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CONTROLLED RECTANGULAR METRIC-LIKE SPACES AND FIXED POINTS OF GRAPH PRESERVING MAPPINGS

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Abstract. We introduce the concept of controlled rectangular metric-like spaces and analyze the existence of fixed points for graph preserving mappings in such spaces. As an application of our main result, we prove a coincidence point result which generalizes several recent results including fixed points of expansive mappings. We also illustrate our main result by presenting some nontrivial examples.

1. Introduction

Fixed point theory is an important branch of mathematics because of its applications in various fields of applied sciences. Banach contraction principle [4] in metric spaces plays a vital role in solving different existence and uniqueness problems in several fields of science and engineering. Many authors successfully generalized this remarkable theorem by modifying the contractive type condition or by weakening the underlying space(see [10, 11, 12, 14, 15, 18, 32, 34, 36, 37] and references therein). Among all these, Bakhtin [3] initiated the concept of b-metric spaces as a generalization of metric spaces and generalized the famous Banach contraction principle in such spaces. Later on, Kamran et al.[25] gave the idea of extended b-metric spaces as an extension of b-metric spaces. Very recently, Mlaiki et al.[27] introduced another new extension of b-metric spaces, called controlled metric type spaces. They also proved the corresponding Banach contraction principle and some other fixed point results in the setting of controlled metric type spaces(see [27, 28, 29, 30]).

The study of fixed point theory combining a graph is another new development in the domain of contractive type single-valued mappings. Starting from these considerations, the study of fixed points and coincidence points of mappings satisfying a certain

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contractive type condition endowed with a graph attracted many researchers, see for examples [5, 6, 7, 16, 17, 23, 31, 33, 35]. In this work, we first introduce the concept of controlled rectangular metric-like spaces, discuss some topological properties and then analyze the existence and uniqueness of fixed points for a graph preserving mapping in this new framework. We also present a coincidence point result by applying our main result. Our results extend, unify and generalize several well-known comparable results in the literature. Finally, we provide some examples to illustrate and justify the validity of our main result.

2. Some Basic Concepts

In this section, we present some basic notations, definitions and results needed in the sequel.

Definition 2.1. [3] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric on X if the following conditions hold:

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

The pair (X, d) is called a b-metric space.

Definition 2.2. [25] Let X be a nonempty set and $\theta : X \times X \to [1, \infty)$. An extended b-metric is a function $d : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$:

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \leq \theta(x, y) [d(x, z) + d(z, y)]$.

The pair (X, d) is called an extended b-metric space.

Definition 2.3. [27] Let X be a nonempty set and $\eta : X \times X \to [1, \infty)$. A function $d : X \times X \to [0, \infty)$ is called a controlled metric type if the following conditions hold:

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(iii) $d(x, y) \leq \eta(x, z) d(x, z) + \eta(z, y) d(z, y)$ for all $x, y, z \in X$.

The pair (X, d) is called a controlled metric type space.

It is worth mentioning that the class of controlled metric type spaces is effectively larger than that of the b-metric spaces [27].

Definition 2.4. [22] Let X be a nonempty set. A function $\sigma : X \times X \to [0, \infty)$ is said to be a dislocated metric (or a metric-like) on X if for any $x, y, z \in X$, the following conditions hold:

(σ_1) $\sigma(x, y) = 0 \Longrightarrow x = y$;
(σ_2) $\sigma(x, y) = \sigma(y, x)$;
(σ_3) $\sigma(x, y) \leq \sigma(x, z) + \sigma(z, y)$.

The pair (X, σ) is then called a dislocated metric (or metric-like) space.

It is valuable to note that a partial metric [26] is also a dislocated metric but the converse is not true, in general. A trivial example of a dislocated metric is given by $\sigma(x, y) = \max\{x, y\}$, for all $x, y \geq 0$.

The following example illustrates the above fact.

Example 2.5. [2] Let $X = \{1, 2, 3\}$ and consider the dislocated metric $\sigma : X \times X \to [0, \infty)$ given by

$$\sigma(1,1) = 0,\ \sigma(2,2) = 1,\ \sigma(3,3) = \frac{2}{3},$$

$$\sigma(1,2) = \sigma(2,1) = \frac{9}{10},\ \sigma(2,3) = \sigma(3,2) = \frac{4}{5},$$

$$\sigma(1,3) = \sigma(3,1) = \frac{7}{10}.$$

Since $\sigma(2,2) \neq 0$, σ is not a metric and since $\sigma(2,2) > \sigma(1,2)$, σ is not a partial metric.

Definition 2.6. [9] Let X be a nonempty set and $d : X \times X \to [0, \infty)$ be a function such that

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x,\ y \in X$;
(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ for all $x,\ y \in X$ and all distinct points $u,\ v \in X \setminus \{x,\ y\}$.

Then d is called a rectangular metric on X and (X, d) is called a rectangular metric space.

Definition 2.7. [19] Let X be a nonempty set and $d : X \times X \to [0, \infty)$ be a function such that

(i) $d(x, y) = 0 \Longrightarrow x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x,\ y \in X$;
(iii) there exists a real number $s \geq 1$ such that

$$d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$$

for all $x,\ y \in X$ and all points $u,\ v \in X \setminus \{x,\ y\}$.

Then d is called a dislocated rectangular b-metric on X and (X, d) is called a dislocated rectangular b-metric space with coefficient s.

We now introduce the concept of controlled rectangular metric-like on a nonempty set X.

Definition 2.8. Let X be a nonempty set and $\eta : X \times X \to [1, \infty)$. A function $d : X \times X \to [0, \infty)$ is called a controlled rectangular metric-like if the following conditions hold:

(i) $d(x, y) = 0 \Longrightarrow x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x,\ y \in X$;
(iii) $d(x, y) \leq \eta(x, u)d(x, u) + \eta(u, v)d(u, v) + \eta(v, y)d(v, y)$ for all $x,\ y \in X$ and all points $u,\ v \in X \setminus \{x,\ y\}$.

Then (X, d) is called a controlled rectangular metric-like space defined by η.

It is easy to observe that every dislocated rectangular b-metric is a controlled rectangular metric-like but not conversely (see Example 2.15).

In a controlled rectangular metric-like space (X, d), we define an open ball $B(x, r)$ for $x \in X$ and $r > 0$ as follows:

$$B(x,r) = \{y \in X : \mid d(x,y) - d(x,x) \mid < r\}.$$

Definition 2.9. Let (X, d) be a controlled rectangular metric-like space. A subset $U \subseteq X$ is said to be open if for every $x \in U$ there exists $r > 0$ such that $B(x, r) \subseteq U$.

A subset $V \subseteq X$ is said to be closed if $X \setminus V$ is open.
The family of all open subsets of X will be denoted by τ_{d_η}.

Theorem 2.10. *τ_{d_η} defines a topology on (X, d).*

Remark 2.11. Let (X, d) be a controlled rectangular metric-like space, (x_n) be a sequence in X and $x \in X$. Then (x_n) converges to x with respect to(henceforth, we write 'w.r.t.' in short for 'with respect to') τ_{d_η} if $\lim_{n\to\infty} d(x_n, x) = d(x, x)$.

Suppose that $\lim_{n\to\infty} d(x_n, x) = d(x, x)$. We shall show that $x_n \to x$ w.r.t. τ_{d_η}. Let $U \in \tau_{d_\eta}$ and $x \in U$. Then there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$. By hypothesis, there exists $n_0 \in \mathbb{N}$ such that $\mid d(x_n, x) - d(x, x) \mid < \epsilon$ for all $n \geq n_0$. This ensures that $x_n \in B(x, \epsilon)$ for all $n \geq n_0$ and hence $x_n \in U$ for all $n \geq n_0$. Therefore, (x_n) converges to x w.r.t. τ_{d_η} on X.

Definition 2.12. Let (X, d) be a controlled rectangular metric-like space and let (x_n) be a sequence in X. Then

(i) (x_n) converges to a point $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = d(x, x)$. This will be denoted as $\lim_{n\to\infty} x_n = x$ or $x_n \to x (n \to \infty)$.

(ii) (x_n) is called a Cauchy sequence if $\lim_{n,m\to\infty} d(x_n, x_m)$ exists and is finite.

(iii) (X, d) is said to be complete if for each Cauchy sequence (x_n) in X, there is some $x \in X$ such that $\lim_{n\to\infty} d(x_n, x) = d(x, x) = \lim_{n,m\to\infty} d(x_n, x_m)$.

Lemma 2.13. *A nonempty subset A of a controlled rectangular metric-like space (X, d) is closed if and only if the following statement holds:*
Every sequence $(x_n) \subseteq A$ converging to some point $x \in X$ implies that $x \in A$.

Proof. Suppose that A is a closed subset of X. Let (x_n) be a sequence in A such that $x_n \to x$ as $n \to \infty$. We shall show that $x \in A$. If possible, suppose that $x \notin A$. So $x \in X \setminus A$ and $X \setminus A$ is open. Then there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq X \setminus A$. Therefore, $B(x, \epsilon) \cap A = \emptyset$. Since $x_n \to x$ as $n \to \infty$, for $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mid d(x_n, x) - d(x, x) \mid < \epsilon$, for all $n \geq n_0$. This gives that $x_n \in B(x, \epsilon)$, for all $n \geq n_0$. Hence $x_n \in B(x, \epsilon) \cap A$, for all $n \geq n_0$, which leads to a contradiction with $B(x, \epsilon) \cap A = \emptyset$. So, $x \in A$.

Conversely, assume that the condition holds, i.e., for any sequence (x_n) in A which converges to x, we have $x \in A$. Let us prove that A is closed. In fact, we have to show that $X \setminus A$ is open. So for any $x \in X \setminus A$, we need to prove that there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq X \setminus A$, i.e., $B(x, \epsilon) \cap A = \emptyset$. If possible, suppose that for any $\epsilon > 0$, we have $B(x, \epsilon) \cap A \neq \emptyset$. So for any $n \geq 1$, choose $x_n \in B(x, \frac{1}{n}) \cap A$. Then $x_n \in A$ for all $n \geq 1$ and $\mid d(x_n, x) - d(x, x) \mid < \frac{1}{n}$ for all $n \geq 1$. Therefore, $\lim_{n\to\infty} d(x_n, x) = d(x, x)$. That is,

$x_n \to x$ as $n \to \infty$ in (X, d). Hence, by assumption $x \in A$, which is a contradiction. So for any $x \in X \setminus A$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq X \setminus A$, i.e., $X \setminus A$ is open and hence A is closed in X. $\square$

Definition 2.14. Let (X, d) be a controlled rectangular metric-like space and let (x_n) be a sequence in X. Then,

(i) (x_n) is called a 0-Cauchy sequence if

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$

(ii) (X, d) is said to be 0-complete if every 0-Cauchy sequence in X converges to a point $x \in X$ such that $d(x, x) = 0$.

It is to be noted that if a controlled rectangular metric-like space (X, d) is complete, then it is 0-complete. The converse assertion is not true, in general. The following example supports the above remark.

Example 2.15. Let us take $X = [0, \infty) \cap \mathbb{Q}$ and consider $d : X \times X \to [0, \infty)$ given by

$$d(x, y) = \begin{cases} |x-y|^2 + x + y, & \text{if } x \neq 0,\ y \neq 0, \\ \\ \frac{x+y}{1+x+y}, & \text{otherwise.} \end{cases}$$

We take $\eta(x, y) = 3(1 + x + y)^2$ for all $x,\ y \in X$.

Then, conditions (i) and (ii) of Definition 2.8 are obvious. We now verify condition (iii) of Definition 2.8. We have to consider the following cases:

Case 1: Suppose that $x \neq 0,\ y \neq 0$ and $u,\ v \in X\backslash\{x,\ y\}$.

(a) If $u \neq 0,\ v \neq 0$ then

$$\begin{aligned} d(x, y) &= |x-y|^2 + x + y \\ &= |x-u+u-v+v-y|^2 + x + y \\ &\leq 3[|x-u|^2 + |u-v|^2 + |v-y|^2] + x + y \\ &< 3(1+x+u)^2\{|x-u|^2 + x + u\} + 3(1+u+v)^2\{|u-v|^2 + u + v\} \\ &\quad +3(1+v+y)^2\{|v-y|^2 + v + y\} \\ &= \eta(x, u)d(x, u) + \eta(u, v)d(u, v) + \eta(v, y)d(v, y). \end{aligned}$$

(b) If $u \neq 0,\ v = 0$ then

$$\begin{aligned} d(x, y) &= |x-y|^2 + x + y \\ &= |x-u+u-v+v-y|^2 + x + y \\ &\leq 3[|x-u|^2 + |u-0|^2 + |0-y|^2] + x + y \\ &< 3(1+x+u)^2\{|x-u|^2 + x + u\} + 3(1+u)^2\frac{u}{1+u} + 3(1+y)^2\frac{y}{1+y} \\ &= \eta(x, u)d(x, u) + \eta(u, 0)d(u, 0) + \eta(0, y)d(0, y). \end{aligned}$$

(c) The case $u = 0,\ v \neq 0$ can be treated similarly as that of (b).

(d) If $u=0$, $v=0$ then

$$\begin{aligned} d(x,y) &= |x-y|^2+x+y \\ &\leq 3[x^2+y^2]+x+y \\ &< 3(1+x)^2\frac{x}{1+x}+3(1+y)^2\frac{y}{1+y} \\ &= \eta(x,0)d(x,0)+\eta(0,0)d(0,0)+\eta(0,y)d(0,y). \end{aligned}$$

Case 2: Suppose that $x=0$, $y\neq 0$ and $u,\, v\in X\backslash\{0,\, y\}$. Then

$$\begin{aligned} d(x,y) &= \frac{y}{1+y} \\ &< y<|v-y|^2+v+y=d(v,y) \\ &< \eta(0,u)d(x,u)+\eta(u,v)d(u,v)+\eta(v,y)d(v,y). \end{aligned}$$

Case 3: The case $x\neq 0$, $y=0$ and $u,\, v\in X\backslash\{0,\, x\}$ is similar to that of *Case* 2.
Case 4: The case $x=0, y=0$ is trivial.

Therefore, (X,d) is a controlled rectangular metric like space. But (X,d) is not a dislocated rectangular b-metric space. If possible, suppose that (X,d) is a dislocated rectangular b-metric space with coefficient $s\geq 1$. Then for all $x,\, y\in X$ and all points $u,v\in X\backslash\{x,\, y\}$,

$$d(x,y)\leq s[d(x,u)+d(u,v)+d(v,y)].$$

For $y>0$, $u=v=0$, we have

$$d(y,y+1)\leq s\big[d(y,0)+d(0,0)+d(0,y+1)\big].$$

That is,

$$2y+2\leq s\left[\frac{y}{1+y}+\frac{y+1}{y+2}\right].$$

Letting $y\to\infty$, we get $\infty\leq 2s$, which is a contradiction.

We now verify that (X,d) is 0-complete, but it is not complete.

Let (x_n) be a 0-Cauchy sequence in (X,d). Then,

$$\lim_{n,m\to\infty} d(x_n,x_m)=0.$$

This gives that $\lim\limits_{n,m\to\infty}[|x_n-x_m|^2+x_n+x_m]=0$, if $x_m,\, x_n\neq 0$ and $\lim\limits_{n,m\to\infty}\dfrac{x_n+x_m}{1+x_n+x_m}=0$, otherwise.
If $x_m,\, x_n\neq 0$, then

$$0\leq x_n\leq |x_n-x_m|^2+x_n+x_m.$$

Taking limit as $n,\, m\to\infty$, we have

$$\lim_{n\to\infty} x_n=0.$$

Otherwise,

$$\lim_{n\to\infty}\frac{x_n}{1+x_n}=0.$$

Thus, in either case $\lim\limits_{n\to\infty} \dfrac{x_n}{1+x_n} = 0$. It is easy to compute that

$$0 \le |d(x_n, 0) - d(0,0)| = \frac{x_n}{1+x_n}.$$

Taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} |d(x_n, 0) - d(0,0)| = 0, \ i.e., \ \lim_{n\to\infty} d(x_n, 0) = d(0,0).$$

Therefore, every 0-Cauchy sequence (x_n) converges to $0 \in X$ with $d(0,0) = 0$ and hence (X, d) is 0-complete.

We now show that (X, d) is not complete.

Let us consider the sequence (x_n), where $x_n = (1 + \frac{1}{n})^n$ in (X, d). As $(1 + \frac{1}{n})^n \to e$ in $\mathbb{R}$, we have $\lim\limits_{n\to\infty} |x_n - e| = 0$. Then,

$$\begin{aligned} |d(x_n, x_m) - 2e| &= ||x_n - x_m|^2 + x_n + x_m - 2e| \\ &= ||(x_n - e) - (x_m - e)|^2 + (x_n - e) + (x_m - e)| \\ &\le [| \, x_n - e \, | + | \, x_m - e \, |]^2 + | \, x_n - e \, | + | \, x_m - e \, | \\ &\to 0 \text{ as } n, m \to \infty. \end{aligned}$$

This implies that, $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 2e$ exists and is finite. Therefore, $\{(1 + \frac{1}{n})^n\}$ is a Cauchy sequence in (X, d). But $\{(1 + \frac{1}{n})^n\}$ does not converge in (X, d). So, (X, d) is not complete.

Moreover, the sequence (x_n) with $x_n = (1+\frac{1}{n})^n$ for each $n \in \mathbb{N}$ is a Cauchy sequence in (X, d) without being a 0-Cauchy sequence.

We next review some basic notions in graph theory.

Let (X, d) be a controlled rectangular metric-like space. We assign a digraph G with the set of vertices $V(G) = X$ and the set $E(G)$ of its edges contains all the loops, i.e., $\Delta \subseteq E(G)$ where $\Delta = \{(x, x) : x \in X\}$. We assume that G has no parallel edges and we can identify G with the pair $(V(G), E(G))$. Let G^{-1} be the graph obtained from G by reversing the direction of edges, i.e., $E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}$. Let $\tilde{G}$ denote the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\tilde{G}$ as a digraph for which the set of its edges is symmetric. Under this convention,

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

Our graph theory notations and terminology are standard and can be found in all graph theory books, like [8, 13, 20].

Definition 2.16. Let (X, d) be a controlled rectangular metric-like space and let $G = (V(G), E(G))$ be a graph. Then the mapping $f : X \to X$ is called edge preserving if

$$x, \, y \in X, \ (x, y) \in E(\tilde{G}) \Rightarrow (fx, fy) \in E(\tilde{G}).$$

Definition 2.17. Let (X, d) be a controlled rectangular metric-like space with a graph $G = (V(G), E(G))$ and let $f, g : X \to X$ be two mappings. Then f is called edge preserving w.r.t. g if

$$x, y \in X, \ (gx, gy) \in E(\tilde{G}) \Rightarrow (fx, fy) \in E(\tilde{G}).$$

3. Main Results

Before presenting our results, we state a property of the graph G, call it property $(*)$.

Property $(*)$**:** If (gx_n) is a sequence in a controlled rectangular metric-like space (X, d) endowed with a digraph G such that $d(gx_n, x) \to 0$ and $(gx_n, gx_{n+1}) \in E(\tilde{G})$ for all $n \geq 1$, then there exists a subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, x) \in E(\tilde{G})$ for all $i \geq 1$.

Taking $g = I$, the identity map on X, the above property reduces to the property $(*')$ which may be stated as follows:

Property $(*')$**:** If (x_n) is a sequence in a controlled rectangular metric-like space (X, d) endowed with a digraph G such that $d(x_n, x) \to 0$ and $(x_n, x_{n+1}) \in E(\tilde{G})$ for all $n \geq 1$, then there exists a subsequence (x_{n_i}) of (x_n) such that $(x_{n_i}, x) \in E(\tilde{G})$ for all $i \geq 1$.

If $(X, d, \preceq)$ is a partially ordered controlled rectangular metric-like space, then by taking $G = G_2$, where the graph G_2 is defined by $E(G_2) = \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\}$, the last property reduces to the property $(\dagger)$ which can be stated as follows:

Property $(\dagger)$**:** If (x_n) is a sequence in a partially ordered controlled rectangular metric-like space $(X, d, \preceq)$ such that $d(x_n, x) \to 0$ and x_n, x_{n+1} are comparable for all $n \geq 1$, then there exists a subsequence (x_{n_i}) of (x_n) such that x_{n_i}, x are comparable for all $i \geq 1$.

Theorem 3.1. *Let (X, d) be a 0-complete controlled rectangular metric-like space defined by the control function $\eta : X \times X \to [1, \infty)$ and let $G = (V(G), E(G))$ be a graph. Let $f : X \to X$ be such that*

$$d(fx, fy) \leq kd(x, y) \tag{3.1}$$

for all $x, y \in X$ with $(x, y) \in E(G)$, where $0 < k < 1$. Suppose that there exists $x_0 \in X$ such that $(x_0, fx_0) \in E(\tilde{G})$ and for $x_n = f^n x_0$, $n = 1, 2, 3, \cdots$, $\eta(x_n, x_n) \leq \eta(x_n, x_{n+1})$ and

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\eta(x_{i+1}, x_{i+2})}{\eta(x_i, x_{i+1})} \eta(x_{i+1}, x_m) < \frac{1}{k}. \tag{3.2}$$

Also, assume that if $\lim_{n \to \infty} d(x_n, x) = 0$ for some $x \in X$ then $\lim_{n \to \infty} \eta(x, x_n)$ and $\lim_{n \to \infty} \eta(x_n, fx)$ both exist and are finite. If f is edge preserving and the graph G has the property $(*')$, then f has a fixed point u(say) in X with $d(u, u) = 0$. Moreover, f has a unique fixed *point in X if the graph G has the following property:*

$(**)$ *If x, y are fixed points of f in X, then $(x,y) \in E(\tilde{G})$.*

Proof. We first note that condition (3.1) holds for all $x, y \in X$ with $(x,y) \in E(\tilde{G})$, as d is symmetric. Suppose that there exists $x_0 \in X$ such that $(x_0, fx_0) \in E(\tilde{G})$. We construct a sequence (x_n) such that $x_n = fx_{n-1}$, $n = 1, 2, 3, \cdots$. If $x_{n-1} = x_n$ for some $n \in \mathbb{N}$, then $x_{n-1} = fx_{n-1}$ and hence x_{n-1} becomes a fixed point of f. Therefore, we assume that, $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$.

As f is edge preserving and $(x_0, x_1) \in E(\tilde{G})$, it follows that $(fx_0, fx_1) \in E(\tilde{G})$ i.e., $(x_1, x_2) \in E(\tilde{G})$. Continuing this process, we can show that $(x_n, x_{n+1}) \in E(\tilde{G})$ for all $n \in \mathbb{N} \cup \{0\}$.

For all $n \in \mathbb{N}$, it follows from condition (3.1) that

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1) \tag{3.3}$$

and

$$d(x_n, x_n) \leq k^n d(x_0, x_0). \tag{3.4}$$

We claim that $x_m \neq x_n$ for all $m, n \in \mathbb{N}$ with $m \neq n$. If possible, suppose that $x_m = x_n$ for some $m, n \in \mathbb{N}$ with $m > n$. Let us put $m - n = p \in \mathbb{N}$. As $x_n \neq x_{n+1}$, it follows that $d(x_n, x_{n+1}) \neq 0$. Now,

$$\begin{aligned} d(x_n, x_{n+1}) = d(x_n, fx_n) &= d(x_m, fx_m) \\ &= d(x_m, x_{m+1}) \\ &= d(x_{n+p}, x_{n+p+1}) \\ &\leq k^p\, d(x_n, x_{n+1}). \end{aligned}$$

This gives that $k^p \geq 1$, a contradiction since $0 < k < 1$. Thus, our claim is justified.

We shall show that (x_n) is a 0-Cauchy sequence in X. In fact, we shall show that $\lim\limits_{n\to\infty} d(x_n, x_{n+p}) = 0$ *for all* $p \in \mathbb{N}$.

Case − I : Suppose p is even, that is, $p = 2m$ for some $m(> 1) \in \mathbb{N}$. Then by using conditions (3.3) and (3.4), we get

$$\begin{aligned}
d(x_n, x_{n+p}) &\leq \eta(x_n, x_{n+1})d(x_n, x_{n+1}) + \eta(x_{n+1}, x_{n+2})d(x_{n+1}, x_{n+2}) \\
&\quad +\eta(x_{n+2}, x_{n+2m})d(x_{n+2}, x_{n+2m}) \\
&\leq \eta(x_n, x_{n+1})d(x_n, x_{n+1}) + \eta(x_{n+1}, x_{n+2})d(x_{n+1}, x_{n+2}) \\
&\quad +\eta(x_{n+2}, x_{n+2m})[\eta(x_{n+2}, x_{n+3})d(x_{n+2}, x_{n+3}) \\
&\quad +\eta(x_{n+3}, x_{n+4})d(x_{n+3}, x_{n+4})] \\
&\quad +\eta(x_{n+2}, x_{n+2m})\eta(x_{n+4}, x_{n+2m})d(x_{n+4}, x_{n+2m}) \\
&\leq \eta(x_n, x_{n+1})d(x_n, x_{n+1}) + \eta(x_{n+1}, x_{n+2})d(x_{n+1}, x_{n+2}) \\
&\quad +\eta(x_{n+2}, x_{n+2m})[\eta(x_{n+2}, x_{n+3})d(x_{n+2}, x_{n+3}) \\
&\quad +\eta(x_{n+3}, x_{n+4})d(x_{n+3}, x_{n+4})] \\
&\quad +\eta(x_{n+2}, x_{n+2m})\eta(x_{n+4}, x_{n+2m})[\eta(x_{n+4}, x_{n+5})d(x_{n+4}, x_{n+5}) \\
&\quad +\eta(x_{n+5}, x_{n+6})d(x_{n+5}, x_{n+6})] \\
&\quad +\eta(x_{n+2}, x_{n+2m})\eta(x_{n+4}, x_{n+2m})\eta(x_{n+6}, x_{n+2m})d(x_{n+6}, x_{n+2m}).
\end{aligned}$$

Continuing in this way, the last term of the above computation at $(m-1)$-th step should be

$$\eta(x_{n+2}, x_{n+2m})\eta(x_{n+4}, x_{n+2m}) \cdots \eta(x_{n+2m-2}, x_{n+2m})d(x_{n+2m-2}, x_{n+2m}).$$

This can be further computed as follows:

$$\begin{aligned}
d(x_{n+2m-2}, x_{n+2m}) &\leq \eta(x_{n+2m-2}, x_{n+2m-1})d(x_{n+2m-2}, x_{n+2m-1}) \\
&\quad +\eta(x_{n+2m-1}, x_{n+2m-1})d(x_{n+2m-1}, x_{n+2m-1}) \\
&\quad +\eta(x_{n+2m-1}, x_{n+2m})d(x_{n+2m-1}, x_{n+2m}).
\end{aligned}$$

Therefore, finally we obtain by using $\eta(x, y) \geq 1$ and $\eta(x_n, x_n) \leq \eta(x_n, x_{n+1})$ that

$$\begin{aligned}
d(x_n, x_{n+p}) &\leq \eta(x_n, x_{n+1})k^n d(x_0, x_1) + \eta(x_{n+1}, x_{n+2})k^{n+1}d(x_0, x_1) \\
&\quad + \sum_{i=n+2}^{n+2m-1} \Big(\prod_{j=n+2}^{i} \eta(x_j, x_{n+p})\Big)\eta(x_i, x_{i+1})k^i d(x_0, x_1) \\
&\quad + \prod_{j=n+2}^{n+2m-2} \eta(x_j, x_{n+p})\eta(x_{n+2m-1}, x_{n+2m-1})k^{n+2m-1}d(x_0, x_0) \\
&\leq \eta(x_n, x_{n+1})k^n d(x_0, x_1) + \eta(x_{n+1}, x_{n+2})k^{n+1}d(x_0, x_1) \\
&\quad + \sum_{i=n+2}^{n+2m-1} \Big(\prod_{j=0}^{i} \eta(x_j, x_{n+p})\Big)\eta(x_i, x_{i+1})k^i d(x_0, x_1) \\
&\quad + \prod_{j=0}^{n+2m-1} \eta(x_j, x_{n+p})\eta(x_{n+2m-1}, x_{n+2m})k^{n+2m-1}d(x_0, x_0).
\end{aligned} \tag{3.5}$$

Let us put

$$\Upsilon_r = \sum_{i=0}^{r} \Big(\prod_{j=0}^{i} \eta(x_j, x_{n+p})\Big)\eta(x_i, x_{i+1})k^i.$$

Then, condition (3.5) reduces to

$$\begin{aligned} d(x_n, x_{n+p}) \leq\ & \eta(x_n, x_{n+1})k^n d(x_0, x_1) + \eta(x_{n+1}, x_{n+2})k^{n+1} d(x_0, x_1) \\ & + d(x_0, x_1)[\Upsilon_{n+2m-1} - \Upsilon_{n+1}] \\ & + \prod_{j=0}^{n+2m-1} \eta(x_j, x_{n+p})\eta(x_{n+2m-1}, x_{n+2m})k^{n+2m-1} d(x_0, x_0). \end{aligned} \tag{3.6}$$

Condition (3.2) and the ratio test for series convergence together assure that the following series is convergent:

$$\sum_{i=0}^{\infty} \left(\prod_{j=0}^{i} \eta(x_j, x_{n+p}) \right) \eta(x_i, x_{i+1})\, k^i.$$

Thus, $\lim_{n\to\infty} \Upsilon_n$ exists and the sequence of real numbers (Υ_n) is Cauchy. As $\eta(x, y) \geq 1$, it follows that

$$\eta(x_i, x_{i+1})\, k^i \leq \left(\prod_{j=0}^{i} \eta(x_j, x_{n+p}) \right) \eta(x_i, x_{i+1})\, k^i.$$

The above series being convergent, we have

$$\lim_{i\to\infty} \eta(x_i, x_{i+1})\, k^i = 0.$$

Taking limit as $n \to \infty$, it follows from condition (3.6) that

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = 0.$$

If $m = 1$, then

$$\begin{aligned} d(x_n, x_{n+p}) &= d(x_n, x_{n+2}) \\ &\leq \eta(x_n, x_{n+1})d(x_n, x_{n+1}) + \eta(x_{n+1}, x_{n+1})d(x_{n+1}, x_{n+1}) \\ &\quad + \eta(x_{n+1}, x_{n+2})d(x_{n+1}, x_{n+2}) \\ &\leq \eta(x_n, x_{n+1})k^n d(x_0, x_1) + \eta(x_{n+1}, x_{n+2})k^{n+1} d(x_0, x_0) \\ &\quad + \eta(x_{n+1}, x_{n+2})k^{n+1} d(x_0, x_1) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus,

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = 0 \text{ for all even } p \in \mathbb{N}. \tag{3.7}$$

Case − II : Suppose p is odd, that is, $p = 2m - 1$ for some $m \in \mathbb{N}$.
If $m = 1$, then

$$d(x_n, x_{n+p}) = d(x_n, x_{n+1}) \leq k^n d(x_0, x_1) \to 0 \text{ as } n \to \infty.$$

If $m > 1$, then proceeding similarly to that of Case-I, we obtain that

$$\begin{aligned}
d(x_n, x_{n+p}) &= d(x_n, x_{n+2m-1}) \\
&\leq \eta(x_n, x_{n+1})d(x_n, x_{n+1}) + \eta(x_{n+1}, x_{n+2})d(x_{n+1}, x_{n+2}) \\
&\quad + \eta(x_{n+2}, x_{n+2m-1})d(x_{n+2}, x_{n+2m-1}) \\
&\vdots \\
&\leq \eta(x_n, x_{n+1})d(x_n, x_{n+1}) + \eta(x_{n+1}, x_{n+2})d(x_{n+1}, x_{n+2}) \\
&\quad + \sum_{i=n+2}^{n+2m-2} \Big(\prod_{j=n+2}^{i} \eta(x_j, x_{n+p}) \Big) \eta(x_i, x_{i+1}) k^i d(x_0, x_1) \\
&\leq \eta(x_n, x_{n+1})d(x_n, x_{n+1}) + \eta(x_{n+1}, x_{n+2})d(x_{n+1}, x_{n+2}) \\
&\quad + \sum_{i=n+2}^{n+2m-2} \Big(\prod_{j=0}^{i} \eta(x_j, x_{n+p}) \Big) \eta(x_i, x_{i+1}) k^i d(x_0, x_1) \\
&\leq \eta(x_n, x_{n+1})\, k^n d(x_0, x_1) + \eta(x_{n+1}, x_{n+2})\, k^{n+1} d(x_0, x_1) \\
&\quad + \sum_{i=n+2}^{n+2m-2} \Big(\prod_{j=0}^{i} \eta(x_j, x_{n+p}) \Big) \eta(x_i, x_{i+1}) k^i d(x_0, x_1) \\
&= \eta(x_n, x_{n+1})\, k^n d(x_0, x_1) + \eta(x_{n+1}, x_{n+2})\, k^{n+1} d(x_0, x_1) \\
&\quad + d(x_0, x_1)[\Upsilon_{n+p-1} - \Upsilon_{n+1}].
\end{aligned}$$

Since $\lim_{n\to\infty} \Upsilon_n$ exists, passing to the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = 0.$$

Thus,

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = 0 \text{ for all odd } p \in \mathbb{N}. \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = 0 \text{ for all } p \in \mathbb{N}.$$

Therefore, (x_n) is a 0-Cauchy sequence in (X, d). As (X, d) is 0-complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} d(x_n, u) = d(u, u) = 0.$$

Consequently, it follows that $\lim_{n\to\infty} \eta(u, x_n)$ and $\lim_{n\to\infty} \eta(x_n, fu)$ both exist and are finite. As $(x_n, x_{n+1}) \in E(\tilde{G})$ for all $n \geq 1$, by property $(\acute{*})$, there exists a subsequence (x_{n_i}) of (x_n) such that $(x_{n_i}, u) \in E(\tilde{G})$ for all $i \geq 1$.

We shall show that u is a fixed point of f.
We assume that $x_{n_i}, x_{n_i+1} \in X \setminus \{u, fu\}$ for large $i \in \mathbb{N}$. Because if $x_{n_i+1} = u$, then $x_{n_i+2} = fx_{n_i+1} = fu$ which implies that $d(u, fu) = d(x_{n_i+1}, x_{n_i+2}) \leq k^{n_i+1}d(x_0, x_1) \to 0$ as $i \to \infty$. This gives that $d(u, fu) = 0$. Consequently, it follows that $fu = u$. On

the other hand, if $x_{n_i+1} = fu$, then $d(u, fu) = d(u, x_{n_i+1}) \to 0$ as $i \to \infty$. Therefore, $d(u, fu) = 0$ and hence $fu = u$.

Now,

$$\begin{aligned} d(u, fu) &\leq \eta(u, x_{n_i})d(u, x_{n_i}) + \eta(x_{n_i}, x_{n_i+1})d(x_{n_i}, x_{n_i+1}) \\ &\quad + \eta(x_{n_i+1}, fu)d(x_{n_i+1}, fu) \\ &\leq \eta(u, x_{n_i})d(u, x_{n_i}) + \eta(x_{n_i}, x_{n_i+1})k^{n_i}d(x_0, x_1) \\ &\quad + \eta(x_{n_i+1}, fu)kd(x_{n_i}, u). \end{aligned}$$

Passing to the limit as $i \to \infty$, we have $d(u, fu) = 0$ and hence $u = fu$.

For uniqueness, we assume that $v \in X$ is another fixed point of f. By property $(**)$, we have $(u, v) \in E(\tilde{G})$. Then,

$$d(u, v) = d(fu, fv) \leq kd(u, v).$$

Since $0 < k < 1$, it follows that $d(u, v) = 0$ and so $u = v$. Hence, f has a unique fixed point u in X and $d(u, u) = 0$. □

Corollary 3.2. Let $(X, d, \preceq)$ be a partially ordered 0-complete controlled rectangular metric-like space defined by the control function $\eta : X \times X \to [1, \infty)$ and let $f : X \to X$ be such that

$$d(fx, fy) \leq kd(x, y)$$

for all $x, y \in X$ with $x \preceq y$ *or* $y \preceq x$, where $0 < k < 1$. Suppose that there exists $x_0 \in X$ such that x_0, fx_0 are comparable and for $x_n = f^n x_0$, $n = 1, 2, 3, \cdots$, $\eta(x_n, x_n) \leq \eta(x_n, x_{n+1})$ and

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\eta(x_{i+1}, x_{i+2})}{\eta(x_i, x_{i+1})} \eta(x_{i+1}, x_m) < \frac{1}{k}.$$

Also, assume that if $\lim_{n \to \infty} d(x_n, x) = 0$ for some $x \in X$ then $\lim_{n \to \infty} \eta(x, x_n)$ and $\lim_{n \to \infty} \eta(x_n, fx)$ both exist and are finite. If f maps comparable elements into comparable elements and the property (†) holds, then f has a fixed point u(say) in X with $d(u, u) = 0$. Moreover, f has a unique fixed point in X if the following property holds:

(‡) If x, y are fixed points of f in X, then x, y are comparable.

Proof. The proof can be obtained from Theorem 3.1 by taking $G = G_2$, where the graph G_2 is defined by $E(G_2) = \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\}$. □

Corollary 3.3. Let (X, d) be a 0-complete dislocated rectangular b-metric space with coefficient $s \geq 1$ and let $f : X \to X$ be a mapping. If there exists $k \in (0, \frac{1}{s})$ such that

$$d(fx, fy) \leq kd(x, y)$$

for all $x, y \in X$, then f has a unique fixed point u(say) in X with $d(u, u) = 0$.

Proof. The proof follows from Theorem 3.1 by considering $\eta(x, y) = s$ for all $x, y \in X$ and $G = G_0$, where G_0 is the complete graph $(X, X \times X)$. □

Remark 3.4. As every complete controlled rectangular metric-like space is 0-complete, the results of this study are obtained under the weaker assumption.

We now present an example to justify the validity of our main result. It should be noticed that the well known Banach contraction theorem cannot explain the existence of fixed point in the following example.

Example 3.5. Let $X = \{2\} \cup \{2 + \frac{1}{n} : n \in \mathbb{N}\} \cup \{2 - \frac{1}{n} : n \in \mathbb{N}\}$ and for $m, n, n_1 \in \mathbb{N}$, $n \neq n_1$, we define $d : X \times X \to [0, \infty)$ as

$$d(2,2) = 0,\ d(2 - \frac{1}{n}, 2 - \frac{1}{n}) = 3,\ d(2 + \frac{1}{n}, 2 + \frac{1}{n}) = 10,$$
$$d(2, 2 + \frac{1}{n}) = 7 = d(2 + \frac{1}{n}, 2),\ d(2 - \frac{1}{n}, 2) = 2 = d(2, 2 - \frac{1}{n}),$$
$$d(2 + \frac{1}{m}, 2 - \frac{1}{n}) = 18 = d(2 - \frac{1}{n}, 2 + \frac{1}{m}),$$
$$d(2 + \frac{1}{n}, 2 + \frac{1}{n_1}) = 11,\ d(2 - \frac{1}{n}, 2 - \frac{1}{n_1}) = \min\{n, n_1\}.$$

Then (X, d) is a 0-complete controlled rectangular metric like space with control function $\eta : X \times X \to [1, \infty)$ given as follows: For $r_1, r_2 \in \mathbb{N}$, we have

$$\eta(x,y) = \begin{cases} \max\{r_1, r_2\}, & \text{if } x = 2 \pm \frac{1}{r_1},\ y = 2 \pm \frac{1}{r_2},\ r_1, r_2 \neq 1 \text{ simultaneously}, \\ r_1, & \text{if } x = 2 \pm \frac{1}{r_1},\ y = 2 \text{ or } x = 2,\ y = 2 \pm \frac{1}{r_1},\ r_1 \neq 1, \\ 2, & \text{otherwise}. \end{cases}$$

We note that (X, d) is not a dislocated rectangular b-metric space. If possible, suppose that (X, d) is a dislocated rectangular b-metric space with coefficient $s \geq 1$. Then for all $x, y \in X$ and all points $u, v \in X \setminus \{x, y\}$,

$$d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)].$$

Taking $x = 2 - \frac{1}{r}$, $y = 2 - \frac{1}{r+1}$, $r > 1$ and $u = v = 2$, we get

$$d(2 - \frac{1}{r}, 2 - \frac{1}{r+1}) \leq s\Big[d(2 - \frac{1}{r}, 2) + d(2, 2) + d(2, 2 - \frac{1}{r+1})\Big].$$

That is,

$$\min\{r, r+1\} = r \leq s\big[2 + 0 + 2\big].$$

Letting $r \to \infty$, we get $\infty \leq 4s$, which is a contradiction.

Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(2, 2 + \frac{1}{n}) : n \in \mathbb{N}\} \cup \{(2, 2 - \frac{1}{n}) : n \in \mathbb{N}\}$. We define $f : X \to X$ as

$$f(x) = \begin{cases} 2, & \text{if } x \in \{2 - \frac{1}{n} : n \in \mathbb{N}\}, \\ 1, & \text{if } x = 3, \\ \frac{x+4}{x+1}, & \text{otherwise}. \end{cases}$$

We see that f does not satisfy Banach contraction theorem with respect to the usual metric d_u defined on X. In fact, for $x = 2$ and $y = 3$, we have $d_u(fx, fy) = d_u(f2, f3) = d_u(2, 1) = 1$ and $d_u(x, y) = d_u(2, 3) = 1$. So, $d_u(fx, fy) = d_u(x, y) > kd_u(x, y)$ for all $0 < k < 1$. Hence, Banach contraction theorem with respect to usual metric cannot be applied to obtain fixed points of f.

We need to verify condition (3.1) for $(x,y) \in E(G)$. We have to consider the following cases:

(1) If $(x,y) = (2,2)$ then $d(fx,fy) = 0 = \frac{1}{3}d(x,y)$.
(2) If $(x,y) = (3,3)$ then $d(fx,fy) = d(1,1) = 3 < \frac{10}{3} = \frac{1}{3}d(x,y)$.
(3) If $(x,y) = (2+\frac{1}{n}, 2+\frac{1}{n})$, $n \geq 2$ then
$d(fx,fy) = d(2-\frac{1}{3n+1}, 2-\frac{1}{3n+1}) = 3 < \frac{10}{3} = \frac{1}{3}d(x,y)$.
(4) If $(x,y) = (2-\frac{1}{n}, 2-\frac{1}{n})$ then $d(fx,fy) = d(2,2) = 0 < \frac{1}{3}d(x,y)$.
(5) If $(x,y) = (2,3)$ then $d(fx,fy) = d(2,1) = 2 < \frac{7}{3} = \frac{1}{3}d(x,y)$.
(6) If $(x,y) = (2, 2+\frac{1}{n})$, $n \geq 2$ then
$d(fx,fy) = d(2, 2-\frac{1}{3n+1}) = 2 < \frac{7}{3} = \frac{1}{3}d(x,y)$.
(7) If $(x,y) = (2, 2-\frac{1}{n})$ then $d(fx,fy) = d(2,2) = 0 < \frac{1}{3}d(x,y)$.

Thus, condition (3.1) of Theorem 3.1 holds true for $k = \frac{1}{3}$.

We note that f is edge preserving and any sequence (x_n) in X with $d(x_n, x) \to 0$ and $(x_n, x_{n+1}) \in E(\tilde{G})$ for all $n \geq 1$ must be a constant sequence. In fact, $x_n = 2$ for all $n \in \mathbb{N}$. So property $(\acute{*})$ of the graph G is obvious. Also there exists $x_0 = 2 - \frac{1}{n}$ such that $(x_0, fx_0) \in E(\tilde{G})$, $x_n = fx_{n-1} = 2$ for all $n \in \mathbb{N}$ and

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\eta(x_{i+1}, x_{i+2})}{\eta(x_i, x_{i+1})}\eta(x_{i+1}, x_m) = 2 < \frac{1}{k} = 3,$$

$$\eta(x_n, x_n) \leq \eta(x_n, x_{n+1}) \text{ for all } n \in \mathbb{N}.$$

In addition, if $d(x_n, x) \to 0$ then $\lim\limits_{n\to\infty} \eta(x_n, fx)$ and $\lim\limits_{n\to\infty} \eta(x, x_n)$ exist and are finite. Furthermore, the graph G has the property $(**)$. Thus, all the conditions of Theorem 3.1 are satisfied. Now Theorem 3.1 can be applied to obtain a unique fixed point $2 \in X$ of f with $d(2,2) = 0$.

The following example shows that Theorem 3.1 cannot assure unique fixed point without property $(**)$ of the graph G.

Example 3.6. Let us take $X = [3,4]$ and define $d : X \times X \to [0,\infty)$ as follows:

For n, n_1, n_2, $r \in \mathbb{N}$, $n_1 \neq n_2$, r is not divisible by 3 and $x \in (3,4)\backslash\{3+\frac{1}{3n} : n \in \mathbb{N}\}$, $y \neq z$, y, $z \in (3,4]\backslash\{3+\frac{1}{3n} : n \in \mathbb{N}\}$,

$$d(3,3) = 0 = d(4,4),\ d(x,x) = 2,\ d(3+\frac{1}{3r}, 3+\frac{1}{3r}) = 9,\ d(3+\frac{1}{9n}, 3+\frac{1}{9n}) = 2,$$
$$d(3, 3+\frac{1}{3r}) = 14 = d(3+\frac{1}{3r}, 3),\ d(3, 3+\frac{1}{9n}) = 3 = d(3+\frac{1}{9n}, 3),$$
$$d(z,y) = 9 = d(y,z),\ d(y,3) = 4 = d(3,y),$$
$$d(3+\frac{1}{3n}, y) = 15 = d(y, 3+\frac{1}{3n}),$$
$$d(3+\frac{1}{3n_1}, 3+\frac{1}{3n_2}) = 3\min\{n_1, n_2\}.$$

For $s,\, t \in \mathbb{N}$, we define $\eta : X \times X \to [1, \infty)$ to be symmetric and

$$\eta(x,y) = \begin{cases} 3\max\{t,s\}, & \text{if } x = 3 + \frac{1}{3t},\ y = 3 + \frac{1}{3s}, \\ 3t, & \text{if } x \in [3,4]\backslash\{3 + \frac{1}{3n} : n \in \mathbb{N}\},\ y = 3 + \frac{1}{3t}, \\ 3, & \text{otherwise.} \end{cases}$$

Then,

$$d(x,y) \le \eta(x,u)d(x,u) + \eta(u,v)d(u,v) + \eta(v,y)d(v,y)$$

for all $x,\, y \in X$ and all points $u,\, v \in X\backslash\{x,\, y\}$.

Therefore, (X,d) is a controlled rectangular metric like space. Moreover, (X,d) is 0-complete. But (X,d) is not a dislocated rectangular b-metric space. If possible, suppose that (X,d) is a dislocated rectangular b-metric space with coefficient $s \ge 1$. Then for all $x,\, y \in X$ and all points $u,\, v \in X\backslash\{x,\, y\}$,

$$d(x,y) \le s[d(x,u) + d(u,v) + d(v,y)].$$

For $x = 3 + \frac{1}{3h}$, where $h \in \mathbb{N}$ is divisible by 3, $y = 3 + \frac{1}{3(h+1)}$ and $u = v = 3$, we have

$$d(3 + \frac{1}{3h}, 3 + \frac{1}{3(h+1)}) \le s\left[d(3 + \frac{1}{3h}, 3) + d(3,3) + d(3, 3 + \frac{1}{3(h+1)})\right].$$

That is,

$$3\min\{h, h+1\} = 3h \le s(3 + 0 + 14).$$

Letting $h \to \infty$, we get $\infty \le 17s$, which is a contradiction.

Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(3, 3 + \frac{1}{3n}) : n \in \mathbb{N}\}$. We define $f : X \to X$ as

$$fx = \begin{cases} \frac{x+6}{3}, & \text{if } x \in \{3 + \frac{1}{3n} : n \in \mathbb{N},\ n \text{ is not divisible by } 3\}, \\ 4, & \text{if } x = 4, \\ 3, & \text{otherwise.} \end{cases}$$

We observe that f is not a contraction operator with respect to the usual metric defined on it. In fact, for $x = \frac{10}{3}$ and $y = \frac{59}{18}$, we have $d(fx, fy) = d(\frac{28}{9}, 3) = \frac{1}{9}$ and $d(x,y) = d(\frac{10}{3}, \frac{59}{18}) = \frac{1}{18}$. So that $d(fx, fy) > kd(x,y)$ for all $0 < k < 1$. Hence, Banach contraction theorem with respect to the usual metric cannot be used to get fixed point of f.

We now verify condition (3.1) for $(x,y) \in E(G)$. The following cases to be considered:

(1) If $x = 3$ then $d(fx, fx) = d(3,3) = 0 = \frac{1}{4}d(x,x)$.

(2) If $x \in \{3 + \frac{1}{3n} : n \in \mathbb{N},\ n \text{ is not divisible by } 3\}$ then
$d(fx, fx) = d(3 + \frac{1}{9n}, 3 + \frac{1}{9n}) = 2 < \frac{9}{4} = \frac{1}{4}d(x,x)$.

(3) If $x \in \{3 + \frac{1}{9n} : n \in \mathbb{N}\}$ then
$d(fx, fx) = d(3,3) = 0 < \frac{1}{2} = \frac{1}{4}d(x,x)$.

(4) If $x = 4$ then $d(fx, fx) = d(4,4) = 0 = \frac{1}{4}d(x,x)$.

(5) If $x \in (3,4)\backslash\{3 + \frac{1}{3n} : n \in \mathbb{N}\}$ then
$d(fx, fx) = d(3,3) = 0 < \frac{1}{2} = \frac{1}{4}d(x,x)$.

(6) If $(x, y) \in \{(3, 3 + \frac{1}{3n}) : n \in \mathbb{N}, n$ is not divisible by 3$\}$ then $d(fx, fy) = d(3, 3 + \frac{1}{9n}) = 3 < \frac{14}{4} = \frac{1}{4}d(x, y)$.
(7) If $(x, y) \in \{(3, 3 + \frac{1}{9n}) : n \in \mathbb{N}\}$ then $d(fx, fy) = d(3, 3) = 0 < \frac{3}{4} = \frac{1}{4}d(x, y)$.

Therefore, condition (3.1) of Theorem 3.1 holds for $k = \frac{1}{4}$.

We note that f is edge preserving and any sequence (x_n) in X such that $d(x_n, x) \to 0$ and $(x_n, x_{n+1}) \in E(\tilde{G})$ for all $n \geq 1$ must be a constant sequence which is either $x_n = 3$ for all $n \in \mathbb{N}$ or $x_n = 4$ for all $n \in \mathbb{N}$. So property $(*)'$ holds trivially. Also there exists $x_0 = 3 + \frac{1}{9n}$ such that $(x_0, fx_0) = (3 + \frac{1}{9n}, 3) \in E(\tilde{G})$ and $x_n = fx_{n-1} = 3$ for all $n \in \mathbb{N}$ with

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\eta(x_{i+1}, x_{i+2})}{\eta(x_i, x_{i+1})} \eta(x_{i+1}, x_m) = 3 < \frac{1}{k} = 4,$$

$$\eta(x_n, x_n) = 3 = \eta(x_n, x_{n+1}), \ \forall n \in \mathbb{N}.$$

Therefore, condition (3.2) of Theorem 3.1 is satisfied. In addition, if $d(x_n, x) \to 0$ then $\lim_{n \to \infty} \eta(x_n, fx)$ and $\lim_{n \to \infty} \eta(x, x_n)$ exist and are finite. By applying Theorem 3.1, we can find two fixed points $3, 4 \in X$ of f with $d(3, 3) = 0$ and $d(4, 4) = 0$. As $(3, 4) \notin E(\tilde{G})$, the graph G does not have the property $(**)$. We observe that f fails to get unique fixed point due to lack of the property $(**)$.

4. Some Coincidence Point Results

Definition 4.1. [1] Let f and g be self mappings of a set X. If $y = fx = gx$ for some x in X, then x is called a coincidence point of f and g and y is called a point of coincidence of f and g.

Definition 4.2. [24] The mappings $f, g : X \to X$ are weakly compatible, if for every $x \in X$, the following holds:

$$g(fx) = f(gx) \textit{ whenever } fx = gx.$$

Proposition 4.3. [1] Let f and g be weakly compatible self maps of a nonempty set X. If f and g have a unique point of coincidence $y = fx = gx$, then y is the unique common fixed point of f and g.

We state the following lemma which is a key result in this section.

Lemma 4.4. [21] *Let X be a nonempty set and $f : X \to X$ a function. Then there exists a subset $U \subseteq X$ such that $f(U) = f(X)$ and $f : U \to X$ is one-to-one.*

As an application of Theorem 3.1, we obtain the following result.

Theorem 4.5. *Let (X, d) be a 0-complete controlled rectangular metric-like space defined by the control function $\eta : X \times X \to [1, \infty)$ and let $f : X \to X$ be such that*

$$d(fx, fy) \leq kd(x, y) \tag{4.1}$$

for all $x,\ y \in X$, where $0 < k < 1$ is a constant. Suppose that there exists $x_0 \in X$ such that for $x_n = f^n x_0,\ n = 1,\ 2,\ 3,\ \cdots,\ \eta(x_n, x_n) \leq \eta(x_n, x_{n+1})$ and

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\eta(x_{i+1}, x_{i+2})}{\eta(x_i, x_{i+1})} \eta(x_{i+1}, x_m) < \frac{1}{k}.$$

Also, assume that if $\lim_{n\to\infty} d(x_n, x) = 0$ for some $x \in X$ then $\lim_{n\to\infty} \eta(x, x_n)$ and $\lim_{n\to\infty} \eta(x_n, fx)$ both exist and are finite. Then f has a unique fixed point u(say) in X with $d(u, u) = 0$.

Proof. The proof follows from Theorem 3.1 by taking $G = G_0$. □

We now present the main result of this section.

Theorem 4.6. *Let (X, d) be a controlled rectangular metric-like space defined by the control function $\eta : X \times X \to [1, \infty)$ and the mappings $f,\ g : X \to X$ satisfy the following conditions:*

$$d(fx, fy) \leq kd(gx, gy) \tag{4.2}$$

for all $x,\ y \in X$, where $0 < k < 1$ is a constant, $f(X) \subseteq g(X)$ and $g(X)$ is a 0-complete subspace of X. Suppose that there exists $x_0 \in X$ such that for $gx_n = fx_{n-1},\ n = 1,\ 2,\ 3,\ \cdots,\ \eta(gx_n, gx_n) \leq \eta(gx_n, gx_{n+1})$ and

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\eta(gx_{i+1}, gx_{i+2})}{\eta(gx_i, gx_{i+1})} \eta(gx_{i+1}, gx_m) < \frac{1}{k}.$$

Also, assume that if $\lim_{n\to\infty} d(gx_n, x) = 0$ for some $x \in g(X)$ then $\lim_{n\to\infty} \eta(x, gx_n)$ and $\lim_{n\to\infty} \eta(gx_n, fx)$ both exist and are finite. Then f and g have a unique point of coincidence u(say) in $g(X)$ with $d(u, u) = 0$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $g(X)$.

Proof. By Lemma 4.4, there exists $U \subseteq X$ such that $g(U) = g(X)$ and $g : U \to X$ is one-to-one. Define $h : g(U) \to g(U)$ by $h(gx) = fx$. This is possible as $f(X) \subseteq g(X)$. Then h is well defined, as g is one-to-one on U.

For all $gx,\ gy \in g(U)$, we obtain from condition (4.2) that

$$d(h(gx), h(gy)) = d(fx, fy) \leq kd(gx, gy).$$

This proves that $h : g(U) \to g(U)$ satisfies condition (4.1) of Theorem 4.5. Since $g(U) = g(X)$ is 0-complete, by Theorem 4.5, there exists a unique $gy_0 \in g(X)$ such that $h(gy_0) = gy_0 = u$, say with $d(u, u) = 0$. That is, $fy_0 = gy_0 = u$. Hence, f and g have unique point of coincidence u in $g(X)$.
If f and g are weakly compatible, then by Proposition 4.3 it follows that f and g have a unique common fixed point in $g(X)$. □

Corollary 4.7. Let (X, d) be a 0-complete controlled rectangular metric-like space defined by the control function $\eta : X \times X \to [1, \infty)$ and let $g : X \to X$ be an onto mapping satisfying

$$d(gx, gy) \geq \beta\, d(x, y)$$

for all x, $y \in X$, where $\beta > 1$ is a constant. Suppose that there exists $x_0 \in X$ such that for $x_{n-1} = gx_n,\ n = 1, 2, 3, \cdots, \eta(x_{n-1}, x_{n-1}) \leq \eta(x_{n-1}, x_n)$ and

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\eta(x_{i+1}, x_{i+2})}{\eta(x_i, x_{i+1})} \eta(x_{i+1}, x_m) < \beta.$$

Also, assume that if $\lim_{n\to\infty} d(x_n, x) = 0$ for some $x \in X$ then $\lim_{n\to\infty} \eta(x, x_n)$ and $\lim_{n\to\infty} \eta(x_n, x)$ both exist and are finite. Then g has a unique fixed point in u(say) in X with $d(u, u) = 0$.

Proof. The proof follows from Theorem 4.6 by taking $f = I$, the identity map on X and $\beta = \frac{1}{k}$. □

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