\varepsilon)(u_\varepsilon+\varepsilon)^{\theta+p\kappa}.$$

Using (3.1), the fact that $u_\varepsilon^r \le (u_\varepsilon+\varepsilon)^r$, $0 < \varepsilon^{\theta+p\kappa} < (u_\varepsilon+\varepsilon)^{\theta+p\kappa}$ (since $u_\varepsilon \ge 0$, $0<\varepsilon<1$, and $\theta+p\kappa>0$) and that $f_\varepsilon \le f$, we get

$$C_1 \int_\Omega \left|\nabla \left[(u_\varepsilon+\varepsilon)^{\kappa+1} - \varepsilon^{\kappa+1}\right]\right|^p = \int_\Omega |\nabla u_\varepsilon|^p (u_\varepsilon+\varepsilon)^{p\kappa} \le 2\lambda \int_\Omega (u_\varepsilon+\varepsilon)^{r+\theta+p\kappa} + 2\int_\Omega f (u_\varepsilon+\varepsilon)^{\theta+p\kappa}. \quad (3.7)$$

By applying Sobolev's inequality to the left-hand side and Hölder's inequality to the right-hand side, and using equation (3.7) along with the fact that $r+\theta<p$, we can derive the following expression:

$$\left(\int_\Omega \left|(u_\varepsilon+\varepsilon)^{\kappa+1} - \varepsilon^{\kappa+1}\right|^{p^*}\right)^{\frac{p}{p^*}} \le C_2 \|f\|_{L^m(\Omega)} \left(\int_\Omega (u_\varepsilon+\varepsilon)^{(\theta+p\kappa)m'}\right)^{\frac{1}{m'}} + C_3 \int_\Omega (u_\varepsilon+\varepsilon)^{r+\theta+p\kappa}. \quad (3.8)$$

It should be noted that (3.6) can be restated as the condition that $p^*(\kappa+1) = (\theta+p\kappa)m'$. As a result of $r<p-\theta$, we have $1 < \frac{p^*}{p} = \frac{(\kappa+1)p^*}{p\kappa+p} < \frac{(\kappa+1)p^*}{r+\theta+p\kappa}$. Thus, using Hölder inequality in the second term of the right-hand side of (3.9) and the fact that $|(t+\varepsilon)^\tau - \varepsilon^\tau|^{p^*} \ge C(t+\varepsilon)^{\tau p^*} - C$, for every $t,\tau \ge 0$ we then have

$$\left(\int_\Omega \left[(u_\varepsilon+\varepsilon)^{(\kappa+1)p^*} - 1\right]\right)^{\frac{p}{p^*}} \le C_5 \|f\|_{L^m(\Omega)} \left(\int_\Omega (u_\varepsilon+\varepsilon)^{(\kappa+1)p^*}\right)^{\frac{1}{m'}} + C_6 \left(\int_\Omega (u_\varepsilon+\varepsilon)^{(\kappa+1)p^*}\right)^{\frac{r+\theta+p\kappa}{(\kappa+1)p^*}}. \quad (3.9)$$

Now we point out that $\frac{p}{p^*} > \frac{1}{m'}$, $\frac{p}{p^*} > \frac{r+\theta+p\kappa}{(\kappa+1)p^*}$ due to the hypotheses on m, θ and r. Hence

$$\int_\Omega (u_\varepsilon+\varepsilon)^{(\kappa+1)p^*} \le C. \quad (3.10)$$

Therefore, by (3.10) and the fact that $(\kappa+1)p^* = (p-\theta)\frac{Nm}{N-pm}$, we get the estimate (3.4). Thus, from (3.7) and the fact that $p\kappa \ge 0$ (since (1.9) holds), it follows that

$$\int_{\{u_\varepsilon \ge 1\}} |\nabla u_\varepsilon|^p \le \int_\Omega |\nabla u_\varepsilon|^p (u_\varepsilon+\varepsilon)^{p\kappa} \le C. \quad (3.11)$$

On the other hand, the use of $T_1(u_\varepsilon)$ as test function in (2.2) yields

$$\int_\Omega a(x, T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \widehat{a}(x, u_\varepsilon, \nabla u_\varepsilon).\nabla T_1(u_\varepsilon) + \int_\Omega b(x) \frac{u_\varepsilon |\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\theta+1}} T_1(u_\varepsilon) = \int_\Omega \frac{\lambda u_\varepsilon^r}{1+\varepsilon u_\varepsilon^r} T_1(u_\varepsilon) + \int_\Omega f_\varepsilon T_1(u_\varepsilon).$$

Using (1.3)-(1.5) and the boundedness of the sequence u_ε in $L^{(p-\theta)\frac{Nm}{N-pm}}(\Omega)$ (recall that $r < (p-\theta)\frac{Nm}{N-pm}$), after dropping nonnegative lower-order term, we get

$$\frac{\alpha}{(1+\rho_2)^\gamma}\int_{\{u_\varepsilon<1\}}|\nabla T_1(u_\varepsilon)|^p \le \lambda\int_\Omega u_\varepsilon^r + \int_\Omega f \le C. \tag{3.12}$$

In sum, from (3.11) and (3.12), we deduce (3.5), as we wanted to prove. □

Lemma 3.4. *Assume that f belongs to $L^m(\Omega)$ with $m \ge \frac{N}{p}$. Then there exists a positive constant C, depending on N, $\|f\|_{L^m(\Omega)}$, γ, and r such that*

$$\|u_\varepsilon\|_{W_0^{1,p}(\Omega)} \le C, \tag{3.13}$$

and

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \le C. \tag{3.14}$$

Proof. Suppose that $p-\theta>0$, then there exists $\varrho>1$ such that

$$\frac{Nr}{p(p-\theta+r)} < \varrho < \frac{N}{p}. \tag{3.15}$$

We know that f belongs also to $L^\varrho(\Omega)$, by Lemma 3.3 the sequence u_ε is bounded in $L^{(p-\theta)\frac{N\varrho}{N-p\varrho}}(\Omega)$. It follows from (3.15) that $\frac{N\varrho(p-\theta)}{r(N-p\varrho)} > \frac{N}{p}$. From where, the right-hand side of (2.2) is bounded in $L^s(\Omega)$, with $s>\frac{N}{p}$. For each $t\ge 0$, we define the function

$$H(t)=\int_0^t \frac{d\tau}{(\rho_2+\tau)^\gamma}.$$

Note that the function H is well defined since $\rho_2>0$. Taking $G_k(H(u_\varepsilon))$ as test function in (2.2). Using (1.3)-(1.5), the fact that $f_\varepsilon \le f$, and dropping the nonnegative lower order term, we obtain

$$\alpha\int_\Omega |\nabla G_k(H(u_\varepsilon))|^p \le \int_\Omega (\lambda u_\varepsilon^r + f)G_k(H(u_\varepsilon)). \tag{3.16}$$

As the right-hand side of equation (3.16) is bounded in $L^s(\Omega)$, where $s>\frac{N}{p}$, the inequality itself serves as the initial step of Stampacchia's proof of L^∞-regularity (see [20]). Therefore, a constant c_1 exists that is independent of ε, and we have:

$$0 \le H(u_\varepsilon) \le c_1.$$

Therefore, the coercivity of H implies (3.14). The estimate of the sequence u_ε in $W_0^{1,p}(\Omega)$ is now very easy. In fact, by taking u_ε as test function in (2.2). Using (1.3)-(1.5), the boundedness of the sequence u_ε in $L^\infty(\Omega)$, and dropping the nonnegative lower-order term, we obtain

$$\int_\Omega |\nabla u_\varepsilon|^p \le c\|u_\varepsilon\|_{L^\infty(\Omega)}\left(\|u_\varepsilon\|^r_{L^\infty(\Omega)} + \|f\|_{L^m(\Omega)}\right) \le C. \tag{3.17}$$

Thus, (3.13) follows from (3.17). □

Lemma 3.5. *Assume that f belongs to $L^m(\Omega)$ with m satisfies* (1.11)*, and let u_ε be a solution of* (2.2)*. Then there exists a positive constant C, depending on N, $\|f\|_{L^m(\Omega)}$, γ, and r such that*

$$\|u_\varepsilon\|_{W_0^{1,\sigma}(\Omega)} \le C, \tag{3.18}$$

and

$$\|T_k(u_\varepsilon)\|_{W_0^{1,p}(\Omega)} \leq C, \quad \forall k > 0, \tag{3.19}$$

where σ *given by* (1.12).

Proof. The demonstration is the same as that of Lemma 3.3 until reaching the a priori estimation (3.4), since the assumption (1.11) implies that $p\kappa > 0$. From (3.7), and (3.4), we obtain

$$\int_\Omega |\nabla u_\varepsilon|^p (u_\varepsilon + \varepsilon)^{p\kappa} \leq C. \tag{3.20}$$

Now, by Hölder's inequality, (1.11), (3.6)-(3.7), (3.20), (3.10) and that $\sigma < p$, $-\kappa \frac{\sigma p}{p-\sigma} = (\kappa+1)p^*$, we obtain

$$\begin{aligned}
\int_\Omega |\nabla u_\varepsilon|^\sigma &= \int_\Omega \frac{|\nabla u_\varepsilon|^\sigma}{(u_\varepsilon+\varepsilon)^{-\kappa\sigma}} (u_\varepsilon + \varepsilon)^{-\kappa\sigma} \\
&\leq \left(\int_\Omega \frac{|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{-p\kappa}} \right)^{\frac{\sigma}{p}} \left(\int_\Omega (u_\varepsilon+\varepsilon)^{-\kappa \frac{\sigma p}{p-\sigma}} \right)^{\frac{p-\sigma}{p}} \\
&\leq \left(\int_\Omega |\nabla u_\varepsilon|^p (u_\varepsilon+\varepsilon)^{p\kappa} \right)^{\frac{\sigma}{p}} \left(\int_\Omega (u_\varepsilon+\varepsilon)^{(\kappa+1)p^*} \right)^{\frac{p-\sigma}{p}} \\
&\leq C.
\end{aligned} \tag{3.21}$$

So, (3.21) imply (3.18).

Moreover, taking $T_k(u_\varepsilon)$ as test function in (2.2). Using (1.2)-(1.5), (3.4) (recall that $r < (p-\theta)\frac{Nm}{N-pm}$), $f_\varepsilon \leq f$, By omitting the lower-order term which is non-negative, we can deduce the following result

$$\frac{\alpha}{(\rho_2 + k)^\gamma} \int_\Omega |\nabla T_k(u_\varepsilon)|^p \leq \int_\Omega [\lambda u_\varepsilon^r + f] T_k(u_\varepsilon) \leq C.$$

Hence, we have completed the demonstration of Lemma 3.5. □

Lemma 3.6. *Let* $f \in L^1(\Omega)$ *be a positive function. and let* u_ε *be a solution of* (2.2). *Then there exists a positive constant* C, *depending on* N, $\|f\|_{L^m(\Omega)}$, γ, *and* r *such that*

$$\|u_\varepsilon\|_{W_0^{1,\eta}(\Omega)} \leq C, \tag{3.22}$$

and

$$\|T_k(u_\varepsilon)\|_{W_0^{1,p}(\Omega)} \leq C, \quad \forall k > 0, \tag{3.23}$$

where η *given by* (1.13).

Proof. The estimate (3.1) and that $2^{\theta+1} u_\varepsilon u_\varepsilon^\theta \geq (u_\varepsilon + \varepsilon)^{\theta+1}$ in $\{u_\varepsilon \geq 1\}$, give

$$\frac{\nu_1}{2^{\theta+1}} \int_{\{u_\varepsilon \geq 1\}} \frac{|\nabla u_\varepsilon|^p}{u_\varepsilon^\theta} \leq \int_{\{u_\varepsilon \geq 1\}} b(x) \frac{u_\varepsilon |\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\theta+1}} \leq \lambda \|u_\varepsilon^r\|_{L^1(\Omega)} + \|f\|_{L^1(\Omega)}. \tag{3.24}$$

Consider a positive real number s such that $1 < s < p$. By applying Hölder's inequality, we can readily obtain that

$$\int_\Omega |\nabla G_1(u_\varepsilon)|^s = \int_{\{u_\varepsilon \geq 1\}} \frac{|\nabla G_1(u_\varepsilon)|^s}{u_\varepsilon^{\frac{\theta s}{p}}} u_\varepsilon^{\frac{\theta s}{p}}$$
$$\leq \left(\int_{\{u_\varepsilon \geq 1\}} \frac{|\nabla G_1(u_\varepsilon)|^p}{u_\varepsilon^{\theta}} \right)^{\frac{s}{p}} \left(\int_{\{u_\varepsilon \geq 1\}} u_\varepsilon^{\frac{\theta s}{p-s}} \right)^{\frac{p-s}{p}}. \tag{3.25}$$

Setting

$$M = \lambda \|u_\varepsilon^r\|_{L^1(\Omega)} + \|f\|_{L^1(\Omega)}. \tag{3.26}$$

In the case of $s = p - \theta$, we have $1 < s < p$. Therefore, using (3.24)-(3.26), we get

$$\int_\Omega |\nabla G_1(u_\varepsilon)|^s \leq C_1 M^{\frac{s}{p}} \left(\int_{\{u_\varepsilon \geq 1\}} u_\varepsilon^s \right)^{\frac{\theta}{p}}$$
$$\leq C_1 \left[M^{\frac{s}{p}} \left(\int_{\{u_\varepsilon \geq 1\}} G_1(u_\varepsilon)^s \right)^{\frac{\theta}{p}} + M^{\frac{s}{p}} \right]. \tag{3.27}$$

Therefore, utilizing Poincaré's inequality on the left-hand side of equation (3.27), and applying Young's inequality on the right-hand side, it leads to

$$\int_\Omega G_1(u_\varepsilon)^s \leq C_2 \left(M + M^{\frac{s}{p}} \right). \tag{3.28}$$

We get by applying Minkowski's inequality, the fact that $|T_1(u_\varepsilon)| \leq 1$,, and the convexity of the real function $t \mapsto t^s$ (since $s > 1$)

$$\int_\Omega u_\varepsilon^s \leq C_3 \left(1 + \int_\Omega G_1(u_\varepsilon)^s \right). \tag{3.29}$$

Observe that, From (3.28) and (3.29), it follows that

$$\int_\Omega u_\varepsilon^s \leq C_4 \left(M + M^{\frac{s}{p}} + 1 \right). \tag{3.30}$$

The fact that $r < p - \theta$ we get $r < s$, and using Hölder's inequality, we have

$$\int_\Omega u_\varepsilon^r \leq C_5 \left(\int_\Omega u_\varepsilon^s \right)^{\frac{r}{s}}. \tag{3.31}$$

Combining (3.26), (3.30), and (3.31), we obtain

$$M - \|f\|_{L^1(\Omega)} \leq C_6 \left(M^{\frac{r}{s}} + M^{\frac{r}{p}} + 1 \right). \tag{3.32}$$

This implies that

$$M \leq C.$$

Then, from (3.26), the sequence u_ε^r is bounded in $L^1(\Omega)$. By using $s = \eta$ as a test function in (3.25) and using the boundedness of sequence u_ε^r in $L^1(\Omega)$, we can see that

$$\int_\Omega |\nabla G_1(u_\varepsilon)|^\eta \leq C_7 \left(\int_{\{u_\varepsilon \geq 1\}} (G_1(u_\varepsilon))^{\frac{\theta \eta}{p-\eta}} \right)^{\frac{p-\eta}{p}} + C_8 \tag{3.33}$$

Equality (1.13) implies that $\eta^* = \frac{\eta\theta}{p-\eta}$. By Sobolev's inequality, we get

$$\left(\int_{\{u_\varepsilon \geq 1\}} (G_1(u_\varepsilon))^{\eta^*}\right)^{\frac{\eta}{\eta^*}} \leq C_9 \int_{\{u_\varepsilon \geq 1\}} |\nabla G_1(u_\varepsilon)|^\eta \leq C_{10} \left(\int_\Omega (G_1(u_\varepsilon))^{\eta^*}\right)^{\frac{\theta}{\eta^*}} + C_{11}. \tag{3.34}$$

The fact that $\theta < 1 < \eta$, the inequality (3.34) implies that $G_1(u_\varepsilon)$ is bounded in $L^{\eta^*}(\Omega)$ and from (3.33), we get

$$\|G_1(u_\varepsilon)\|_{W_0^{1,\eta}(\Omega)} \leq C. \tag{3.35}$$

Considering the following function $T_k(u_\varepsilon)$ as test function in (2.2), we obtain

$$\int_{\{u_\varepsilon \leq k\}} |\nabla T_k(u_\varepsilon)|^p \leq C_{12} k(k+1)^\gamma \quad \forall n \in \mathbb{N}. \tag{3.36}$$

By (3.35), (3.36) (taking $k = 1$ in (3.36)) and the fact that $u_\varepsilon = G_1(u_\varepsilon) + T_1(u_\varepsilon)$, we get the estimate (3.22).

Moreover (3.36) implies that

$$\|T_k(u_\varepsilon)\|_{W_0^{1,p}(\Omega)} \leq C_{13} k(k+1)^\gamma, \quad \forall n \geq 1.$$

This finishes the proof of Lemma 3.6. □

4. Proofs of the main results

We start by proving Theorem 1.2.

4.1. Proof of Theorem 1.2. As consequence of Lemma 3.3, the sequence $(u_\varepsilon)_n$ is bounded in $W_0^{1,p}(\Omega) \cap L^{(p-\theta)\frac{Nm}{N-pm}}(\Omega)$. Therefore, there exists a function $u \in W_0^{1,p}(\Omega)$ such that (up to a subsequence)

$$\begin{cases} u_\varepsilon \rightharpoonup u & \text{in } W_0^{1,p}(\Omega), \\ u_\varepsilon \to u & \text{a.e. in } \Omega. \end{cases} \tag{4.1}$$

Now, we will prove that

$$\nabla u_\varepsilon \to \nabla u \quad \text{a.e. in } \Omega. \tag{4.2}$$

Let $h, k > 0$. We use $T_h(u_\varepsilon - T_k(u))$ as a test function in (2.2), by hypothesis (1.4) and estimate (3.1), we get

$$\int_\Omega \frac{\alpha}{(d(x) + T_{\frac{1}{\varepsilon}}(u_\varepsilon))^\gamma} \widehat{a}(x, u_\varepsilon, \nabla u_\varepsilon) \nabla T_h(u_\varepsilon - T_k(u)) \leq 2h \int_\Omega (\lambda u_\varepsilon^r + f).$$

Recalling $r < (p-\theta)\frac{Nm}{N-pm}$, u_ε is bounded in $L^{(p-\theta)\frac{Nm}{N-pm}}(\Omega)$, we find that

$$\int_\Omega \frac{\alpha}{(d(x) + T_{\frac{1}{\varepsilon}}(u_\varepsilon))^\gamma} \widehat{a}(x, u_\varepsilon, \nabla u_\varepsilon) \nabla T_h(u_\varepsilon - T_k(u)) \leq 2Ch \tag{4.3}$$

This gives, by (1.4), (1.7), and (4.3),

$$
\begin{aligned}
&\int_{\{|u_\varepsilon - T_k(u)|\le h, |u|\le k\}} \frac{|\nabla(u_\varepsilon - T_k(u))|^p}{(\rho_2+u_\varepsilon)^\gamma} \\
&\le \int_{\{|u_\varepsilon - T_k(u)|\le h, |u|\le k\}} C_p \frac{(\widehat{a}(x,u_\varepsilon,\nabla u_\varepsilon) - \widehat{a}(x,u_\varepsilon,\nabla T_k(u)))\nabla T_h(u_\varepsilon - T_k(u))}{(\rho_2+u_\varepsilon)^\gamma} \\
&\le C_p\frac{Ch}{\alpha} - C_p\int_{\{|u_\varepsilon - T_k(u)|\le h, |u|\le k\}} \frac{\widehat{a}(x,u_\varepsilon,\nabla T_k(u))\nabla T_h(u_\varepsilon - u)}{(\rho_2+u_\varepsilon)^\gamma} \\
&= C_p\frac{Ch}{\alpha} - C_p\int_{\{|u_\varepsilon - T_k(u)|\le h, |u|\le k\}} \frac{\widehat{a}(x,T_{h+k}(u_\varepsilon),\nabla T_k(u))\nabla T_h(u_\varepsilon - u)}{(\rho_2+u_\varepsilon)^\gamma}.
\end{aligned}
\tag{4.4}
$$

Combining (1.6) and (4.1), we obtain

$$
\begin{cases}
\nabla T_h(u_\varepsilon - u) \rightharpoonup 0, & \text{in } L^p(\Omega), \\
\widehat{a}(x,T_{k+h}(u_\varepsilon),\nabla T_k(u)) \to \widehat{a}(x,u,\nabla T_k(u)), & \text{in } L^{p'}(\Omega).
\end{cases}
\tag{4.5}
$$

According (4.4) and (4.5), we have

$$
\limsup_{\varepsilon\to 0}\int_{\{|u_\varepsilon - T_k(u)|\le h, |u|\le k\}} |\nabla(u_\varepsilon - T_k(u))|^p \le Ch(\rho_2+k+h)^\gamma.
\tag{4.6}
$$

Let now τ be such that $1<\tau<p$. It is clear that

$$
\begin{aligned}
\int_\Omega |\nabla(u_\varepsilon - u)|^\tau \le &\int_{\{|u_\varepsilon - u|\le h, |u|\le k\}} |\nabla(u_\varepsilon - u)|^\tau \\
&+ \int_{\{|u_\varepsilon - u|\le h, |u|> k\}} |\nabla(u_\varepsilon - u)|^\tau \\
&+ \int_{\{|u_\varepsilon - u|> h\}} |\nabla(u_\varepsilon - u)|^\tau.
\end{aligned}
\tag{4.7}
$$

Since the sequence $u_\varepsilon - u$ is bounded in $W_0^{1,\tau}(\Omega)$ (since $\tau<p$), then using Hölder's inequality with exponent $\frac{p}{\tau}$ on the two last terms of right-hand side of (4.7), we obtain

$$
\begin{aligned}
\int_\Omega |\nabla(u_\varepsilon - u)|^\tau \le\ & C\left(\int_{\{|u_\varepsilon - u|\le h, |u|\le k\}} |\nabla(u_\varepsilon - u)|^p\right)^{\frac{\tau}{p}} \\
&+ C(\mu(\{|u|>k\}))^{1-\frac{\tau}{p}} \\
&+ C(\mu(\{|u_\varepsilon - u|>h\}))^{1-\frac{\tau}{p}}.
\end{aligned}
\tag{4.8}
$$

Thus, we deduce from (4.6) and (4.8), that

$$
\limsup_{h\to 0}\limsup_{\varepsilon\to 0}\int_\Omega |\nabla(u_\varepsilon - u)|^\tau \le C(\mu(\{|u|>k\}))^{1-\frac{\tau}{p}}, \quad \forall k>0.
$$

At the limit as $k\to+\infty$, $\mu(\{|u|>k\})$ converges to 0. Therefore (up to subsequences), $\nabla u_\varepsilon \to \nabla u$ a.e. in Ω.

Remark 4.1. The method employed to demonstrate equation (4.2) under the conditions of Theorem 1.2 is identical to the approach used in Theorems 1.3 through 1.5.

Next result shows that an approach allows to establish that the weak limit u of the sequence of approximate solutions u_ε is strictly positive.

Lemma 4.2. *Let $0 < \theta < 1$. Let u_ε and u be as in* (4.2)*. Then $u > 0$ in Ω.*

Proof. We define, for $t \geq 0$,

$$H_\varepsilon(t) = \int_0^t \frac{\tau(\rho_2 + T_{\frac{1}{\varepsilon}}(\tau))^\gamma}{\alpha(\tau+\varepsilon)^{\theta+1}} d\tau, \quad H_0(t) = \int_0^t \frac{(\rho_2+\tau)^\gamma}{\alpha \tau^\theta} d\tau.$$

Observe that H_i(with $i = \epsilon$ or 0) is well-defined, since $\theta < 1$. Let $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a positive function. We choose $e^{-\nu_2 H_\varepsilon(u_\varepsilon)}\phi$ as a test function in (2.2). Using hypothesis (1.2)-(1.5), we obtain

$$\begin{aligned}
&\int_\Omega a(x, T_{\frac{1}{\varepsilon}}(u_\varepsilon))\widehat{a}(x, u_\varepsilon, \nabla u_\varepsilon).e^{-\nu_2 H_\varepsilon(u_\varepsilon)}\nabla\phi - \int_\Omega \left(\frac{\lambda u_\varepsilon^r}{1+\varepsilon u_\varepsilon^r} + f_\varepsilon\right) e^{-\nu_2 H_\varepsilon(u_\varepsilon)}\phi \\
&\geq -\int_\Omega b(x)\frac{u_\varepsilon |\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\theta+1}} e^{-\nu_2 H_\varepsilon(u_\varepsilon)}\phi - \int_\Omega a(x, T_{\frac{1}{\varepsilon}}(u_\varepsilon))\widehat{a}(x, u_\varepsilon, \nabla u_\varepsilon).\nabla\big(e^{-\nu_2 H_\varepsilon(u_\varepsilon)}\big)\phi \\
&\geq -\int_\Omega \nu_2 \frac{u_\varepsilon |\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\theta+1}}\phi e^{-\nu_2 H_\varepsilon(u_\varepsilon)} + \int_\Omega \nu_2 \frac{u_\varepsilon |\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\theta+1}}\phi e^{-\nu_2 H_\varepsilon(u_\varepsilon)} \\
&\geq 0.
\end{aligned}$$

Since $f_\varepsilon \geq T_1(f), \forall \varepsilon < 1$, we get

$$\beta \int_\Omega \widehat{a}(x, u_\varepsilon, \nabla u_\varepsilon).\nabla\phi e^{-\nu_2 H_\varepsilon(u_\varepsilon)} \geq \int_\Omega T_1(f) e^{-\nu_2 H_\varepsilon(u_\varepsilon)}\phi. \tag{4.9}$$

We shall now introduce a function for $\lambda > 0$ as follows:

$$\psi_\lambda(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1, \\ -\frac{1}{\lambda}(t - 1 - \lambda) & \text{if } 1 \leq t < \lambda + 1, \\ 0 & \text{if } \lambda + 1 \leq t, \end{cases}$$

Let $\varphi \geq 0$ be in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, and choose $\psi_\lambda(u_\varepsilon)\varphi$ as test function in (4.9), Then we have

$$\begin{aligned}
&\int_\Omega T_1(f)\varphi\psi_\lambda(u_\varepsilon)e^{-\nu_2 H_\varepsilon(u_\varepsilon)}\chi_{\{0\leq u_\varepsilon \leq \lambda+1\}} \\
&\qquad \leq \beta \int_\Omega \widehat{a}(x, u_\varepsilon, \nabla u_\varepsilon).\nabla\varphi\psi_\lambda(u_\varepsilon)e^{-\nu_2 H_\varepsilon(u_\varepsilon)}\chi_{\{0\leq u_\varepsilon \leq \lambda+1\}}.
\end{aligned} \tag{4.10}$$

Then, letting $\lambda \to 0$, Lebesgue's Theorem yield

$$\psi_\lambda(u_\varepsilon)\chi_{\{0\leq u_\varepsilon\leq\lambda+1\}} \to \chi_{\{0\leq u_\varepsilon\leq 1\}} \quad \text{in } L^1(\Omega). \tag{4.11}$$

Equation (4.10) and (4.11) imply that

$$\begin{aligned}
&\beta \int_{\{0\leq u_\varepsilon\leq 1\}} \widehat{a}(x, T_1(u_\varepsilon), \nabla T_1(u_\varepsilon)).\nabla\varphi e^{-\nu_2 H_\varepsilon(T_1(u_\varepsilon))} \\
&\qquad \geq \int_{\{0\leq u_\varepsilon\leq 1\}} T_1(f)\varphi e^{-\nu_2 H_\varepsilon(T_1(u_\varepsilon))}.
\end{aligned} \tag{4.12}$$

According to (1.6),(4.1), and (4.2), we have

$$\widehat{a}(x, T_1(u_\varepsilon), \nabla T_1(u_\varepsilon)) \rightharpoonup \widehat{a}(x, T_1(u), \nabla T_1(u)) \quad \text{in } L^{p'}(\Omega). \tag{4.13}$$

Now we pass to the limit as $n \to +\infty$ in (4.11), we deduce from (4.13), that

$$\int_\Omega \widehat{a}(x, T_1(u), \nabla T_1(u)).\nabla\varphi e^{-\nu_2 H_0(T_1(u))}$$
$$\geq \frac{1}{\beta}\int_{\{0\leq u\leq 1\}} T_1(f)\varphi e^{-\nu_2 H_0(T_1(u))}, \tag{4.14}$$

Then, by density, for every nonnegative φ in $W_0^{1,p}(\Omega)$. Let us consider the function $\vartheta : [0, +\infty) \to [0, +\infty)$ defined by

$$\vartheta(t) = \int_0^t e^{-\nu_2 H_0(s)}ds,$$

we have that $v = \vartheta(T_1(u)) \in W_0^{1,p}(\Omega)$; furthermore, since

$$\Phi_0\left(T_1(u)\right) \geq \Phi_0(1) = e^{-\nu_2 H_0(1)} > 0,$$

the inequality (4.14) is equivalent to that

$$\int_\Omega M(x, v, \nabla v).\nabla\varphi \geq \frac{1}{\beta}\int_\Omega g(x)\varphi,$$

where

$$\begin{cases} M(x,s,\xi) = e^{-\nu_2 H_0(T_1(u))}\widehat{a}\left(x, sT_1(u)\left(\int_0^{T_1(u)} e^{-\nu_2 H_0(s)}ds\right)^{-1}, \frac{\xi}{e^{-\nu_2 H_0(T_1(u))}}\right), \\ g(x) = \frac{1}{\beta}T_1(f)e^{-\nu_2 H_0(1)}\chi_{\{0\leq u(x)\leq 1\}}, \quad v = \vartheta(T_1(u)). \end{cases}$$

Using the comparison principle in $W_0^{1,p}(\Omega)$ we get $v(x) \geq z(x)$, where z is the bounded weak solution of

$$\begin{cases} z \in W_0^{1,p}(\Omega), \\ -\mathrm{div}(M(x, z, \nabla z)) = g(x). \end{cases}$$

The fact that the function g is nonnegative and not identically zero, and the weak Harnack inequality (see [21]) gives $z > 0$ in Ω and thus $v > 0$. Since $T_1(u) \geq v$, we conclude that $T_1(u) > 0$ in Ω, that is $u > 0$ in Ω. □

Corollary 4.3. Let $0 < \theta < 1$ and u be the weak limit of the sequence u_ε. We have $\frac{|\nabla u|^p}{u^\theta} \in L^1(\Omega)$.

Proof. From (3.1), and (1.2), we obtain

$$\nu_1\int_\Omega \frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\theta+1}} \leq \lambda\int_\Omega u_\varepsilon^r + \int_\Omega f. \tag{4.15}$$

Taking limit as $\varepsilon \to 0$, by Fatou Lemma, as well as the weak convergence of u_ε to u in $H_0^1(\Omega)$, and the strict positivity of u, we deduce that

$$\nu_1\int_\Omega \frac{|\nabla u|^p}{u^\theta} \leq \lambda\int_\Omega u^r + \int_\Omega f. \tag{4.16}$$

This completes the proof. □

Our objective now is to demonstrate that u is a weak solution of problem (1.1). This is the purpose of the following lemma.

Lemma 4.4. *Let u be the weak limit of the sequence u_ε. Then u satisfies*

$$\int_\Omega a(x,u)\widehat{a}(x,u,\nabla u).\nabla\varphi + \int_\Omega \frac{b(x)|\nabla u|^p}{u^\theta}\varphi = \int_\Omega (\lambda u^r + f)\varphi, \tag{4.17}$$

for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Proof. We do as in the proof of Theorem 2.6 in [6]. For any $k > 0$, let us define

$$\hbar_k(s) = \begin{cases} 1 & \text{if } s \le k \\ k+1-s & \text{if } k < s \le k+1 \\ 0 & \text{if } s > k+1. \end{cases}$$

For some $0 \le \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, let us take the function

$$v_\varepsilon = e^{-\nu_2 \frac{H_\varepsilon(u_\varepsilon)}{\alpha}} e^{\nu_2 \frac{H_{1/j}(T_j(u))}{\alpha}} \hbar_k(u_\varepsilon)\varphi. \tag{4.18}$$

The function v_ε belongs also to $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, then it's a test function for (2.2), and using it we get

$$\begin{aligned}
&\int_\Omega a(x,T_{\frac{1}{\varepsilon}}(u_\varepsilon))\widehat{a}(x,u_\varepsilon,\nabla u_\varepsilon).\nabla\varphi e^{-\nu_2 \frac{H_\varepsilon(u_\varepsilon)}{\alpha}} e^{\nu_2 \frac{H_{1/j}(T_j(u))}{\alpha}} \hbar_k(u_\varepsilon) \\
&\quad + \frac{\nu_2}{\alpha}\int_\Omega \frac{a(x,T_{\frac{1}{\varepsilon}}(u_\varepsilon))\widehat{a}(x,u_\varepsilon,\nabla u_\varepsilon)\nabla T_j(u)}{(T_j(u)+1/j)^\theta(\rho_2+T_j(u))^{-\gamma}} e^{-\nu_2 \frac{H_\varepsilon(u_\varepsilon)}{\alpha}} e^{\nu_2 \frac{H_{1/j}(T_j(u))}{\alpha}} \hbar_k(u_\varepsilon)\varphi \\
&= \int_\Omega \left[\frac{\lambda u_\varepsilon^r}{1+\varepsilon u_\varepsilon^r} + f_\varepsilon\right] e^{-\nu_2 \frac{H_\varepsilon(u_\varepsilon)}{\alpha}} e^{\nu_2 \frac{H_{1/j}(T_j(u))}{\alpha}} \hbar_k(u_\varepsilon)\varphi \\
&\quad + \int_\Omega \frac{\nu_2}{\alpha}\left[\frac{a(x,T_{\frac{1}{\varepsilon}}(u_\varepsilon))\widehat{a}(x,u_\varepsilon,\nabla u_\varepsilon).\nabla u_\varepsilon}{(u_\varepsilon+\varepsilon)^\theta(\rho_2+u_\varepsilon)^{-\gamma}} - b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\theta+1}}\right] \\
&\quad \times e^{-\nu_2 \frac{H_\varepsilon(u_\varepsilon)}{\alpha}} e^{\nu_2 \frac{H_{1/j}(T_j(u))}{\alpha}} \hbar_k(u_\varepsilon)\varphi \\
&\quad + \int_{k<u_\varepsilon<k+1} a(x,T_{\frac{1}{\varepsilon}}(u_\varepsilon))\widehat{a}(x,u_\varepsilon,\nabla u_\varepsilon).\nabla u_\varepsilon e^{-\nu_2 \frac{H_\varepsilon(u_\varepsilon)}{\alpha}} e^{\nu_2 \frac{H_{1/j}(T_j(u))}{\alpha}} \hbar_k(u_\varepsilon)\varphi.
\end{aligned} \tag{4.19}$$

We begin by noting that due to (1.2)-(1.4), the function in the second integral on the right-hand side is non-negative. Thus, we may drop the last positive term. Utilizing Fatou's Lemma along with the weak convergence of u_ε to u in $W_0^{1,p}(\Omega)$ on the right-hand side and the weak convergence of $a(x,T_{\frac{1}{\varepsilon}}(u_\varepsilon))\widehat{a}(x,u_\varepsilon,\nabla u_\varepsilon)$ to $a(x,u)\widehat{a}(x,u,\nabla u)$ in $(L^{p'}(\Omega))^N$ on the left-hand side, we can proceed to take the limit as ε approaches zero in inequality (4.19), which yields:

$$\begin{aligned}
&\int_\Omega a(x,u)\widehat{a}(x,u,\nabla u).\nabla\varphi e^{-\nu_2\frac{H_0(u)}{\alpha}}e^{\nu_2\frac{H_{1/j}(T_j(u))}{\alpha}}\hbar_k(u)\\
&\quad+\frac{\nu_2}{\alpha}\int_\Omega\frac{a(x,u)\widehat{a}(x,u,\nabla u)\nabla T_j(u)}{(T_j(u)+1/j)^\theta(\rho_2+T_j(u))^{-\gamma}}e^{-\nu_2\frac{H_0(u)}{\alpha}}e^{\nu_2\frac{H_{1/j}(T_j(u))}{\alpha}}\hbar_k(u)\varphi\\
&\geq\int_\Omega\Big[\lambda u^r+f\Big]e^{-\nu_2\frac{H_0(u)}{\alpha}}e^{\nu_2\frac{H_{1/j}(T_j(u))}{\alpha}}\hbar_k(u_\varepsilon)\varphi\\
&\quad+\int_\Omega\frac{\nu_2}{\alpha}\left[\frac{a(x,u)\widehat{a}(x,u),\nabla u).\nabla u}{u^\theta(\rho_2+u)^{-\gamma}}-b(x)\frac{|\nabla u|^p}{u^\theta}\right]e^{-\nu_2\frac{H_0(u)}{\alpha}}e^{\nu_2\frac{H_{1/j}(T_j(u))}{\alpha}}\hbar_k(u)\varphi.
\end{aligned}\tag{4.20}$$

By (1.2), (4.16), notice that $e^{-\nu_2\frac{H_0(u)}{\alpha}}e^{\nu_2\frac{H_{1/j}(T_j(u))}{\alpha}}\leq 1$ (since $H_{1/j}(T_j(u))\leq H_{1/j}(u)\leq H_0(u)$), furthermore, we have $\hbar_k(u)=0$ for $u>k+1$. Thus, we can apply Lebesgue's convergence Theorem to take the limit as j approaches infinity in (4.20) and conclude that:

$$\begin{aligned}
&\int_\Omega a(x,u)\widehat{a}(x,u,\nabla u).\nabla\varphi\hbar_k(u)+\frac{\nu_2}{\alpha}\int_\Omega\frac{a(x,u)\widehat{a}(x,u,\nabla u)\nabla u}{u^\theta(\rho_2+u)^{-\gamma}}\hbar_k(u)\varphi\\
&\geq\int_\Omega\Big[\lambda u^r+f\Big]\hbar_k(u)\varphi\\
&\quad+\int_\Omega\frac{\nu_2}{\alpha}\left[\frac{a(x,u)\widehat{a}(x,u),\nabla u).\nabla u}{u^\theta(\rho_2+u)^{-\gamma}}-b(x)\frac{|\nabla u|^p}{u^\theta}\right]\hbar_k(u)\varphi.
\end{aligned}\tag{4.21}$$

Then, since $\frac{a(x,u)\widehat{a}(x,u),\nabla u).\nabla u}{u^\theta(\rho_2+u)^{-\gamma}}\hbar_k(u)$ belongs to $L^1(\Omega)$ (by (1.3), (1.5), (4.16), and the fact that $\hbar_k(u)=0$, when $u>k+1$), we have

$$\begin{aligned}
&\int_\Omega a(x,u)\widehat{a}(x,u,\nabla u).\nabla\varphi\hbar_k(u)+\int_\Omega b(x)\frac{|\nabla u|^p}{u^\theta}\hbar_k(u)\varphi\\
&\quad\geq\int_\Omega\Big[\lambda u^r+f\Big]\hbar_k(u)\varphi
\end{aligned}\tag{4.22}$$

As we allow k to approach infinity (noting that $\hbar_k(u)$ approaches 1), we arrive at:

$$\int_\Omega a(x,u)\widehat{a}(x,u,\nabla u).\nabla\varphi+\int_\Omega b(x)\frac{|\nabla u|^p}{u^\theta}\varphi\geq\int_\Omega\Big[\lambda u^r+f\Big]\varphi.\tag{4.23}$$

To establish the reverse inequality, we select a non-negative test function $\varphi\in W_0^{1,p}(\Omega)$ for (2.2), which yields:

$$\begin{aligned}
&\int_\Omega a(x,T_{\frac{1}{\varepsilon}}(u_\varepsilon))\widehat{a}(x,u_\varepsilon,\nabla u_\varepsilon).\nabla\phi+\int_\Omega b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^\theta}\phi\\
&\quad\leq\int_\Omega\Big[\lambda u_\varepsilon^r+f_\varepsilon\Big]\varphi.
\end{aligned}\tag{4.24}$$

By utilizing the weak convergence of the sequence and taking the limit in (4.24), we have $a(x,T_{\frac{1}{\varepsilon}}(u_\varepsilon))\widehat{a}(x,u_\varepsilon,\nabla u_\varepsilon)$ to $a(x,u)\widehat{a}(x,u,\nabla u)$ in $L^{p'}(\Omega)$, Fatou's Lemma, and the strong convergence of u_ε in $L^r(\Omega)$ (due to the fact that u_ε is bounded in $L^{(p-\theta)\frac{Nm}{N-pm}}(\Omega)$

and $r < (p-\theta)\frac{Nm}{N-pm}$), it follows that

$$\int_\Omega a(x,u)\widehat{a}(x,u,\nabla u).\nabla\varphi + \int_\Omega b(x)\frac{|\nabla u|^p}{u^\theta}\varphi \leq \int_\Omega \Big[\lambda u^r + f\Big]\phi. \tag{4.25}$$

Combining (4.23) and (4.25), we deduce that

$$\int_\Omega a(x,u)\widehat{a}(x,u,\nabla u).\nabla\varphi + \int_\Omega b(x)\frac{|\nabla u|^p}{u^\theta}\varphi = \int_\Omega \Big[\lambda u^r + f\Big]\varphi,$$

this holds for every non-negative test function $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. Consequently, we have established that (1.8) is valid for all non-negative test functions. By selecting φ^+ and φ^- and summing the two resulting equalities, we can extend the validity of (1.8) to general test functions ϕ. □

4.2. Proof of Theorem 1.3. According to Lemma 3.4, the approximate solutions u_ε is bounded in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. Thus, there exists a function $u \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ such that u_ε converges weakly in $W_0^{1,p}(\Omega)$ to u (up to subsequences, with $u > 0$ in Ω and $\frac{|\nabla u|^p}{u^\theta} \in L^1(\Omega)$, as shown by Lemma 4.2 and Corollary 4.3. Moreover, by (4.2), we know that ∇u_ε converges almost everywhere to ∇u in Ω. To prove that u is a weak solution of problem (1.1), we use a similar approach to that of Lemma 4.4. Specifically, we test (2.2) with the function $e^{-\nu_2\frac{H_\varepsilon(u_\varepsilon)}{\alpha}}e^{B\frac{H_\varepsilon(u_\varepsilon)}{\alpha}}\varphi$ instead of the test function given in (4.18), since in this case the function u is bounded.

4.3. Proof of Theorem 1.4. Using Lemma 3.5, we can show that u_ε and $T_k(u_\varepsilon)$ are bounded in $W_0^{1,\sigma}(\Omega)\cap L^\infty(\Omega)$ and $W_0^{1,p}(\Omega)$, respectively, for every $k > 0$. Therefore, there exists a function $u \in W_0^{1,\sigma}(\Omega)\cap L^\infty(\Omega)$ such that, up to subsequences, u_ε and $T_k(u_\varepsilon)$ converge weakly to u and $T_k(u)$, respectively, in $W_0^{1,\sigma}(\Omega)$ and $W_0^{1,p}(\Omega)$, and almost everywhere in Ω. By repeating the argument from the proof of Lemma 4.2, we can deduce that $u > 0$ in Ω, and the Corollary 4.3 implies that $\frac{|\nabla u|^p}{u^\theta}$ belongs to $L^1(\Omega)$. The proof of (4.2) still holds, and therefore, ∇u_ε converges almost everywhere to ∇u in Ω. To complete the proof of Theorem 1.4, it is left to show that u is a distributional solution of the problem (1.1). This is the aim of the following lemma.

Lemma 4.5. *Let u be the weak limit of the sequence u_ε. Then u satisfies*

$$\int_\Omega a(x,u)\widehat{a}(x,u,\nabla u).\nabla\varphi + \int_\Omega \frac{b(x)|\nabla u|^p}{u^\theta}\varphi = \int_\Omega (\lambda u^r + f)\varphi, \quad \forall\varphi \in C_0^1(\Omega).$$

Proof. The proof of Lemma 4.5 is also similar to that of Lemma 4.4. □

4.4. Proof of Theorem 1.5. Lemma 3.6 asserts that the sequence u_ϵ is bounded in the space $W_0^{1,\eta}(\Omega)$ and the sequence $T_k(u_\epsilon)$ is bounded in the space $W_0^{1,p}(\Omega)$ for every $k > 0$. Thus, there exists a function $u \in W_0^{1,\eta}(\Omega)$ such that, after passing to a subsequence, u_ϵ converges weakly in $W_0^{1,\eta}(\Omega)$ and almost everywhere in Ω to u, and $T_k(u_\epsilon)$ converges weakly in $W_0^{1,p}(\Omega)$ and almost everywhere in Ω to $T_k(u)$ for every $k > 0$. Moreover, ∇u_ϵ converges almost everywhere in Ω to ∇u using the same technique as in the proof of (4.2). The method used in the proof of Lemma 4.2 can still be applied to show that u is positive in Ω. By Corollary 4.3, we have $\frac{|\nabla u|^p}{u^\theta} \in L^1(\Omega)$. As $T_k(u_\epsilon)$ weakly converges in $W_0^{1,p}(\Omega)$ and strongly converges to u in $L^r(\Omega)$ (because the sequence u_ϵ is bounded in $W_0^{1,\eta}(\Omega)$ and $r < p - \theta < \eta$), we can apply the same limit argument as in the proof of Theorem 1.4 to conclude that u is a distributional solution of the problem (1.1).

Disclosure statement

No potential conflict of interest was reported by the author(s).

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