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K-FUSION FRAMES IN QUATERNIONIC HILBERT SPACES

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Abstract. In [26] authors have defined and studied K-fusion frames in Hilbert spaces. Motivated by [26], we define K-fusion frames in Quaternionic Hilbert spaces and study their perturbations. Further, we define the woven K-fusion frames in Quaternionic Hilbert spaces. Finally, we study the Paley-Wiener type perturbation for weaving K-fusion frames in quaternionic Hilbert spaces.

1. Introduction

Frames [15] in Hilbert spaces were introduced in 1952 while studying the non-harmonic Fourier Series. But their potential was realized by the researchers after the work done by Daubechies et el. [5], due to its vast applications in various fields like signal and image processing, sigma-delta quantization, filter bank theory, sampling theory and wireless communication, for details one may refer to [11]. Over the past few years, many generalizations of frames were introduced and studied. Two of the generalization which is much appreciated by the researchers are fusion frames defined and studied by P. Casazza and G. kutyniok [12] and K-frames introduced and studied by *Găvruta* [7]. X. C. Xiao, M. L. Ding and Yu Can Zhu [25] have defined and studied the K-fusion frames in Hilbert spaces. In [6] Poumai and Kaushik have studied results concerning Riesz bases and frames in Banach spaces. In [23] Bemrose et al. have defined and studied the properties of weaving frames in Hilbert spaces which are used in distributed signal processing. For details regarding fusion frames and woven fusion frames one may refer to [9, 13, 14] and for details regarding weaving K-frames and their properties one may refer to [1, 2, 24, 27].

Hamilton discovered the field of quaternion which is a generalization of complex numbers. It is a four-dimensional non-commutative real algebra. Quaternions are used to study rotation in the higher dimensional Euclidean spaces, theory of relativity, Newtonian and quantum mechanics and general relativity in which Lorentz

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transformation is given in terms of quaternions. For details regarding quaternions see [18]. In [16] R. Ghiloni et al. have extended the continuous functional calculus in the case of quaternionic Hilbert spaces.

Khokulan et al. [8] have defined and studied frames in finite dimensional quaternionic Hilbert spaces. In [22], Sharma and Goel have studied frames in separable right quaternionic Hilbert spaces. In [3], H. Ellouz studied K-frames in right quaternionic Hilbert spaces. Recently in [17] Ruchi et al. have defined OPV-frames in right quaternionic Hilbert spaces and woven frames in right quaternionic Hilbert spaces were studied in [21]. For details regarding fusion frames, woven fusion frames and woven K-frames in quaternionic Hilbert spaces, one may refer to [20, 10].
In this paper, we have defined and studied K-fusion frames in quaternionic Hilbert spaces. Further, we have given some perturbation results for a K-fusion frames similar to the results in Hilbert spaces. Finally, woven K-fusion frames defined and studied in quaternionic Hilbert spaces.

2. Preliminaries

Throughout this paper, $\mathfrak{Q}$ denote the non-commutative field of quaternions and $\mathcal{I}$ be a countable set. $\mathbb{H}^R(\mathfrak{Q})$, $\mathbb{H}_1^R(\mathfrak{Q})$ and $\mathbb{H}_2^R(\mathfrak{Q})$ represent separable right quaternionic Hilbert spaces. The set of all bounded (right $\mathfrak{Q}$-linear) operators from $\mathbb{H}_1^R(\mathfrak{Q})$ to $\mathbb{H}_2^R(\mathfrak{Q})$ denoted by $\mathfrak{B}(\mathbb{H}_1^R(\mathfrak{Q}), \mathbb{H}_2^R(\mathfrak{Q}))$.

Definition 2.1. [19] Let $\{W_i^R\}_{i\in\mathcal{I}}$ be a sequence of right closed subspaces in a separable right quaternionic Hilbert space $\mathbb{H}^R(\mathfrak{Q})$ and $\{w_i\}_{i\in\mathcal{I}}$ be a family of weights `i.e.` $w_i > 0$, $\forall\, i \in \mathcal{I}$. Then $\{(W_i^R, w_i)\}_{i\in\mathcal{I}}$ is called a fusion frame (frame of subspaces), if there exists constants $0 < r_1 \le r_2 < \infty$ called fusion frame bounds such that

$$r_1||u||^2 \le \sum_{i\in\mathcal{I}} w_i^2 ||\pi_{W_i^R}(u)||^2 \le r_2||u||^2, \,\forall\, u \in \mathbb{H}^R(\mathfrak{Q}). \tag{2.1}$$

We say the family of subspaces $\{W_i^R\}_{i\in\mathcal{I}}$ form a fusion frame with respect to $\{w_i\}_{i\in\mathcal{I}}$, if it satisfies (2.1). Moreover, if $r_1 = r_2$ then we say the family $\{W_i^R\}_{i\in\mathcal{I}}$ is a tight fusion frame of $\mathbb{H}^R(\mathfrak{Q})$ and Parseval fusion frame if $r_1 = r_2 = 1$. If $\mathbb{H}^R(\mathfrak{Q}) = \bigoplus_{i\in\mathcal{I}} W_i^R$, then $\{W_i^R\}_{i\in\mathcal{I}}$ is called an orthonormal basis of subspaces. If $w_i = w_j = w$, then $\{W_i^R\}_{i\in\mathcal{I}}$, is called a w-uniform fusion frame. If only upper bound condition holds then we say the family $\{W_i^R\}_{i\in\mathcal{I}}$, forms a Bessel sequence of subspaces of $\mathbb{H}^R(\mathfrak{Q})$ with respect to $\{w_i\}_{i\in\mathcal{I}}$ with bound r_2.

Let $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ is a Bessel sequence of $\mathbb{H}^R(\mathfrak{Q})$. Then the set

$$\left(\sum_{i\in\mathcal{I}} \bigoplus W_i^R\right)_{l^2(\mathfrak{Q})} = \left\{\{u_i\}_{i\in\mathcal{I}} : u_i \in W_i^R, \sum_{i\in\mathcal{I}} ||u_i||^2 < \infty\right\},$$

forms a right quaternionic Hilbert space under the inner product given by

$$\langle \{u_i\}_{i\in\mathcal{I}} | \{p_i\}_{i\in\mathcal{I}} \rangle = \sum_{i\in\mathcal{I}} \langle u_i | p_i \rangle, \quad \{u_i\}_{i\in\mathcal{I}},\ \{p_i\}_{i\in\mathcal{I}} \in \left(\sum_{i\in\mathcal{I}} \bigoplus W_i^R\right)_{l^2(\mathfrak{Q})}$$

Theorem 2.2. [20] *For every $i \in \mathcal{I}$, let $w_i > 0$, $\{u_{ij}\}_{j\in J_i}$ is a frame sequence in $\mathbb{H}^R(\mathfrak{Q})$ with frame bounds r_{1i} and r_{2i}, respectively. Let $W_i^R = \overline{span}_{j\in J_i}\{u_{ij}\}$ and $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for subspace W_i^R. Suppose that*

$$0 < r_1 \left(= \inf_{i\in\mathcal{I}} r_{1i}\right) \leq r_2 \left(= \sup_{i\in\mathcal{I}} r_{2i}\right) < \infty.$$

The statements below are the same.

(a) *$\{u_{ij}w_i\}_{i\in\mathcal{I},j\in J_i}$ forms a frame of $\mathbb{H}^R(\mathfrak{Q})$.*
(b) *$\{e_{ij}w_i\}_{i\in\mathcal{I},j\in J_i}$ forms a frame of $\mathbb{H}^R(\mathfrak{Q})$.*
(c) *$\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ forms a fusion frame of $\mathbb{H}^R(\mathfrak{Q})$.*

Lemma 2.3. [4] *Let $L_1 \in \mathfrak{B}(\mathbb{H}_1^R(\mathfrak{Q}), \mathbb{H}^R(\mathfrak{Q}))$, $L_2 \in \mathfrak{B}(\mathbb{H}_2^R(\mathfrak{Q}), \mathbb{H}^R(\mathfrak{Q}))$ be bounded right linear operators. The statements below are the same.*

(a) *$R(L_1) \subset R(L_2)$.*
(b) *$L_1L_1^* \leq \lambda^2 L_2L_2^*$, for some $\lambda \geq 0$.*
(c) *there exist a bounded right linear operator $M \in \mathfrak{B}(\mathbb{H}_1^R(\mathfrak{Q}), \mathbb{H}_2^R(\mathfrak{Q}))$, such that $L_1 = L_2M$.*

Definition 2.4. [19] A family of elements $\{u_i\}_{i\in\mathcal{I}}$ of $\mathbb{H}^R(\mathfrak{Q})$ is an atomic system for $K \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$, if

(i) The series $\sum\limits_{i\in\mathcal{I}} u_iq_i$ converges, where $\{q_i\}_{i\in\mathcal{I}} \in {}^2(\mathfrak{Q})$.
(ii) There exists $r > 0$ such that for any $u \in \mathbb{H}^R(\mathfrak{Q})$, there exists $a_u = \{q_i\}_{i\in\mathcal{I}} \in l^2(\mathfrak{Q})$, such that $||a_u||_{l^2(\mathfrak{Q})} \leq r||u||$ and $K(u) = \sum\limits_{i\in\mathcal{I}} u_iq_i$.

Proposition 2.5. [19] Let $\{u_i\}_{i\in\mathcal{I}} \subset \mathbb{H}^R(\mathfrak{Q})$. The statements below are the same.

(i) $\{u_i\}_{i\in\mathcal{I}}$ be an atomic system for $K \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$.
(ii) there exists $r_1, r_2 > 0$ such that $r_1||K^*u||^2 \leq \sum\limits_{i\in\mathcal{I}} |\langle u_i|u\rangle|^2 \leq r_2||u||^2$, $\forall\, u \in \mathbb{H}^R(\mathfrak{Q})$.
(iii) $\{u_i\}_{i\in\mathcal{I}}$ is a Bessel sequence and there exists a Bessel sequence $\{p_i\}_{i\in\mathcal{I}}$ such that $K(u) = \sum\limits_{i\in\mathcal{I}} u_i\langle p_i|u\rangle$, $\forall\, u \in \mathbb{H}^R(\mathfrak{Q})$.

Definition 2.6. [19] A family $\{u_i\}_{i\in\mathcal{I}}$ forms a K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with K-bounds r_1 and r_2, if $\{u_i\}_{i\in\mathcal{I}}$ satisfies (ii) of Proposition 2.5.

By Proposition 2.5, if $\{u_i\}_{i\in\mathcal{I}}$ satisfies any of the three conditions then it is a K-frame of $\mathbb{H}^R(\mathfrak{Q})$.

Proposition 2.7. [3] Let $T \in \mathfrak{B}(\mathbb{H}_1^R(\mathfrak{Q}), \mathbb{H}_2^R(\mathfrak{Q}))$ be a closed right linear operator. Then there exists a right linear operator $T^\dagger \in \mathfrak{B}(\mathbb{H}_2^R(\mathfrak{Q}), \mathbb{H}_1^R(\mathfrak{Q}))$ such that

$$N(T^\dagger) = R(T)^\perp,\;\; R(T^\dagger) = N(T)^\perp,\;\; TT^\dagger u = u,\;\; u \in R(T).$$

Definition 2.8. [20] Let $\mathcal{H} = \{({H_i}^R, v_i)\}_{i\in\mathcal{I}}$ and $\mathcal{W} = \{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ are fusion frames for $\mathbb{H}^R(\mathfrak{Q})$. Then $\mathcal{H}$ and $\mathcal{W}$ forms a woven fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ if there exists $0 < r_1 \leq r_2 < \infty$ such that for any subset σ of $\mathcal{I}$, the family $\{({H_i}^R, v_i)\}_{i\in\sigma} \cup \{({W_i}^R, w_i)\}_{i\in\sigma^c}$ is a fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ with lower and upper fusion frame bounds r_1 and r_2, respectively. Further, if only upper bound condition is hold then $\mathcal{H}$ and $\mathcal{W}$ are said to form a woven Bessel sequence of subspaces for $\mathbb{H}^R(\mathfrak{Q})$ with bound r_2.

Proposition 2.9. [20] Let $\{({H_i}^R, v_i)\}_{i\in\mathcal{I}}$ and $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ are two Bessel sequences of subspaces of $\mathbb{H}^R(\mathfrak{Q})$ with Bessel bounds r_1, r_2. Then the family $\{({H_i}^R, v_i)\}_{i\in\mathcal{I}}$ and $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ form a woven Bessel sequence of subspaces with Bessel bound $r_1 + r_2$.

3. Main Results

In this section, we have defined and studied the K-fusion frames in right quaternionic Hilbert spaces.

Definition 3.1. Let $\{W_i^R\}_{i\in\mathcal{I}}$ be a sequence of closed right subspaces of $\mathbb{H}^R(\mathfrak{Q})$, and $\{w_i\}_{i\in\mathcal{I}}$ be a family of weights. Let $K \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$, the family $W = \{(W_i^R, w_i)\}_{i\in\mathcal{I}}$ forms a K-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$, if there exists constants $0 < r_1 \leq r_2 < \infty$ satisfying,

$$r_1||K^*u||^2 \leq \sum_{i\in\mathcal{I}} w_i^2||\pi_{W_i^R}(u)||^2 \leq r_2||u||^2,\ u \in \mathbb{H}^R(\mathfrak{Q}). \tag{3.2}$$

In [20], we have defined and studied the (right)K-fusion frame operator S_W, the (right)synthesis operator T_W and (right)analysis operator T_W^* corresponding to a Bessel sequence of subspaces $W = \{(W_i^R, w_i)\}_{i\in\mathcal{I}}$ of $\mathbb{H}^R(\mathfrak{Q})$, where

$$T_{\mathcal{W}}(\{u_i\}_{i\in\mathcal{I}}) = \sum_{i\in\mathcal{I}} u_i w_i,\ \{u_i\}_{i\in\mathcal{I}} \in \left(\sum_{i\in\mathcal{I}} \bigoplus W_i^R\right)_{l^2(\mathfrak{Q})},$$

$$T_{\mathcal{W}}^*(u) = \{\pi_{W_i^R}(u)w_i\}_{i\in\mathcal{I}},\ u \in \mathbb{H}^R(\mathfrak{Q}),$$

$$S_{\mathcal{W}}(u) = T_{\mathcal{W}}T_{\mathcal{W}}^*(u) = \sum_{i\in\mathcal{I}} \pi_{W_i^R}(u)w_i^2,\ u \in \mathbb{H}^R(\mathfrak{Q}).$$

Next, we have studied an example of K-fusion frames.

Example 3.2. Let $\mathbb{H}^R(\mathfrak{Q})$ be a three dimensional quaternionic Hilbert space with an orthonormal basis $\{e_1, e_2, e_3\}$. Let $W_1^R = \text{span}\{e_1, e_2\}$ and $W_2^R = \text{span}\{e_3\}$ and $K \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ be such that

$$K(e_1) = e_1, K(e_2) = e_1 \text{ and } K(e_3) = e_2.$$

The adjoint operator K^* corresponding to K is

$$K^*(e_1) = e_1 + e_2,\ K^*(e_2) = e_3 \text{ and } K^*(e_3) = 0.$$

Take $w_1 = w_2 = 1$, we have

$$\frac{1}{2}||K^*u||^2 \leq \sum_{i=1}^{2} w_i^2||\pi_{W_i^R}(u)||^2 \leq 2||u|^2,\ u \in \mathbb{H}^R(\mathfrak{Q}).$$

In the next lemma, we have obtained an expression for the right pseudo inverse operator corresponding to a closed right linear operator.

Lemma 3.3. *Let $T \in \mathfrak{B}(\mathbb{H}_1^R(\mathfrak{Q}), \mathbb{H}_2^R(\mathfrak{Q}))$ be a closed right linear operator. Then $T^\dagger = T^*(TT^*)^{-1}$.*

Proof. We can prove this Lemma by using Proposition 2.7 and Lemma 2.5.2 [11]. □

In the next proposition, we have studied properties of a right linear operator.

Proposition 3.4. Let W^R be a closed right subspace of $\mathbb{H}^R(\mathfrak{Q})$ and $T \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$. The statements below are the same.

(i) $\pi_{\overline{TW^R}}T = T\pi_{W^R}$;

(ii) $T^*TW^R \subset W^R$.

Proof. $(i) \implies (ii)$: Let $u \in (W^R)^\perp$, we have $\pi_{\overline{TW^R}}Tu = T\pi_{W^R}u = 0$, hence $Tu \in (\overline{TW^R})^\perp = (TW^R)^\perp$. Since $\langle Tu|Tw\rangle = 0, \forall\, w \in W^R$ if and only if $\langle u|T^*Tw\rangle = 0$, $\forall\, w \in W^R$ if and only if $u \in (T^*TW^R)^\perp$.
$(ii) \implies (i)$: If $w \in W^R$ then $\pi_{\overline{TW^R}}Tw = Tw$ and $T\pi_{W^R}w = Tw$. Further if, $u \in (W^R)^\perp$ then $T\pi_{W^R}u = T0 = 0$. From the given condition, we have $u \in (T^*TW^R)^\perp$. As before, we have $Tu \in (\overline{TW^R})^\perp$ and hence $\pi_{\overline{TW^R}}Tu = 0$. □

Lemma 3.5. *Let W^R be a closed right subspace of $\mathbb{H}^R(\mathfrak{Q})$ and $T \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$. Then we have*

$$\pi_{W^R}T^* = \pi_{W^R}T^*\pi_{\overline{TW^R}}.$$

Proof. As,

$$u = \pi_{\overline{TW^R}}u + x,\ x \in (\overline{TW^R})^\perp = (TW^R)^\perp,\ \ u \in \mathbb{H}^R(\mathfrak{Q}).$$

Therefore $T^*u = T^*\pi_{\overline{TW^R}}u + T^*x$, but for $w \in W^R$, we have $\langle T^*x|w\rangle = \langle x|Tw\rangle = 0$, thus $T^*u \in (W^R)^\perp$. Therefore,

$$\pi_{W^R}T^*u = \pi_{W^R}T^*\pi_{\overline{TW^R}}u + \pi_{W^R}T^*x = \pi_{W^R}T^*\pi_{\overline{TW^R}}u.$$

Hence by Proposition 3.4 result will follow. □

Proposition 3.6. Let $K \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ be a closed right linear operator and $W = \{(W_i^R, w_i)\}_{i\in\mathcal{I}}$ be a family of K-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$ with bounds r_1, r_2, respectively. Then

(i) $r_1KK^* \leq S_W \leq r_2I$;

(ii) $r_1||K^\dagger||^{-2}||u|| \leq ||S_Wu|| \leq r_2||u||$, $\forall\, u \in R(K)$;

(iii) $r_2^{-1}||u|| \leq ||S_W^{-1}u|| \leq r_1^{-1}||K^\dagger||^2||u||$, $\forall\, u \in S_W(R(K))$.

Proof. (i) From K- fusion frame inequality, we have

$$\langle r_1KK^*u|u\rangle \leq \langle S_Wu|u\rangle \leq \langle r_2u|u\rangle,\ u \in \mathbb{H}^R(\mathfrak{Q}).$$

Thus, $r_1KK^* \leq S_W \leq r_2I$.

(ii) Since $R(K)$ is closed, there exists a right linear operator $K^\dagger$ such that $KK^\dagger = I_{R(K)}$. Hence

$$\begin{aligned} ||u|| &= ||(K^\dagger_{R(K)})^*K^*u|| \\ &\leq ||K^\dagger||\ ||K^*u||, u \in R(K). \end{aligned}$$

The rest will follow from K-fusion frame inequality.

(iii) It will follow from part (ii).

□

In the next result, we have studied a necessary and sufficient condition under which a Bessel sequence of subspaces of $\mathbb{H}^R(\mathfrak{Q})$ forms a K-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$.

Proposition 3.7. Let $W = \{(W_i^R, w_i)\}_{i\in\mathcal{I}}$ is a Bessel sequence of subspaces of $\mathbb{H}^R(\mathfrak{Q})$. Then
$W = \{(W_i^R, w_i)\}_{i\in\mathcal{I}}$ is a K-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$ if and only if $r_1KK^* \leq S_W$ for some $r_1 > 0$.

Proof. Let $W = \{(W_i^R, w_i)\}_{i\in\mathcal{I}}$ forms a K-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$, from part (i) of Proposition 3.6, the necessary condition is true. For sufficient condition, let $r_1KK^* \leq S_W$ for some $r_1 > 0$, consider

$$\begin{aligned} r_1||K^*u||^2 &= \langle r_1KK^*u|u\rangle \\ &\leq \langle S_W u|u\rangle \\ &= \sum_{i\in\mathcal{I}} w_i^2||\pi_{W_i^R}(u)||^2,\ u\in\mathbb{H}^R(\mathfrak{Q}). \end{aligned}$$

□

Corollary 3.8. Let $W = \{(W_i^R, w_i)\}_{i\in\mathcal{I}}$ be a family of K-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$ and $T \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ such that $R(T) \subset R(K)$. Then $W = \{(W_i^R, w_i)\}_{i\in\mathcal{I}}$ is a T-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$.

Proof. Since $R(T) \subset R(K)$ by Lemma 2.3, we have $TT^* \leq \lambda^2KK^*$ for some $\lambda > 0$. Also, there exists $r_1 > 0$, such that $r_1KK^* \leq S_W$. Hence the result will follow from Proposition 3.7. □

Corollary 3.9. Let $W = \{(W_i^R, w_i)\}_{i\in\mathcal{I}}$ be a Bessel sequence of subspaces of $\mathbb{H}^R(\mathfrak{Q})$ with fusion frame operator S_W. Then $W = \{(W_i^R, w_i)\}_{i\in\mathcal{I}}$ is a K-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$ if and only if $K = S_W^{1/2}U$, for some $U \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$.

Proof. $W = \{(W_i^R, w_i)\}_{i\in\mathcal{I}}$ be a K-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$ if and only if $r_1KK^* \leq S_W = \sqrt{S_W}(\sqrt{S_W})^*$. Hence, by Lemma 2.3 and Proposition 3.7 the necessary and sufficient condition will follow easily. □

In the next result, we have studied a condition under which only surjective right linear operator maps a K-fusion frame to a K-fusion frame.

Theorem 3.10. *Let $K, T \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ be right linear operators such that $R(K)$ is dense in $\mathbb{H}^R(\mathfrak{Q})$ and $R(T)$ is closed. Suppose, $W = \{(W_i^R, w_i)\}_{i\in\mathcal{I}}$ forms a K-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$. If $TW = \{(TW_i^R, w_i)\}_{i\in\mathcal{I}}$ forms a K-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$. Then T is a surjective right linear operator on $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. Suppose, $TW = \{(TW_i^R, w_i)\}_{i\in\mathcal{I}}$ is a K-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$ with bounds r_1 and r_2, respectively. Let $\{f_{ij}\}_{j\in J_i}$ is a frame of W_i^R, $i \in \mathcal{I}$ with bounds A_i and B_i, where
$0 < A = \inf_{i\in\mathcal{I}}A_i \leq \sup_{i\in\mathcal{I}}B_i = B < \infty$. By Theorem 2.2, $\{T(f_{ij})w_i\}_{j\in J_i, i\in\mathcal{I}}$ is a K-frame of $\mathbb{H}^R(\mathfrak{Q})$ with bounds r_1A and r_2B respectively. Consider,

$$r_1A||K^*u||^2 \leq \sum_{i\in\mathcal{I}}\sum_{j\in J_i} |\langle Tf_{ij}w_i|u\rangle|^2 \leq r_2B||u||^2,\ u\in\mathbb{H}^R(\mathfrak{Q}).$$

Thus $N(T^*) \subset N(K^*)$, since K^* is injective as $R(K)$ is dense. Therefore T^* is injective and $R(T) = \overline{R(T)} = N(T^*)^\perp = \mathbb{H}^R(\mathfrak{Q})$. □

In the next theorem, we have studied a sufficient condition under which the bounded image of a K-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$ forms a K-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$.

Theorem 3.11. *Let $W = \{(W_i^R, w_i)\}_{i\in\mathcal{I}}$ be a K-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$. Further, let $T \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ be a closed right linear operator with $TK = KT$ and $T^\dagger\overline{T(W_i^R)} \subset W_i^R$. If T is a surjective right linear operator. Then, $TW = \{(TW_i^R, w_i)\}_{i\in\mathcal{I}}$ forms a K-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. As, T is a surjective right operator with closed range implies, $TT^\dagger = I_{\mathbb{H}^R(\mathfrak{Q})}$. We will show that $T(W_i^R)$ be a closed right subspace of $\mathbb{H}^R(\mathfrak{Q})$. Since $T^\dagger\overline{T(W_i^R)} \subset W_i^R$, as T is a surjective right linear operator we have $TT^\dagger\overline{T(W_i^R)} \subset T(W_i^R)$. By Lemma 3.3 we have $TT^*(TT^*)^{-1}\overline{T(W_i^R)} \subset T(W_i^R)$ and hence $\overline{T(W_i^R)} \subset T(W_i^R)$. Therefore $T(W_i^R)$ is a closed right subspace of $\mathbb{H}^R(\mathfrak{Q})$. By Lemma 3.5,

$$\begin{aligned} ||\pi_{W_i^R}T^*u|| &= ||\pi_{W_i^R}T^*\pi_{TW_i^R}u|| \\ &\leq ||T^*||\,||\pi_{TW_i^R}u||,\ u \in \mathbb{H}^R(\mathfrak{Q}). \end{aligned}$$

$$||\pi_{TW_i^R}u|| \geq ||T^*||^{-1}||\pi_{W_i^R}T^*u||,\ u \in \mathbb{H}^R(\mathfrak{Q}).$$

As, $I_{\mathbb{H}^R(\mathfrak{Q})} = I^*_{\mathbb{H}^R(\mathfrak{Q})} = (T^\dagger)^*T^*$, for any $u \in R(T) = \mathbb{H}^R(\mathfrak{Q})$, $K^*u = {T^\dagger}^*T^*K^*u$.

Thus, we have

$$\begin{aligned} ||K^*u|| &= ||{T^\dagger}^*T^*K^*u|| \\ &\leq ||{T^\dagger}^*||\,||T^*K^*u||, \end{aligned}$$

since $TK = KT$ we have

$$\begin{aligned} ||{T^\dagger}^*||^{-1}||K^*u|| &\leq ||T^*K^*u|| && (3.3) \\ &= ||K^*T^*u|| && (3.4) \end{aligned}$$

Now for any $u \in \mathbb{H}^R(\mathfrak{Q})$,

$$\begin{aligned} \sum_{i\in\mathcal{I}} w_i^2||\pi_{TW_i^R}u||^2 &\geq ||{T^\dagger}^*||^{-2}\sum_{i\in\mathcal{I}} w_i^2||\pi_{W_i^R}T^*u||^2 \\ &\geq r_1||T^*||^{-2}||{T^\dagger}^*||^{-2}||K^*u||^2. \end{aligned}$$

Applying Lemma 3.5 with $T^\dagger$ instead of T we have $\pi_{TW_i^R}{T^\dagger}^* = \pi_{TW_i^R}{T^\dagger}^*\pi_{W_i^R}$, therefore

$$\begin{aligned} ||\pi_{TW_i^R}u|| &= ||\pi_{TW_i^R}{T^\dagger}^*\pi_{W_i^R}T^*u|| \\ &\leq ||\pi_{TW_i^R}||\,||{T^\dagger}^*||\,||\pi_{W_i^R}T^*u|| \\ &\leq ||{T^\dagger}^*||\,||\pi_{W_i^R}T^*u||, u \in \mathbb{H}^R(\mathfrak{Q}). \end{aligned}$$

Hence we have

$$\begin{aligned} \sum_{i\in\mathcal{I}} w_i^2||\pi_{TW_i^R}u||^2 &\leq ||{T^\dagger}^*||^2\sum_{i\in\mathcal{I}} w_i^2||\pi_{W_i^R}T^*u||^2 \\ &\leq ||{T^\dagger}^*||^2 r_2||T^*u||^2 \\ &\leq r_2||{T^\dagger}^*||\,||T||^2\,||u||^2,\ u \in \mathbb{H}^R(\mathfrak{Q}). \end{aligned}$$

Thus, $\{(TW_i^R, w_i)\}_{i\in\mathcal{I}}$ forms a K-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$. □

Corollary 3.12. Let $W = \{(W_i^R, w_i)\}_{i\in\mathcal{I}}$ is a K-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$. Further, let $T \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ such that $R(T)$ is closed and $TK = KT$, $T^\dagger\overline{T(W_i^R)} \subset W_i^R$, also T^* be an isometry. Then, $TW = \{(TW_i^R, w_i)\}_{i\in\mathcal{I}}$ forms a K-fusion frame of $R(T)$.

Proof. As, T^* is an isometry,hence

$$\begin{aligned} ||K^*T^*u||^2 &= ||T^*K^*u||^2 && (3.5)\\ &= ||K^*u||^2,\ u \in \mathbb{H}^R(\mathfrak{Q}). && (3.6) \end{aligned}$$

Now follow the step of Theorem 3.11, but instead (3.3) use (3.5). □

In next result, we have proved that if a operator and its adjoint operator, both maps a K-fusion frame to a K-fusion frame then the operator must be an invertible right linear operator.

Theorem 3.13. *Let $K, T \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ such that $R(K)$ is dense in $\mathbb{H}^R(\mathfrak{Q})$ and $R(T)$ is closed, with $TK = KT$. Further, let $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ be a K-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$. If $\{(TW_i^R, w_i)\}_{i\in\mathcal{I}}$ and $\{(T^*W_i^R, w_i)\}_{i\in\mathcal{I}}$ forms K-fusion frames of $\mathbb{H}^R(\mathfrak{Q})$. Then T is an invertible right linear operator on $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. By Theorem 3.10, T is a surjective right linear operator. Let $\{(T^*W_i^R, w_i)\}_{i\in\mathcal{I}}$ is a K-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$ with upper and lower bounds r_1 and r_2, respectively.

$$r_1||K^*u||^2 \leq \sum_{i\in\mathcal{I}} w_i^2||\pi_{T^*W_i^R}u||^2 \leq r_2||u||^2,\ u \in \mathbb{H}^R(\mathfrak{Q}).$$

Hence $N(T) \subset N(K^*)$, since $R(K)$ is dense implies K^* is an injective right linear operator. Thus T is an injective right linear operator. Therefore by the Bounded Inverse Theorem for quaternionic Hilbert spaces, T is an invertible right linear operator. □

In the next two results, we have obtained conditions under which a given K_i-fusion frames forms a fusion frame with respect to different operators.

Theorem 3.14. *Let $W = \{(W_i^R, w_i)\}_{i\in\mathcal{I}}$ satisfy (3.2) with right linear operators $K_1, K_2 \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$. Then W forms a K_1K_2-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. Let r_1, r_2 be bounds corresponding to K_1-fusion frame inequalities.

$$\frac{r_1}{||K_2^*||^2}||(K_1K_2)^*u||^2 \leq r_1||K_1^*u||^2 \leq \sum_{i\in\mathcal{I}} w_i^2||\pi_{W_i^R}u||^2 \leq r_2||u||^2,\ \forall\, u \in \mathbb{H}^R(\mathfrak{Q}).$$

□

Theorem 3.15. *Let $W = \{(W_i^R, w_i)\}_{i\in\mathcal{I}}$ satisfy (3.2) with right linear operators $K_1, K_2 \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$. Let $\alpha, \beta \in \mathbb{R}$ be such that $(\alpha, \beta) \neq (0, 0)$. Then, W forms a $(\alpha K_1 + \beta K_2)$-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. Let r_1, r_2 and R_1, R_2 be bounds corresponding to K_1-fusion frame and K_2-fusion frame respectively. Consider,

$$\begin{aligned} ||(\alpha K_1 + \beta K_2)^*u||^2 &\leq |\alpha|^2||K_1^*u||^2 + |\beta|^2||K_2^*u||^2\\ &\leq \Big(\frac{|\alpha|^2}{r_1} + \frac{|\beta|^2}{R_1}\Big)\sum_{i\in\mathcal{I}} w_i^2||\pi_{W_i^R}u||^2,\ u \in \mathbb{H}^R(\mathfrak{Q}). \end{aligned}$$

Thus, we have

$$\frac{r_1R_1||(\alpha K_1+\beta K_2)^*u||^2}{R_1|\alpha|^2+r_1|\beta|^2}\leq\sum_{i\in\mathcal{I}}w_i^2||\pi_{W_i^R}u||^2\leq\frac{(r_2+R_2)}{2}||u||^2,\ u\in\mathbb{H}^R(\mathfrak{Q}).$$

□

In the next result, we have studied result based on the perturbation of K-fusion frame in right quaternionic Hilbert space $\mathbb{H}^R(\mathfrak{Q})$.

Theorem 3.16. *Let $W=\{(W_i^R,w_i)\}_{i\in\mathcal{I}}$ is a K-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$ with bounds r_1, r_2 respectively. Further let, $\{V_i^R\}_{i\in\mathcal{I}}$ is a right closed subspaces of $\mathbb{H}^R(\mathfrak{Q})$ and suppose exists constants $\lambda_1,\lambda_2,\mu\geq 0$ satisfying max$\{\lambda_1+\frac{\mu}{\sqrt{r_1}},\lambda_2\}<1$ and*

$$\begin{aligned}\Big(\sum_{i\in\mathcal{I}}w_i^2||(\pi_{W_i^R}-\pi_{V_i^R})u||^2\Big)^{1/2}&\leq\lambda_1\Big(\sum_{i\in\mathcal{I}}w_i^2||\pi_{W_i^R}u||^2\Big)^{1/2}\\&+\lambda_2\Big(\sum_{i\in\mathcal{I}}w_i^2||\pi_{V_i^R}u||^2\Big)^{1/2}+\mu||K^*u||^2,u\in\mathbb{H}^R(\mathfrak{Q}).\end{aligned}$$

Then $\{(V_i^R,w_i)\}_{i\in\mathcal{I}}$ forms a K-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$ with bounds

$$r_1\Big(1-\frac{\lambda_1+\lambda_2+\frac{\mu}{\sqrt{r_1}}}{1+\lambda_2}\Big)^2 \text{ and } r_2\Big(1+\frac{\lambda_1+\lambda_2+\frac{\mu}{\sqrt{r_1}}}{1-\lambda_2}\Big)^2,\ \text{respectively.}$$

Proof. Since $r_1||K^*u||^2\leq\sum_{i\in\mathcal{I}}w_i^2||\pi_{W_i^R}u||^2$, implies $-||K^*u||\geq-\frac{1}{\sqrt{r_1}}\Big(\sum_{i\in\mathcal{I}}w_i^2||\pi_{W_i^R}u||^2\Big)^{1/2}$. Consider,

$$\begin{aligned}\Big(\sum_{i\in\mathcal{I}}w_i^2||\pi_{V_i^R}u||^2\Big)^{1/2}&=\Big(\sum_{i\in\mathcal{I}}w_i^2(||\pi_{V_i^R}u||^2-||\pi_{W_i^R}u||^2+||\pi_{W_i^R}u||^2)\Big)^{1/2}\\&\geq\Big(\sum_{i\in\mathcal{I}}w_i^2||\pi_{W_i^R}u||^2\Big)^{1/2}-\Big(\sum_{i\in\mathcal{I}}w_i^2||(\pi_{W_i^R}-\pi_{V_i^R})u||^2\Big)^{1/2}\\&\geq(1-\lambda_1-\frac{\mu}{\sqrt{r_1}})\Big(\sum_{i\in\mathcal{I}}w_i^2||\pi_{W_i^R}u||^2\Big)^{1/2}-\lambda_2\Big(\sum_{i\in\mathcal{I}}w_i^2||\pi_{V_i^R}u||^2\Big)^{1/2}.\end{aligned}$$

Hence,

$$\Big(\sum_{i\in\mathcal{I}}w_i^2||\pi_{V_i^R}u||^2\Big)^{1/2}\geq\sqrt{r_1}\Big(1-\frac{\lambda_1+\lambda_2+\frac{\mu}{\sqrt{r_1}}}{1+\lambda_2}\Big)||K^*u||,\ u\in\mathbb{H}^R(\mathfrak{Q}).$$

For upper bound condition, consider

$$\begin{aligned}\Big(\sum_{i\in\mathcal{I}}w_i^2||\pi_{V_i^R}u||^2\Big)^{1/2}&\leq\Big(\sum_{i\in\mathcal{I}}w_i^2||\pi_{W_i^R}u||^2\Big)^{1/2}+\Big(\sum_{i\in\mathcal{I}}w_i^2||(\pi_{W_i^R}-\pi_{V_i^R})u||^2\Big)^{1/2}\\&\leq(1+\lambda_1+\frac{\mu}{\sqrt{r_1}})\Big(\sum_{i\in\mathcal{I}}w_i^2||\pi_{W_i^R}u||^2\Big)^{1/2}\\&+\lambda_2\Big(\sum_{i\in\mathcal{I}}w_i^2||\pi_{V_i^R}u||^2\Big)^{1/2},\ u\in\mathbb{H}^R(\mathfrak{Q}).\end{aligned}$$

Thus,

$$\Big(\sum_{i\in\mathcal{I}} w_i^2||\pi_{V_i^R}u||^2\Big)^{1/2} \leq \sqrt{r_2}\Big(1+\frac{\lambda_1+\lambda_2+\frac{\mu}{\sqrt{r_1}}}{1-\lambda_2}\Big)||u||,\ u\in\mathbb{H}^R(\mathfrak{Q}).$$

□

Next, we have defined and studied the concept of woven K-fusion frames of $\mathbb{H}^R(\mathfrak{Q})$.

Definition 3.17. Let, $\{(W_i^R, w_i)\}_{i\in\mathcal{I}}$, $\{(V_i^R, v_i)\}_{i\in\mathcal{I}}$ be the K-fusion frames of $\mathbb{H}^R(\mathfrak{Q})$. If there exists $0 < r_1 \leq r_2 < \infty$, such that for any subset $\sigma \subset \mathcal{I}$, the family $\{(W_i^R, w_i)\}_{i\in\sigma} \cup \{(V_i^R, v_i)\}_{i\in\sigma^c}$ form a K-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$, with bounds r_1, r_2 i.e.

$$r_1||K^*u||^2 \leq \sum_{i\in\sigma} w_i^2||\pi_{W_i^R}u||^2 + \sum_{i\in\sigma^c} v_i^2||\pi_{V_i^R}u||^2 \leq r_2||u||^2,\ u\in\mathbb{H}^R(\mathfrak{Q}).$$

Then we say given family of K-fusion frames forms a woven K-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$.

If only upper bound condition hold, we say the K-fusion frames $\{(W_i^R, w_i)\}_{i\in\mathcal{I}}$, $\{(V_i^R, v_i)\}_{i\in\mathcal{I}}$ forms a woven Bessel sequence of subspaces of $\mathbb{H}^R(\mathfrak{Q})$. In [20], we have defined and studied synthesis, analysis and frame operator corresponding to a woven fusion frames for details see [20].

Example 3.18. Let $\mathbb{H}^R(\mathfrak{Q})$ be a three dimensional right quaternionic Hilbert space with the orthonormal basis $\{e_i\}_{i=1}^3$, $v_i = 1$ and $w_i = 2\,\forall i \in \{1,2,3\}$. Let $W_i^R = H_i^R =$ span$\{e_i\}$. Let $K \in \mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ be a right linear operator such that

$$K(e_1) = e_1, K(e_2) = e_1 \text{ and } K(e_3) = e_2.$$

The adjoint operator K^* corresponding to K is

$$K^*(e_1) = e_1 + e_2,\ K^*(e_2) = e_3 \text{ and } K^*(e_3) = 0.$$

Therefore, for any $u \in \mathbb{H}^R(\mathfrak{Q})$ and for any subset $\sigma \subset \{1,2,3\}$, we have

$$\frac{1}{3}||K^*u||^2 \leq \sum_{i\in\sigma}|\langle e_i|u\rangle|^2 + \sum_{i\in\sigma^c} 4|\langle e_i|u\rangle|^2 = ||u||^2 + 3\sum_{i\in\sigma^c}\langle e_i|u\rangle|^2 \leq 4||u||^2,\ u\in\mathbb{H}^R(\mathfrak{Q}).$$

Thus $\{(\mathbb{H}^R(\mathfrak{Q}), v_i)\}_{i\in\mathcal{I}}$ and $\{({W_i}^R, w_i)\}_{i\in\mathcal{I}}$ form a woven K-fusion frame for $\mathbb{H}^R(\mathfrak{Q})$.

In the next result, we have studied the necessary and sufficient condition under which given two Bessel sequences of subspaces of $\mathbb{H}^R(\mathfrak{Q})$ form a woven K-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$.

Theorem 3.19. *Suppose $\{(W_i^R, w_i)\}_{i\in\mathcal{I}}$, $\{(V_i^R, v_i)\}_{i\in\mathcal{I}}$ are Bessel sequences of subspaces of $\mathbb{H}^R(\mathfrak{Q})$. The statements below are the same.*

(i) $\{(W_i^R, w_i)\}_{i\in\mathcal{I}}$ and $\{(V_i^R, v_i)\}_{i\in\mathcal{I}}$ form a woven K-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$.

(ii) There exists $\alpha > 0$ such that for every subset $\sigma \subset \mathcal{I}$, there exits a bounded right linear operator $U_\sigma : \Big(\sum_{i\in\mathcal{I}}\bigoplus X_i^R\Big)_{l^2(\mathfrak{Q})} \to \mathbb{H}^R(\mathfrak{Q})$, (where $X_i^R = W_i^R$ for $i \in \sigma$ and $X_i^R = V_i^R, i \in \sigma^c$) such that

$$U_\sigma(\{u_i\}_{i\in\mathcal{I}}) = \sum_{i\in\sigma}\pi_{W_i^R}(u_i)w_i + \sum_{i\in\sigma^c}\pi_{V_i^R}(u_i)v_i,$$

$$\text{where } \{u_i\}_{i\in\mathcal{I}} \in \left(\sum_{i\in\mathcal{I}} \bigoplus X_i^R\right)_{l^2(\mathfrak{Q})} \text{ and } \alpha KK^* \leq U_\sigma U_\sigma^*.$$

Proof. $(i) \implies (ii)$: Let r_1 be lower bound. For any $\sigma \subset \mathcal{I}$, let T_σ be the right synthesis operator corresponding to $\{(W_i^R w_i)\}_{i\in\sigma} \cup \{(V_i^R w_i)\}_{i\in\sigma^c}$. Choose $\alpha = r_1$, and $U_\sigma = T_\sigma$, then for any $\{u_i\}_{i\in\mathcal{I}} \in \left(\sum_{i\in\mathcal{I}} \bigoplus X_i^R\right)_{l^2(\mathfrak{Q})}$,

$$\begin{aligned}
U_\sigma(\{u_i\}_{i\in\mathcal{I}}) &= T_\sigma(\{u_i\}_{i\in\mathcal{I}}) \\
&= \sum_{i\in\sigma} u_i w_i + \sum_{i\in\sigma^c} u_i v_i \\
&= \sum_{i\in\sigma} \pi_{W_i^R}(u_i) w_i + \sum_{i\in\sigma^c} \pi_{V_i^R}(u_i) v_i.
\end{aligned}$$

Also,

$$\begin{aligned}
\alpha\langle KK^*u|u\rangle &= r_1||K^*u||^2 \\
&\leq ||T_\sigma^* u||^2 \\
&= ||U_\sigma^* u||^2,\ u \in \mathbb{H}^R(\mathfrak{Q}).
\end{aligned}$$

Implies $\alpha KK^* \leq U_\sigma U_\sigma^*$.
$(ii) \implies (i)$: The upper bound condition hold by Proposition 3.5. We can easily prove that the adjoint of U_σ is given by

$$U_\sigma^*(u) = \{\pi_{W_i^R}(u) w_i\}_{i\in\sigma} \cup \{\pi_{V_i^R}(u) v_i\}_{i\in\sigma^c},\ u \in \mathbb{H}^R(\mathfrak{Q}).$$

Since $\alpha KK^* \leq U_\sigma U_\sigma^*$, we have

$$\begin{aligned}
\alpha||K^*u||^2 &= \alpha\langle KK^*u|u\rangle \\
&\leq \langle U_\sigma U_\sigma^* u|u\rangle \\
&= ||U_\sigma^* u||^2 \\
&= \sum_{i\in\sigma} w_i^2||\pi_{W_i^R} u||^2 + \sum_{i\in\sigma^c} v_i^2||\pi_{V_i^R} u||^2,\ u \in \mathbb{H}^R(\mathfrak{Q}).
\end{aligned}$$

□

In the next result, we have studied the necessary and sufficient condition under given K-fusion frames of $\mathbb{H}^R(\mathfrak{Q})$ form a woven K-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$.

Theorem 3.20. *Suppose, $\{(W_i^R, w_i)\}_{i\in\mathcal{I}}$ and $\{(V_i^R, v_i)\}_{i\in\mathcal{I}}$ are K-fusion frames of $\mathbb{H}^R(\mathfrak{Q})$. Then, $\{(W_i^R, w_i)\}_{i\in\mathcal{I}}$ and $\{(V_i^R, v_i)\}_{i\in\mathcal{I}}$ forms a woven K-fusion frames of $\mathbb{H}^R(\mathfrak{Q})$ if and only if there exists $\alpha > 0$ such that for any $\sigma \subset \mathcal{I}$ and for any $u \in \mathbb{H}^R(\mathfrak{Q})$ there exists a sequence $\{u_i^\sigma\}_{i\in\mathcal{I}} \in \left(\sum_{i\in\mathcal{I}} \bigoplus X_i^R\right)_{l^2(\mathfrak{Q})}$, (where $X_i^R = W_i^R$ for $i \in \sigma$ and $X_i^R = V_i^R, i \in \sigma^c$), such that*

$$\left(\sum_{i\in\mathcal{I}} ||u_i^\sigma||^2\right)^{1/2} \leq \alpha||u|| \text{ and } Ku = \sum_{i\in\sigma} \pi_{W_i^R}(u_i^\sigma) w_i + \sum_{i\in\sigma^c} \pi_{V_i^R}(u_i^\sigma) v_i.$$

Proof. Let, r_1 and r_2 be the bounds corresponding to given woven K-fusion frame $\{(W_i^R, w_i)\}_{i\in\mathcal{I}}$ and $\{(V_i^R, v_i)\}_{i\in\mathcal{I}}$. For any subset $\sigma\subset\mathcal{I}$, let T_σ be right synthesis operator corresponding to K-fusion frame $\{(W_i^R, w_i)\}_{i\in\sigma}\cup\{(V_i^R, v_i)\}_{i\in\sigma^c}$. Then $r_1\langle KK^*u|u\rangle = r_1||K^*u||^2 \leq \sum_{i\in\sigma} w_i^2||\pi_{W_i^R}u||^2 + \sum_{i\in\sigma^c} v_i^2||\pi_{V_i^R}u||^2 = ||T^*u||^2$, $u\in\mathbb{H}^R(\mathfrak{Q})$. Thus, $r_1KK^*\leq T_\sigma T_{\sigma^c}^*$. By Lemma 2.3, there exists a bounded right linear operator

$U_\sigma : \mathbb{H}^R(\mathfrak{Q})\to\left(\sum\limits_{i\in\mathcal{I}}\bigoplus X_i^R\right)_{l^2(\mathfrak{Q})}$ such that $||U_\sigma||^2\leq\frac{1}{\sqrt{r_1}}$ and $K = T_\sigma U_\sigma$. Hence, for any $u\in\mathbb{H}^R(\mathfrak{Q})$, the sequence $U_\sigma(u) = \{u_i^\sigma\}_{i\in\mathcal{I}}\in\left(\sum\limits_{i\in\mathcal{I}}\bigoplus X_i^R\right)_{l^2(\mathfrak{Q})}$ be such that

$$\left(\sum_{i\in\mathcal{I}}||u_i^\sigma||^2\right)^{1/2} = ||U_\sigma u||\leq||U_\sigma||\;||u||\leq\frac{1}{r_1^{1/4}}||u||,$$

and

$$\begin{aligned} Ku &= T_\sigma U_\sigma u\\ &= T_\sigma(\{u_i^\sigma\}_{i\in\mathcal{I}})\\ &= \sum_{i\in\sigma}\pi_{W_i^R}(u_i^\sigma)w_i + \sum_{i\in\sigma^c}\pi_{V_i^R}(u_i^\sigma)v_i. \end{aligned}$$

Conversely, suppose the given condition holds. Then the upper bound condition follows from Proposition 3.5. For any $y\in\mathbb{H}^R(\mathfrak{Q})$,

$$\begin{aligned} ||K^*y||^2 &= \sup_{||u||=1}|\langle K^*y|u\rangle|^2\\ &= \sup_{||u||=1}|\langle y|Ku\rangle|^2\\ &= \sup_{||u||=1}|\langle y|\sum_{i\in\sigma}\pi_{W_i^R}(u_i^\sigma)w_i + \sum_{i\in\sigma^c}\pi_{V_i^R}(u_i^\sigma)v_i\rangle|^2\\ &= \sup_{||u||=1}|\sum_{i\in\sigma}\langle\pi_{W_i^R}(y)w_i|u_i^\sigma\rangle + \sum_{i\in\sigma^c}\pi_{V_i^R}(y)v_i|u_i^\sigma\rangle|^2\\ &\leq \sup_{||u||=1}\{|\sum_{i\in\sigma}\langle\pi_{W_i^R}(y)w_i|u_i^\sigma\rangle| + |\sum_{i\in\sigma^c}\pi_{V_i^R}(y)v_i|u_i^\sigma\rangle\}^2\\ &\leq \sup_{||u||=1}\{\sum_{i\in\sigma}w_i||\pi_{W_i^R}y||\;||u_i^\sigma|| + \sum_{i\in\sigma^c}v_i||\pi_{V_i^R}y||\;||u_i^\sigma||\}^2\\ &\leq \sup_{||u||=1}2\{(\sum_{i\in\sigma}w_i||\pi_{W_i^R}y||\;||u_i^\sigma||)^2 + (\sum_{i\in\sigma^c}v_i||\pi_{V_i^R}y||\;||u_i^\sigma||)^2\}\\ &\leq 2\sup_{||u||=1}\{\sum_{i\in\sigma}w_i^2||\pi_{W_i^R}y||^2||u_i^\sigma||^2 + \sum_{i\in\sigma^c}v_i^2||\pi_{V_i^R}y||^2||u_i^\sigma||^2\}\\ &\leq 2\sup_{||u||=1}\{\sum_{i\in\sigma}w_i^2||\pi_{W_i^R}y||^2\alpha^2||u||^2 + \sum_{i\in\sigma^c}v_i^2||\pi_{V_i^R}y||^2\alpha^2||u||^2\}\\ &\leq 2\alpha^2\{\sum_{i\in\sigma}w_i^2||\pi_{W_i^R}y||^2 + \sum_{i\in\sigma^c}v_i^2||\pi_{V_i^R}y||^2\}. \end{aligned}$$

□

In the next result, we have studied a condition under given K-fusion frames of $\mathbb{H}^R(\mathfrak{Q})$ form a woven K-fusion frame of $\mathbb{H}^R(\mathfrak{Q})$.

Theorem 3.21. *Suppose* $\{(W_i^R, w_i)\}_{i\in\mathcal{I}}$ *and* $\{(V_i^R, v_i)\}_{i\in\mathcal{I}}$ *are* K*-fusion frames of* $\mathbb{H}^R(\mathfrak{Q})$ *with lower and upper bounds* r_1, R_1 *and* r_2, R_2 *respectively. If there exist constants* $\mu \geq 0, 0 \leq \lambda < \frac{1}{2}$ *such that* $r_1(\frac{1}{2} - \lambda) - \mu > 0$ *and*

$$\sum_{i\in\mathcal{I}} ||(\pi_{W_i^R} w_i - \pi_{V_i^R} v_i)u||^2 \leq \lambda \sum_{i\in\mathcal{I}} w_i^2 ||\pi_{W_i^R} u||^2 + \mu ||K^* u||^2,\ u \in \mathbb{H}^R(\mathfrak{Q}).$$

Then $\{(W_i^R, w_i)\}_{i\in\mathcal{I}}$ *and* $\{(V_i^R, v_i)\}_{i\in\mathcal{I}}$ *form a woven* K*-fusion frames of* $\mathbb{H}^R(\mathfrak{Q})$.

Proof. The upper bound condition follows from Proposition 3.5. For any subset $\sigma \subset \mathcal{I}$, consider

$$\begin{aligned}
\sum_{i\in\sigma} w_i^2 ||\pi_{W_i^R} u||^2 + \sum_{i\in\sigma^c} v_i^2 ||\pi_{V_i^R} u||^2 &\geq \sum_{i\in\sigma} w_i^2 ||\pi_{W_i^R} u||^2 + \frac{1}{2}\sum_{i\in\sigma^c} w_i^2 ||\pi_{W_i^R} u||^2 \\
&- \sum_{i\in\sigma^c} ||w_i \pi_{W_i^R} u - v_i \pi_{V_i^R} u||^2 \\
&= \frac{1}{2}\sum_{i\in\mathcal{I}} w_i^2 ||\pi_{W_i^R} u||^2 + \frac{1}{2}\sum_{i\in\sigma} w_i^2 ||\pi_{W_i^R} u||^2 \\
&- \sum_{i\in\sigma^c} ||w_i \pi_{W_i^R} u - v_i \pi_{V_i^R} u||^2 \\
&\geq \frac{1}{2}\sum_{i\in\mathcal{I}} w_i^2 ||\pi_{W_i^R} u||^2 - \sum_{i\in\sigma^c} ||w_i \pi_{W_i^R} u - v_i \pi_{V_i^R} u||^2 \\
&\geq \frac{1}{2}\sum_{i\in\mathcal{I}} w_i^2 ||\pi_{W_i^R} u||^2 \\
&- \lambda \sum_{i\in\mathcal{I}} w_i^2 ||\pi_{W_i^R} u||^2 - \mu ||K^* u||^2 \\
&= (\frac{1}{2} - \lambda)\sum_{i\in\mathcal{I}} w_i^2 ||\pi_{W_i^R} u||^2 - \mu ||K^* u||^2 \\
&\geq \Big((\frac{1}{2} - \lambda) r_1 - \mu\Big) ||K^* u||^2,\ u \in \mathbb{H}^R(\mathfrak{Q}).
\end{aligned}$$

□

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