Poincare Journal of Analysis & Applications
Vol. 10, No. 1 (2023), 105-117
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DOI: 10.46753/pjaa.2023.v010i01.008

SOME PICARD TYPE THEOREMS AND CORRESPONDING NORMALITY CRITERIA IN SEVERAL COMPLEX VARIABLES

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Date of Receiving	:	07. 12. 2022
Date of Revision	:	02. 03. 2023
Date of Acceptance	:	18. 04. 2023

Abstract. In this paper, besides a counterexample to Bloch's principle, normality criteria leading to counterexamples to the converse of Bloch's principle in several complex variables are proved. Some Picard-type theorems and their corresponding normality criteria in $\mathbb{C}^n$ are also obtained.

1. Introduction and Auxiliary Results

Let D to be a domain in $\mathbb{C}^n$ and $\mathcal{F}$ to be a family of holomorphic functions in D. $\mathcal{F}$ is said to be normal in D if each sequence in $\mathcal{F}$ contains a subsequence that converges locally uniformly in D. The aim of this paper is to obtain normality criteria leading to counterexamples to the converse of Bloch's principle in several complex variables. Bloch's principle in one complex variable is extensively studied and the reader may refer to [1, 2, 3, 4, 9, 10, 14, 15, 16], in order. Bloch's principle states that a family $\mathcal{F}$ of holomorphic functions of several complex variables in a domain $D \subset \mathbb{C}^n$ satisfying a certain property $\mathbf{P}$ in D is likely to be normal in D if any entire function possessing the property $\mathbf{P}$ in $\mathbb{C}^n$ reduces to a constant. In one complex variable neither Bloch's principle nor its converse holds in general, for example one may refer to [15, 9]. For the construction of counterexamples to the converse, one needs to find a normality criterion with a certain property and then find out a non-constant entire function satisfying this property in $\mathbb{C}^n$; this sounds interesting particularly from the experience of one complex variable case and this is what we have explored in the present paper in several complex variables case too.

We shall denote by $\mathcal{H}(D)$, the class of holomorphic functions $f : D \to \mathbb{C}$, where $D \subset \mathbb{C}^n$ is a domain, and the open unit ball in $\mathbb{C}^n$ shall be denoted by $\mathbb{D}^n := \{z \in \mathbb{C}^n : \|z\| < 1\}$.

2010 *Mathematics Subject Classification.* 32A19, 32A10.
Key words and phrases. Normal families, Picard type theorems, Bloch's principle.
Communicated by. Gopal Datt
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Let $v = (v_1, v_2, \ldots, v_n) \in \mathbb{C}^n$. Then for every $\psi \in \mathcal{C}^2(D)$, at each point $z \in D$ we define a Hermitian form

$$L_z(\psi,\ v) := \sum_{k,l=1}^{n} \frac{\partial^2(\psi)}{\partial z_k \partial \bar{z}_l}(z) v_k \bar{v}_l \tag{1.1}$$

and is called the *Levi form* of the function ψ at z.
The spherical derivative of $f \in \mathcal{H}(D)$ (see [6]) is defined as

$$f^{\#}(z) := \sup_{\|v\|=1} \sqrt{L_z(\log(1+|f|^2),\ v)}. \tag{1.2}$$

Since $L_z(\log(1+|f|^2),\ v) \geq 0,\ f^{\#}(z)$ given by (1.2) is well defined and for $n = 1$, (1.2) reduces to

$$f^{\#}(z) := \frac{|f'(z)|}{1+|f(z)|^2}$$

which is the spherical derivative of a holomorphic function of one complex variable. Thus (1.2) gives the natural extension of the spherical derivative to $\mathbb{C}^n$.
Further, from (1.2), one easily finds that

$$f^{\sharp}(z) = \frac{\sup\limits_{\|v\|=1} |(Df(z), v)|}{1+|f(z)|^2}, \tag{1.3}$$

where

$$(Df(z), v) = \sum_{j=1}^{n} \frac{\partial f(z)}{\partial z_j}.v_j$$

Following known results shall be required to accomplish the proofs of main results of this paper, spread over various sections:

Marty's Theorem in $\mathbb{C}^n$ ([6]): *A family $\mathcal{F} \subset \mathcal{H}(D)$ is normal in D if and only if for each compact set $K \subset D$ there exists a constant $M = M(K)$ such that $f^{\#}(z) \leq M$ for all $f \in \mathcal{F},\ z \in K$.*

Zalcman's Lemma in $\mathbb{C}^n$ ([7]): *A family $\mathcal{F} \subseteq \mathcal{H}(D)$ is not normal at a point $z_0 \in D$ if and only if for each $\alpha \in (-1, \infty)$ there exist sequences $\{f_j\} \subset \mathcal{F},\ \{z_j\} \subset D : z_j \to z_0,$ and $\{r_j\} \subset (0, 1] : r_j \to 0$ such that the zoomed sequence*

$$g_j(z) := r_j^{\alpha} f_j(z_j + r_j z)$$

converges locally uniformly to a non-constant entire function g in $\mathbb{C}^n$ satisfying $g^{\sharp}(z) \leq g^{\sharp}(0) = 1$.

Definition 1.1. Let $f \in \mathcal{H}(\mathbb{C}^n)$ and $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$. The total derivative of f, which we shall denote by $D_0 f$, is defined as

$$D_0 f(z) = \sum_{j=1}^{n} z_j f_{z_j},$$

where f_{z_j} is the partial derivative of f with respect to z_j.

Liu and Cao [11] obtained an extension of Zalcman's lemma concerning the total derivative in several complex variables as follows:

Lemma 1.2. *Let $\mathcal{F} \subset \mathcal{H}(\mathbb{D}^n)$ and suppose that $f(z) \neq 0$ for every $f \in \mathcal{F}$ and for all $z \in \Delta_\delta(0)$ for some $\delta \in (0,1)$ and that $f(z) = 0 \implies |D_0 f(z)| \leq A$ for some $A > 0$. If $\mathcal{F}$ is not normal in $\mathbb{D}^n$, then for all $k \in (-1,1]$, there exist a real number $r \in (0,1]$, and sequences $\{z_j\} \subseteq \mathbb{D}^n : 0 < \|z_j\| < r$, $\{f_j\} \subseteq \mathcal{F}$, and $\{\rho_j\} \subset (0,1] : \rho_j \to 0$ such that*

$$g_j(\zeta) = \rho_j^{-k} f_j(z_j e^{\rho_j \zeta}), \ (\zeta \in \mathbb{C})$$

converges locally uniformly to a non-constant entire function g in $\mathbb{C}$ satisfying $g^\sharp(\zeta) \leq g^\sharp(0) = A + 1$.

Definition 1.3. For $f \in \mathcal{H}(\mathbb{C}^n)$ and $a \in \mathbb{C}^n$, we can write $f(z) = \sum\limits_{i=0}^{\infty} P_i(z-a)$, where $P_i(z)$ is either a homogeneous polynomial of degree i or identically zero. The zero multiplicity of f at a is defined as $V_f(a) = min\{i : P_i \neq 0\}$.

We have a kind of variant of Lemma 1.2:

Lemma 1.4. *Let $\mathcal{F} \subset \mathcal{H}(\mathbb{D}^n)$, $b \in \mathbb{C}$ and $f \in \mathcal{F}$. suppose that for each $a \in f^{-1}(\{b\}) \cap \mathbb{D}^n$, zero multiplicity of f at a is at least k. Then if $\mathcal{F}$ is not normal, then for each $\alpha \in (0,k)$, there exist a real number $r \in (0,1)$ and sequences $\{z_j\} \subset \mathbb{D}^n : \|z_j\| < r$, $\{f_j\} \subset \mathcal{F}$, and $\{\rho_j\} \subset (0,1] : \rho_j \to 0$ such that*

$$g_j(\zeta) = \rho_j^{-\alpha} f_j(z_j + \rho_j \zeta)$$

converges locally uniformly to a non-constant entire function g in $\mathbb{C}^n$ satisfying $g^\sharp(\zeta) \leq g^\sharp(0) = 1$.

The proof of Lemma 1.4 is exactly on the lines of proof of Theorem 5 in [5] with Lemma 1 there replaced by

Lemma 1.5. *Let $f \in \mathcal{H}(\mathbb{D}^n)$ and $b \in \mathbb{C}$. Suppose that for each $a \in f^{-1}(\{b\}) \cap \mathbb{D}^n$, the zero multiplicity of f at a be at least k, and $\alpha \in (-1,k)$. Let $\Omega := \{z \in \mathbb{C}^n : \|z\| < r < 1\} \times (0,1]$ and define a function $F : \Omega \to \mathbb{R}$ as*

$$F(z,t) = \frac{\left(1 - \frac{\|z\|^2}{r^2}\right)^{1+\alpha} t^{1+\alpha} f^{\#}(z) \left(1 + |f(z)|^2\right)}{\left(1 - \frac{\|z\|^2}{r^2}\right)^{2\alpha} t^{2\alpha} + |f(z)|^2}.$$

If $F(z,1) > 1$ for some $z : \|z\| < r$, then there exist $z_0 \in \mathbb{C}^n : \|z_0\| < r < 1$ and $t \in (0,1)$ such that

$$\sup_{\|z\|<r} F(z,t_0) = F(z_0,t_0) = 1.$$

Since the proof of Lemma 1.5 is a modification of the proof of Lemma 1 in [5], here we only need to establish that

$$\lim_{\left(1 - \frac{\|z\|^2}{r^2}\right) t \to 0} F(z,t) = 0, \tag{1.4}$$

as a significant amount of modifications is required to establish this limit.

We may assume that $z_j \to z_0$, where $\|z_0\| \leq r$.
If $f(z_0) \neq 0$, then

$$\lim_{j\to\infty} F(z_j, t_j) \leq 0.$$

Suppose $f(z_0) = 0$. Then $f(z) = \sum_{m=s}^{\infty} P_m(z - z_0),\ s \geq k$, where $P_m(z)$ is either a homogeneous polynomial of degree m or identically zero.

$$\begin{aligned} |f(z_j)| &= |p_s(z_j - z_0) + p_{s+1}(z_j - z_0) + \ldots| \\ &= \|z_j - z_0\|^s \left[\frac{|p_s(z_j - z_0) + p_{s+1}(z_j - z_0) + \ldots|}{\|z_j - z_0\|^s}\right] \\ &= |a_s|\|z_j - z_0\|^s[1 + o(1)]. \end{aligned}$$

Similarly, we find that

$$\left|\frac{\partial}{\partial z_i} f(z_j)\right| = |b_{s_i}|\|z_j - z_0\|^{s-1}[1 + o(1)].$$

We may assume that

$$\lim_{j\to\infty} \frac{\left(1 - \frac{\|z_j\|^2}{r^2}\right) t_j}{\|z_j - z_0\|} = B$$

exists. If $B \geq 1$, then

$$\begin{aligned} \lim_{j\to\infty} F(z_j, t_j) &\leq \lim_{j\to\infty} \left(1 - \frac{\|z_j\|^2}{r^2}\right)^{1-\alpha} t_j^{1-\alpha} f^{\#}(z_j)\left(1 + |f(z_j)|^2\right) \\ &\leq \frac{\|\left(\frac{\partial}{\partial z_1} f(z_j), \ldots, \frac{\partial}{\partial z_n} f(z_j)\right)\|}{\left(1 - \frac{\|z_j\|^2}{r^2}\right)^{\alpha-1} t_j^{\alpha-1}} \\ &\leq \lim_{j\to\infty} \frac{|\frac{\partial}{\partial z_1} f(z_j)| + \ldots + |\frac{\partial}{\partial z_n} f(z_j)|}{\left(1 - \frac{\|z_j\|^2}{r^2}\right)^{\alpha-1} t_j^{\alpha-1}} \\ &\leq \lim_{j\to\infty} \frac{|b_{s_1}|\|z_j - z_0\|^{s-1} + \ldots |b_{s_n}|\|z_j - z_0\|^{s-1}}{\left(1 - \frac{\|z_j\|^2}{r^2}\right)^{\alpha-1} t_j^{\alpha-1}} \\ &\leq \lim_{j\to\infty} \frac{n \max_{i=1}^n |b_{s_i}|\|z_j - z_0\|^{s-1}}{\left(1 - \frac{\|z_j\|^2}{r^2}\right)^{\alpha-1} t_j^{\alpha-1}} \\ &= \lim_{j\to\infty} \frac{n \max_{i=1}^n |b_{s_i}|\|z_j - z_0\|^{s-\alpha}\|z_j - z_0\|^{\alpha-1}}{\left(1 - \frac{\|z_j\|^2}{r^2}\right)^{\alpha-1} t_j^{\alpha-1}} \\ &= 0 \end{aligned}$$

If $B \leq 1$, then

$$F(z_j, t_j) = \frac{\left(1 - \frac{\|z_j\|^2}{r^2}\right)^{\alpha} t_j^{\alpha}|f(z_j)|}{\left(1 - \frac{\|z_j\|^2}{r^2}\right)^{2\alpha} t_j^{2\alpha} + |f(z_j)|^2} \left(1 - \frac{\|z_j\|^2}{r^2}\right) t_j \frac{f^{\#}(z_j)\left(1 + |f(z_j)|^2\right)}{|f(z_j)|}.$$

The first factor on the right hand is obviously bounded by $\frac{1}{2}$.
Moreover, if $B \neq 0$, then we have

$$\begin{aligned}
\lim_{j\to\infty} \frac{\left(1-\frac{\|z_j\|^2}{r^2}\right)^{\alpha} t_j^{\alpha}|f(z_j)|}{\left(1-\frac{\|z_j\|^2}{r^2}\right)^{2\alpha} t_j^{2\alpha}+|f(z_j)|^2} &\leq \lim_{j\to\infty} \frac{B^{\alpha}\|z_j-z_0\|^{\alpha}|a_s|\|z_j-z_0\|^{s}}{B^{2\alpha}\|z_j-z_0\|^{2\alpha}} \\
&= \lim_{j\to\infty} \frac{|a_s|\|z_j-z_0\|^{s-\alpha}}{B^{\alpha}} \\
&= 0
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\lim_{j\to\infty} \left(1-\frac{\|z_j\|^2}{r^2}\right) t_j \quad & \frac{f^{\#}(z_j)\left(1+|f(z_j)|^2\right)}{|f(z_j)|} \\
\leq \quad & \left(1-\frac{\|z_j\|^2}{r^2}\right) t_j . \frac{n\max_{i=1}^{n}|b_{s_i}|\|z_j-z_0\|^{s-1}}{|a_s|\|z_j-z_0\|^{s}} \\
= \quad & \lim_{j\to\infty} \left(1-\frac{\|z_j\|^2}{r^2}\right) t_j . \frac{n\max_{i=1}^{n}|b_{s_i}|}{|a_s|\|z_j-z_0\|} \\
= \quad & B\frac{n\max_{i=1}^{n}|b_{s_i}|}{|a_s|}
\end{aligned}$$

It follows that $\lim\limits_{j\to\infty} F(z_j,t_j)=0$.

Let f be meromorphic function in $\mathbb{C}$ and $a \in \mathbb{C}_\infty$. Then a is called totally ramified value of f if $f-a$ has no simple zeros. Following result known as *Nevanlinna's Theorem* (see [1]) plays a crucial role in the proofs of Theorem 2.3, Theorem 3.3 and Theorem 3.4 to follow:

Theorem 1.6. *Let f be a non-constant meromorphic function $a_1,...,a_q \in \mathbb{C}_\infty$ and $m_1,...,m_q \in \mathbb{N}$. Suppose that all a_j-points of f have multiplicity at least m_j, for $j=1,...,q$. Then*

$$\sum_{j=1}^{q}\left(1-\frac{1}{m_j}\right) \leq 2.$$

If f omits the value a_j, then $m_j=\infty$.

2. Counterexamples to Bloch's Principle and its Converse

2.1. A Counterexample to Bloch's Principle. If $D \subset \mathbb{C}^n$ and $f \in \mathcal{H}(D)$ satisfies a certain property $\mathbf{P}$ in D, then we write it as $(f,D) \in \mathbf{P}$. Recall that Bloch's principle states that a family $\mathcal{F} \subset \mathcal{H}(D) : (f,D) \in \mathbf{P}$ is likely to be normal in D if $(f,\mathbb{C}^n) \in \mathbf{P}$ implies that f is a constant function. As in the case of one complex variable, Bloch's principle fails to hold in $\mathbb{C}^n$ as well.
Let $\mathbf{P}$ be the property of holomorphic function f in D defined as

$$P(f)(z)=\prod_{i=1}^{n}\left(\frac{\partial f(z)}{\partial z_i}-a\right)\left(\frac{\partial f(z)}{\partial z_i}-b\right)\left(\frac{\partial f(z)}{\partial z_i}-f(z)\right)$$

omits the value zero.

Proposition 2.1. If $f \in \mathcal{H}(\mathbb{C}^n) : (f, \mathbb{C}^n) \in \mathbf{P}$, then $f \equiv$ constant.

Proof. Suppose $f \in \mathcal{H}(\mathbb{C}^n)$ possesses the property $\mathbf{P}$. Let

$$w = (a_1, \ldots, a_n),\ w' = (b_1, \ldots, b_n) \in \mathbb{C}^n$$

and define

$$h_i(z_i) = f(b_1, \ldots, b_{i-1}, z_i, a_{i+1}, \ldots, a_n),\ i = 1, 2, \ldots n.$$

Then $\frac{d}{dz_i} h_i(z_i)$ omits the values a and b and so $\frac{d}{dz_i} h_i(z_i) = c_i$, a constant. This implies that $h_i(z_i) = c_i z_i + d_i$, but $\frac{d}{dz_i} h_i(z_i) - h_i(z_i)$ omits zero and so $c_i - c_i z_i - d_i \neq 0$ which implies $c_i = 0$. This shows that each $h_i(z_i)$ is constant and hence f is constant. □

Counterexample 2.2. Following counterexample to the Bloch's principle is constructed from the ideas of Rubel[15].

Take r sufficiently small so that

$$\{(z_1, \ldots, z_n) : z_1 + \ldots + z_n = 1\} \cap D^n(0, r) = \emptyset.$$

Consider

$$\mathcal{F} = \{f_m(z_1, \ldots, z_n) = m(z_1 + \ldots + z_n) : (z_1, \ldots, z_n) \in D^n(0, r), m \geq 3\}.$$

Then

$$\frac{\partial f(z)}{\partial z_i} = m \neq 1, 2.$$

Also,

$$\frac{\partial f(z)}{\partial z_i} - f(z) = m(1 - (z_1 + \ldots + z_n)) \neq 0\ \forall z = (z_1, \ldots, z_n) \in D^n(0, r).$$

Thus each f_m satisfies the property $\mathbf{P}$ in $D^n(0, r)$ with $a = 1$ and $b = 2$ but $\mathcal{F}$ is not normal in $D^n(0, r)$ showing that Bloch's principle does not hold in general.

2.2. Counterexamples to the Converse of Bloch's Principle. The converse of Bloch's principle precisely states that if

$$\mathcal{F} := \{f \in \mathcal{H}(D) : (f, D) \in \mathbf{P}\}$$

is normal in D, then any $f \in \mathcal{H}(\mathbb{C}^n) : (f, \mathbb{C}^n) \in \mathbf{P}$ reduces to a constant.
Let the property $\mathbf{P}$ satisfied by a function $f \in \mathcal{H}(D)$ be defined as follows:

(i) there exist three distinct complex numbers a_1, a_2, a_3 and three complex numbers b_1, b_2, b_3 satisfying $f(z) = a_i \implies D_0 f(z) = b_i$;
(ii) $f(z) \neq 0\ \forall\ z \in \Delta_\delta(0)$ for some $\delta > 0$.

This property leads to the following normality criterion:

Theorem 2.3. *A subfamily $\mathcal{F}$ of $\mathcal{H}(D)$ is normal in D if $(f, D) \in \mathbf{P}$ for all $f \in \mathcal{F}$.*

Proof. Since normality is a local property, we may assume D to be the open unit ball $\mathbb{D}^n := \{z \in \mathbb{C}^n : \|z\| < 1\}$. Suppose $\mathcal{F}$ is not normal in $\mathbb{D}^n$. Then by Lemma 1.2, there exist $r \in (0,1),\ z_j \in \mathbb{D}^n : 0 < \|z_j\| < r,\ \{f_j\} \subseteq \mathcal{F},\ \{\rho_j\} \subset (0,1] : \rho_j \to 0$ such that $g_j(\zeta) = f_j(z_j e^{\rho_j \zeta})$ converges locally uniformly to non-constant entire function g in $\mathbb{C}$. Let $g(\zeta_0) = a_i$. Then by Hurwitz's theorem there exists a sequence $\{\zeta_j\}$ converging to ζ_0 such that for sufficiently large $j,\ g_j(\zeta_j) = a_i$ which implies $f_j(z_j e^{\rho_j \zeta_j}) = a_i$ and so by given hypothesis $D_0 f_j(z_j e^{\rho_j \zeta_j}) = b_i$. Therefore,

$$g'(\zeta_0) = \lim_{j\to\infty} g_j'(\zeta_j) = \lim_{j\to\infty} \rho_j D_0 f_j(z_j e^{\rho_j \zeta_j}) = 0.$$

Thus each a_i-points of g has multiplicity at least two. Let m_i $(i = 1,2,3)$ be the multiplicities of a_i-points of g. Then $m_i \geq 2$ for $i = 1,2,3$. Let $a_4 = \infty$. Then g being an entire function omits a_4 and hence $m_4 = \infty$. By simple calculation, we find that

$$\sum_{i=1}^{4}\left(1 - \frac{1}{m_i}\right) > 2,$$

which contradicts Theorem 1.6. ☐

Theorem 2.3 leads to a counterexample to the converse of Bloch's principle.

Counterexample 2.4. Consider the function

$$f(z_1, z_2, \ldots, z_n) = z_1 + z_2 + \ldots + z_n + 4,$$

$a_1 = 1,\ a_2 = 2,\ a_3 = 3$ and $b_1 = -3,\ b_2 = -2,\ b_3 = -1$.
Then $f(z) = a_j \implies D_0 f(z) = b_j$ $(j = 1,2,3)$. Also $f(z) \neq 0\ \forall\ z \in \Delta_\delta(0)$ for some $\delta > 0$. Thus there exists a non-constant entire function in $\mathbb{C}^n$ satisfying the property $\mathbf{P}$ and by Theorem 2.3, $\mathcal{F} := \{f \in \mathcal{H}(D) : (f,D) \in \mathbf{P}\}$ is normal in D.

The following weak version of Lappan's normality criterion in several complex variables (see [5],Theorem 1) also leads to a counterexample to the converse of Bloch's principle:

Theorem 2.5. *Let $D \subseteq \mathbb{C}^n$ and $\mathcal{F} \subset \mathcal{H}(D)$. Let E be a subset of $\mathbb{C}$ containing at least three points such that for each compact set $K \subset D$, there exists a positive constant $M = M(K)$ for which*

$$\sup_{\|v\|=1} |(Df(z), v)| \leq M \ \textit{whenever}\ f(z) \in E,\ z \in K,\ f \in \mathcal{F}. \tag{2.1}$$

Then $\mathcal{F}$ is normal in D.

The proof of Theorem 2.5 is obtained on the lines of the proof of Theorem 1 in [5] hence omitted.

Counterexample 2.6. Let $D \subset \mathbb{C}^n$ be a domain. For $f \in \mathcal{H}(D)$, $(f,D) \in \mathbf{P}$ if there exists a set $E \subset \mathbb{C}$ containing at least three points such that for each compact subset $K \subset D$, there exists a positive constant $M = M(K)$ for which

$$\sup_{\|v\|=1} |(Df(z), v)| \leq M \ \text{whenever}\ f(z) \in E,\ z \in K. \tag{2.2}$$

Consider the function

$$f(z_1, z_2, \ldots, z_n) = z_1 + z_2 + \ldots + z_n.$$

Then

$$\sup_{\|v\|=1} |(Df(z), v)| \leq \sup_{\|v\|=1} [|v_1| + |v_2| + \cdots + |v_n|] \leq n.$$

Thus f is a non-constant entire function in $\mathbb{C}^n$ satisfying the property $\mathbf{P}$ in $\mathbb{C}^n$ and by Theorem 2.5, $\mathcal{F} := \{f \in \mathcal{H}(D) : (f, D) \in \mathbf{P}\}$ is normal in D. This is a another counterexample to the converse of Bloch's principle in several complex variables.

3. Picard-type Theorems and their Corresponding Normality Criteria

By Bloch's principle one can expect to have a normality criterion corresponding to every Picard-type theorem. Here we prove a couple of Picard-type theorems and obtain the corresponding normality criteria.

In the discussions to follow, we shall denote the operator $\left(\frac{\partial^k}{\partial z_1^k}, \frac{\partial^k}{\partial z_2^k}, \ldots, \frac{\partial^k}{\partial z_n^k}\right)$ by D^k.

Theorem 3.1. *Let $f \in \mathcal{H}(\mathbb{C}^n)$ and k be a positive integer. If $f(z) \neq 0$, $(D^k f(z), v) \neq 1$ for all $v = (v_1, ..., v_n) : \|v\| = 1$, then f is constant.*

Proof. Suppose f is non-constant. Let

$$w = (a_1, \ldots, a_n),\ w' = (b_1, \ldots, b_n) \in \mathbb{C}^n$$

and define

$$h_i(z_i) := f(b_1, \ldots, b_{i-1}, z_i, a_{i+1}, \ldots, a_n),\ i = 1, 2, \ldots n.$$

By given hypothesis, it follows that $h_i(z) \neq 0$ for all $z \in \mathbb{C}$.

Claim: $\frac{\partial^k}{\partial z_i^k} f(b_1, \ldots, b_{i-1}, z_i, a_{i+1}, \ldots, a_n) \not\equiv 1$

Suppose

$$\frac{\partial^k}{\partial z_i^k} f(b_1, \ldots, b_{i-1}, z_i, a_{i+1}, \ldots, a_n) \equiv 1.$$

Then $h_i(z)$ is a polynomial of degree k and so $h_i(a) = 0$ for some $a \in \mathbb{C}$, a contradiction and this establishes the claim.

Since $h_i(z) \neq 0$ for all $z \in \mathbb{C}$. So, $\frac{\partial^k}{\partial z_i^k} h_i(a) = 1$ for some $a \in \mathbb{C}$ which implies

$$\frac{\partial^k}{\partial z_i^k} f(b_1, \ldots, b_{i-1}, a, a_{i+1}, \ldots, a_n) = 1.$$

which implies that

$$\sum_{j=1}^{n} \frac{\partial^k}{\partial z_j^k} f(b_1, \ldots, b_{i-1}, a, a_{i+1}, \ldots, a_n) e_j = 1$$

where $e_j = \begin{cases} 1, & \text{for } j = i \\ 0, & \text{for } j \neq i. \end{cases}$

That is,

$$(D^k f(b_1, \ldots, b_{i-1}, a, a_{i+1}, \ldots, a_n), v) = 1, \text{ where } v = (0, \ldots, 1, \ldots, 0),$$

which is a contradiction. □

The corresponding normality criterion of Theorem 3.1 is

Theorem 3.2. *Let D be a domain in $\mathbb{C}^n$. Let $\mathcal{F}$ be a family of holomorphic functions in a domain D and k be a positive integer. If $f(z) \neq 0$, $(D^k f(z), v) \neq 1$ for all $f \in \mathcal{F}$, $v = (v_1, ..., v_n) : \|v\| = 1$, then $\mathcal{F}$ is normal in D.*

Proof. Suppose $\mathcal{F}$ is not normal in D. Then by Lemma 1.4, there exist sequences $f_j \in \mathcal{F}$, $z_j \to z_0$, $\rho_j \to 0$, such that the sequence $\{g_j\}$ defined as $g_j(\zeta) = \frac{f_j(z_j+\rho_j\zeta)}{\rho_j^k}$ converges locally uniformly in $\mathbb{C}^n$ to a non-constant entire function g.
Let $w = (a_1, \ldots, a_n)$, $w' = (b_1, \ldots, b_n) \in \mathbb{C}^n$ and define

$$h_i(z_i) := g(b_1, \ldots, b_{i-1}, z_i, a_{i+1}, \ldots, a_n), \ i = 1, 2, \ldots n.$$

By Hurwitz's theorem, it follows that $h_i(z) \neq 0$ for all $z \in \mathbb{C}$.
By Weierstrass's theorem, we have

$$\frac{\partial^k}{\partial z_i^k} g_j(b_1, \ldots, b_{i-1}, z_i, a_{i+1}, \ldots, a_n) \to \frac{\partial^k}{\partial z_i^k} g(b_1, \ldots, b_{i-1}, z_i, a_{i+1}, \ldots, a_n).$$

Claim: $\frac{\partial^k}{\partial z_i^k} g(b_1, \ldots, b_{i-1}, z_i, a_{i+1}, \ldots, a_n) \not\equiv 1$

Suppose $\frac{\partial^k}{\partial z_i^k} g(b_1, \ldots, b_{i-1}, z_i, a_{i+1}, \ldots, a_n) \equiv 1$. Then $h_i(z)$ is a polynomial of degree k and so $h_i(a) = 0$ for some $a \in \mathbb{C}$, a contradiction and this establishes the claim.
Since $h_i(z) \neq 0$ for all $z \in \mathbb{C}$. So, $\frac{\partial^k}{\partial z_i^k} h_i(a) = 1$ for some $a \in \mathbb{C}$ which implies

$$\frac{\partial^k}{\partial z_i^k} g(b_1, \ldots, b_{i-1}, a, a_{i+1}, \ldots, a_n) = 1.$$

Since $\frac{\partial^k}{\partial z_i^k} g$ is non-constant. So, by Hurwitz's theorem there is a sequence $w_j = (b_1, \ldots, b_{i-1}, a_j, a_{i+1}, \ldots, a_n) \to w_0 = (b_1, \ldots, b_{i-1}, a, a_{i+1}, \ldots, a_n)$ such that for sufficiently large j

$$\frac{\partial^k}{\partial z_i^k} g_j(w_j) = 1$$

which implies

$$\sum_{l=1}^{n} \frac{\partial^k}{\partial z_l^k} f_j(z_j + \rho_j w_j) e_l = 1$$

$$\text{where } e_l = \begin{cases} 1, & \text{for } l = i \\ 0, & \text{for } l \neq i. \end{cases}$$

That is, $(D^k f_j(z_j + \rho_j w_j), v) = 1$, where $v = (0, \ldots, 1, \ldots, 0)$, which is a contradiction and hence $\mathcal{F}$ is normal in D. □

The next Picard-type theorem is

Theorem 3.3. *Let $f \in \mathcal{H}(\mathbb{C}^n)$ and a and b be two distinct finite complex numbers. If $f \neq a$ and for each $\alpha \in f^{-1}(\{b\})$, zero multiplicity of f at α is at least two, then f is constant.*

Proof. Let $w = (a_1, \ldots, a_n)$, $w' = (b_1, \ldots, b_n) \in \mathbb{C}^n$ and define

$$h_i(z_i) = f(b_1, \ldots, b_{i-1}, z_i, a_{i+1}, \ldots, a_n), \ i = 1, 2, \ldots n.$$

By the given hypothesis h_i omits a. Suppose $h_i(\alpha_i) = b$. Then

$$f(b_1, \dots, b_{i-1}, \alpha_i, a_{i+1}, \dots, a_n) = b$$

and so $B_i = (b_1, \dots, b_{i-1}, \alpha_i, a_{i+1}, \dots, a_n) \in f^{-1}(\{b\})$. Since for each $\alpha \in f^{-1}(\{b\})$, zero multiplicity of f at α is at least two, we have

$$f(z) = \sum_{m=2}^{\infty} P_m(z - B_i),$$

where P_m is either identically zero or a homogeneous polynomial of degree m.
Now

$$\frac{\partial}{\partial z_i} f(z) = \sum_{m=1}^{\infty} q_m(z - B_i)$$

where q_m is either identically zero or a homogeneous polynomial of degree m.
Thus $\frac{\partial}{\partial z_i} f(B_i) = 0$ which implies $\frac{d}{dz_i} h_i(\alpha_i) = 0$. Thus $b-$points of h_i have multiplicity at least two and so by Theorem 1.6, we conclude that each $h_i(z_i)$ is constant and hence f is constant. $\square$

The corresponding normality criterion of Theorem 3.3 is

Theorem 3.4. *Let D be a domain in $\mathbb{C}^n$. Let $\mathcal{F} \subset \mathcal{H}(D)$, a and b be two distinct finite complex numbers. If for each $f \in \mathcal{F}$, $f(z) \neq a$ and for each $\alpha \in f^{-1}(\{b\}) \cap D$, zero multiplicity of f at α is at least two, then $\mathcal{F}$ is normal in D.*

Proof. Suppose $\mathcal{F}$ is not normal in D. Then by Lemma 1.4, there exist sequences $f_j \in \mathcal{F}$, $z_j \to z_0$, $\rho_j \to 0$, such that the sequence $\{g_j\}$ defined as $g_j(\zeta) = f_j(z_j + \rho_j\zeta)$ converges locally uniformly in $\mathbb{C}^n$ to a non-constant entire function g. Let $w = (a_1, \dots, a_n)$, $w' = (b_1, \dots, b_n) \in \mathbb{C}^n$ and define

$$h_i(z_i) = g(b_1, \dots, b_{i-1}, z_i, a_{i+1}, \dots, a_n),\ i = 1, 2, \dots n.$$

By Hurwitz's theorem, it follows that $h_i(z) \neq 0$ for all $z \in \mathbb{C}$.
Suppose $h_i(\alpha_i) = b$ which implies $g(b_1, \dots, b_{i-1}, \alpha_i, a_{i+1}, \dots, a_n) = b$. By Hurwitz's theorem there exist sequences $\beta_{j,i} = (b_1, \dots, b_{i-1}, \alpha_{j,i}, a_{i+1}, \dots, a_n) \to \beta = (b_1, \dots, b_{i-1}, \alpha_i, a_{i+1}, \dots, a_n)$ such that $f_j(z_j + \rho_j\beta_{j,i}) = b$ which implies $z_j + \rho_j\beta_{j,i} \in f_j^{-1}(\{b\}) \cap D$, for sufficiently large j. Since for $\alpha \in f_j^{-1}(\{b\}) \cap D$, zero multiplicity of f_j at α is at least two, we have $f_j(z) = \sum\limits_{m=2}^{\infty} P_{j_m}(z - (z_j + \rho_j\beta_{j,i}))$, where P_{j_m} is either identically zero or a homogeneous polynomial of degree m.
Thus $\frac{\partial}{\partial z_i}(f_j(z)) = \sum\limits_{m=1}^{\infty} q_{j_m}(z - (z_j + \rho_j\beta_{j,i}))$, where q_{j_m} is either identically zero or a homogeneous polynomial of degree m.
Thus

$$\frac{\partial}{\partial z_i}(f_j(z_j + \rho_j\beta_{j,i})) = 0$$

which implies

$$\frac{\partial}{\partial z_i} g_j(b_1, \dots, b_{i-1}, \alpha_{j,i}, a_{i+1}, \dots, a_n) = 0$$

Taking $j \to \infty$, $\frac{\partial}{\partial z_i} g(b_1, \ldots, b_{i-1}, \alpha_i, a_{i+1}, \ldots, a_n) = 0$ which implies $\frac{d}{dz_i} h_i(\alpha_i) = 0$. Thus $b-$ points of h_i have multiplicity atleast two and so by Theorem 1.6 each $h_i(z_i)$ is constant and hence g is constant, a contradiction. $\square$

The fundamental normality test for holomorphic functions of several complex variables is a simple consequence of Theorem 3.4.

4. Extension of Some Known Results from $\mathbb{C}$ to $\mathbb{C}^n$

In 2012, J. Grahl and S. Nevo[13] proved the following normality criterion:

Theorem 4.1. *Let some $\epsilon > 0$ be given and set $\mathcal{F} := \{f \in \mathcal{H}(\Omega) : f^\sharp(z) \geq \epsilon$ for all $z \in \Omega\}$. Then $\mathcal{F}$ is normal in Ω.*

We find a several complex variables analogue of Theorem 4.1:

Theorem 4.2. *Let D be a domain in $\mathbb{C}^n$. Let $k \geq 0$ be an integer, $c > 0$, $\alpha > 1$ constant. Then*

$$\mathcal{F} = \left\{ f \in \mathcal{H}(D) : \sup_{\|v\|=1} \frac{|(D^k f(z), v)|}{1 + |f(z)|^\alpha} > c \right\}$$

is normal in D.

Proof. Suppose $\mathcal{F}$ is not normal in D. Taking $\beta > \frac{k}{\alpha - 1}$, by Zalcman's lemma in $\mathbb{C}^n$, there exist sequences $f_j \in \mathcal{F}$, $z_j \to z_0$, $\rho_j \to 0$, such that the sequence $\{g_j\}$ defined as

$$g_j(\zeta) = \rho_j^\beta f_j(z_j + \rho_j \zeta)$$

converges locally uniformly in $\mathbb{C}^n$ to a non-constant entire function g. Let $\zeta_0 \in \mathbb{C}^n$ such that $g(\zeta_0) \neq 0$. Clearly $f_j(z_j + \rho_j \zeta_0) \to \infty$ as $j \to \infty$. Also

$$\sup_{\|v\|=1} |(D^k g_j(\zeta_0), v)| = \sup_{\|v\|=1} \rho_j^{\beta+k} |(D^k f_j(z_j + \rho_j \zeta_0), v)| \to \sup_{\|v\|=1} |(D^k g(\zeta_0), v)|.$$

Since

$$\sup_{\|v\|=1} |(D^k f_j(z_j + \rho_j \zeta_0), v)| > c |f_j(z_j + \rho_j \zeta_0)|^\alpha,$$

we have

$$\begin{aligned}
\sup_{\|v\|=1} |(D^k g_j(\zeta_0), v)| &= \sup_{\|v\|=1} \rho_j^{\beta+k} |(D^k f_j(z_j + \rho_j \zeta_0), v)| \\
&> \rho_j^{\beta+k} c |f_j(z_j + \rho_j \zeta_0)|^\alpha \\
&= c(\rho_j^\beta |f_j(z_j + \rho_j \zeta_0)|)^\alpha . \rho_j^{\beta+k-\beta\alpha} \to \infty \text{ as } j \to \infty.
\end{aligned}$$

which is a contradiction to the fact that g is holomorphic at ζ_0. Therefore, $\mathcal{F}$ has to be normal. $\square$

Remark 4.3. For $k = 1$ and $\alpha = 2$, we obtain several complex variable version of Theorem 4.1.

Further, we obtain an improvement of a Marty's Theorem in $\mathbb{C}^n$:

Theorem 4.4. *Let D be a domain in $\mathbb{C}^n$, $\mathcal{F} \subset \mathcal{H}(D)$ and α be a positive real number. If*

$$F_\alpha = \left\{ \sup_{\|v\|=1} \frac{|(Df(z), v)|}{1 + |f(z)|^\alpha} : f \in \mathcal{F} \right\}$$

is locally uniformly bounded in D, then $\mathcal{F}$ is normal in D.

Proof. Suppose $\mathcal{F}$ is not normal in D. Then by Lemma 1.4, there exist sequences $f_j \in \mathcal{F}$, $z_j \to z_0$, $\rho_j \to 0$, such that the sequence $\{g_j\}$ defined as $g_j(\zeta) = \rho_j^{-1/2} f_j(z_j + \rho_j \zeta)$ converges locally uniformly in $\mathbb{C}^n$ to a non-constant entire function g.
By Weierstrass' Theorem,

$$\frac{\partial}{\partial z_i} g_j(\zeta) \to \frac{\partial}{\partial z_i} g(\zeta),\ i = 1, 2, \ldots, n.$$

Now

$$\begin{aligned} \sup_{\|v\|=1} \sum_{l=1}^{n} \left| \frac{\partial}{\partial z_l} g_j(\zeta).v_l \right| &= \rho_j^{1/2} \sup_{\|v\|=1} \sum_{l=1}^{n} \left| \frac{\partial}{\partial z_l} f_j(z_j + \rho_j\zeta).v_l \right| \\ &= \rho_j^{1/2} \sup_{\|v\|=1} |(Df_j(z_j + \rho_j\zeta), v)| \\ &\leq \rho_j^{1/2} M[1 + |f_j(z_j + \rho_j\zeta)|^\alpha] \\ &= \rho_j^{1/2} M[1 + |\rho_j^{1/2} g_j(\zeta)|^\alpha] \to 0 \text{ as } j \to \infty \end{aligned}$$

Thus

$$\sup_{\|v\|=1} \sum_{l=1}^{n} \left| \frac{\partial}{\partial z_l} g_j(\zeta).v_l \right| = 0.$$

That is,

$$\frac{\partial g}{\partial z_i}(\zeta) = 0,\ i = 1, \ldots, n,$$

which implies g is a constant, a contradiction. □

Acknowledgement

The work of the second author is supported by the CSIR, India (File No. 09/100(0217)/2018-EMR-I.

Statements and Declarations

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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