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COMPACTIFICATION ON $*$-TOPOLOGY

KULCHHUM KHATUN, AHMAD AL-OMARI, AND SHYAMAPADA MODAK[†]

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Abstract. In the study of compactifications, the Hausdorff space plays an important role. This paper has been shown that what is the change of compactification due to the ideal on topological spaces. To do this Hausdorff space has been redefined in terms of ideal. Compactness and homeomorphism are also significant for studying the ideal Hausdorff space. Product of ideal Hausdorff spaces is not as usual product of Hausdorff spaces and it has also been discussed in this paper.

1. Literature review

The study of Hausdorff space was introduced in 1914 and it was introduced from several parts of views. This space has remarkable role for the study of metrizability, compactness, paracompactness, manifold and topological dimension theory etc. The study of Hausdorff space will be interesting when the ideal ([5], [11]) play a role on it. For this study we consider following: An ideal on a set K is a sub collection of $\wp(K)$ (power set of K) satisfy hereditary property and finite additivity property. λ_f, the collection of all finite subsets of K; λ_{co}, the collection of all countable subsets of K are the examples of ideals on K. If we go through the topological space (K, V), where V is a topology on K, then we get the following more ideals:
$\lambda_{ci\phi}$, the collection of all nowhere dense subsets of K; λ_{mg}, the collection of all relatively compact subsets of K are ideals on K. The generalization of limit point of a subset of K is defined as a local function on the topological space (K, V, λ), although this space has already been used in terms of ideal topological space [10]. The set-valued set function $g : \wp(K) \to \wp(K)$ defined as $A^g(K, \lambda)$(or simply A^g)=$\{k \in K : \ v \cap A \notin \lambda\}$, where $v \in V(x)$, the collection of all open sets containing x. This is termed as local function [4, 5]. Its associated function is, $\psi : \wp(K) \to \wp(K)$ ([9]) and defined as $\psi(A) = K \setminus (K \setminus A)^g$, that is $g \sim^K \psi$ ([7]).

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[†]*Corresponding author*

Several topologies have been induced from g and ψ operators. One of the most important topology is g-topology (sometimes it is known as $*$-topology [2]) and it will be denoted as V^g. We shall improve some results related to the ideal Hausdorff spaces. Compactness and homeomorphic image of an ideal Hausdorff space are also a part of this paper. Further, we discussed the product of two ideal Hausdorff spaces in front of the ideal $\lambda_1\lambda_2 = \{Q \subseteq A \times B \mid A \in \lambda_1, B \in \lambda_2\}$ where λ_1 is an ideal on K_1 and λ_2 is an ideal on K_2. In this connection, for the ideals $\lambda_1, \lambda_2, \ldots and\ \lambda_k$ on $K_1, K_2, \ldots and\ K_k$ respectively,
$\lambda_1\lambda_2 \cdots \lambda_k = \{L \subseteq A_1 \times A_2 \times \ldots \times A_k \mid A_1 \in \lambda_1, A_2 \in \lambda_2, \ldots, A_k \in \lambda_k\}$ is an ideal on $K_1 \times K_2 \times \ldots \times K_k$. Further more, we shall try to find the answer of the question "what is the one-point compactification of (K, V^g)?"

2. Ideal Hausdorff spaces

Let V be a topology on a set K and λ be an ideal on the same set K. Then the triplicate (K, V, λ) is called a topological space with an ideal λ and it is denoted as $\mathcal{P}$.

Definition 2.1. A space $\mathcal{P}$ is called ideal Hausdorff if for each pair p_1, p_2 of distinct points of K, there are two open sets $v_1,\ v_2$ in V such that $p_1 \in v_1,\ p_2 \in v_2$ and $v_1 \cap v_2 \in \lambda$.

This idea has already been introduced by Suriyakala and Vembu in [10].

If $\lambda = \{\emptyset\}$, then Hausdorff space and ideal Hausdorff space are synonym. If $\lambda = 2^K$, then every topological space (whose original set is K) is an ideal Hausdorff space. As these consequences, we will study the ideal Hausdorff space when the ideal is a proper ideal. For converse relation of Hausdorffness implies ideal Hausdorffness, we shall give following counter example.

Example 2.2. Let $K = \{\tan a, \tan b, \tan c\}$, $V = \{\emptyset, \{\tan b\}, \{\tan a, \tan b\}, \{\tan b, \tan c\}, K\}$, $\lambda = \{\emptyset, \{\tan b\}\}$
In this example (K, V, λ) is an ideal Hausdorff space but not a Hausdorff space.

We know that $\mathbb{R}_K$ [8] is a Hausdorff space and hence it is an ideal Hausdorff space. Followings are the examples on infinite sets and these examples meet the role of ideal on ideal Hausdorff space.

Consider $\mathbb{R}$ and λ_{co} is the collection of all countable subsets of $\mathbb{R}$. Consider the countable complement topology V_{com} on $\mathbb{R}$. Then $(\mathbb{R}, V_{com}, \lambda_{co})$ is an ideal Hausdorff space. Note that λ_{co} is the collection of all closed sets of the countable complement topology V_{com}. Moreover the topological space $(\mathbb{R}, V_{com})$ is not a Hausdorff space.

Further, if we consider $\mathbb{R}$ and λ_f is the collection of all finite subsets of $\mathbb{R}$. Consider the countable complement topology V_{foc} on $\mathbb{R}$. Then $(\mathbb{R}, V_{foc}, \lambda_f)$ is an ideal Hausdorff space. Note that λ_f is the collection of all closed sets of the countable complement topology V_{foc}. Moreover the topological space $(\mathbb{R}, V_{foc})$ is not a Hausdorff space.
One of the more condition on λ in which every ideal Hausdorff space is a Hausdorff space is $V \cap \lambda = \{\emptyset\}$. That is, ideal Hausdorff space implies Hausdorff space, when $V \cap \lambda = \{\emptyset\}$. But its converse relation is not true, in general. As for example, if we consider $\mathbb{R}$ with usual topology $\mathcal{U}$ and $\lambda = \wp(\mathbb{R})$, then $\mathbb{R}$ with usual topology is an ideal Hausdorff space as well as Hausdorff space, but $\wp(\mathbb{R}) \cap \mathcal{U} \neq \{\emptyset\}$. Thus, we have for $V \cap \lambda = \{\emptyset\}$, Hausdorff

space and ideal Hausdorff space coincide. Modak and Bandyopadhyay [6] introduced a new topology with the help of ψ-operator using the condition $V \cap \lambda = \{\emptyset\}$. Thus the topology $(V^g)^\alpha$ and V^ψ [1] are the same topologies on K in respect of the condition $V \cap \lambda = \{\emptyset\}$.

Theorem 2.3. *Let (K, V, λ) be an ideal Hausdorff space. Then for $e_1, e_2 \in K$ with $e_1 \neq e_2$, there is an open set $v \in V$ containing e_2 such that e_1 is not a member of the local function of v.*

Lemma 2.4. *Every compact subspace of an ideal Hausdorff space is a g-closed (or $*$-closed) set.*

Proof. Suppose L is a compact subspace of an ideal Hausdorff space (K, V, λ). Let $k_0 \in K \setminus L$. Given that (K, V, λ) is an ideal Hausdorff space, then for each point e of L, there exist open sets S_e containing k_0 and T_e containing e such that $S_e \cap T_e \in \lambda$. The collection $\{T_e \mid \ e \in L\}$ is a covering of L by sets open in K. As L is compact, then finitely many of them $T_{e_1},\ T_{e_2}, ..., T_{e_l}$ cover L. Thus the open set $T = T_{e_1} \cup T_{e_2} \cup ... \cup T_{e_l}$ contains L. Further, the space is an ideal Hausdorff space so, we have following open sets $S_{e_1},\ S_{e_2},\ \dots$ and S_{e_l} containing k_0 such that $T_{e_1} \cap S_{e_1} \in \lambda$, $T_{e_2} \cap S_{e_2} \in \lambda$,... and $T_{e_l} \cap S_{e_l} \in \lambda$. Put $S = S_{e_1} \cap S_{e_2} \cap ... \cap S_{e_l}$ and it is an open set containing k_0.

Now $L \cap S \subseteq T \cap S = [T_{e_1} \cup T_{e_2} \cup ... \cup T_{e_l}] \cap [S_{e_1} \cap S_{e_2} \cap ... \cap S_{e_l}] = [T_{e_1} \cap (S_{e_1} \cap S_{e_2} \cap ... \cap S_{e_l})] \cup [T_{e_2} \cap (S_{e_1} \cap S_{e_2} \cap ... \cap S_{e_l})] \cup ... \cup [T_{e_l} \cap (S_{e_1} \cap S_{e_2} \cap ... \cap S_{e_l})] \subseteq (T_{e_1} \cap S_{e_1}) \cup (T_{e_2} \cap S_{e_2}) \cup ... \cup (T_{e_l} \cap S_{e_l}) \in \lambda$. Thus $k_0 \notin L^g$, and hence $L = L \cup L^g = cl^g(L)$. Therefore L is a g-closed set. $\square$

Theorem 2.5. *Let (K', V', λ') be an ideal Hausdorff space and (K, V) be another topological space. Let $p : K \to K'$ be a surjective and continuous map, then $(K, V, p^{-1}(\lambda'))$ is an ideal Hausdorff space.*

Proof. Let $e_1, e_2 \in K'$ and $e_1 \neq e_2$. Given that (K', V', λ') is an ideal Hausdorff space. So, there exist two open sets S_{e_1} containing e_1 and S_{e_2} containing e_2 in (K', V') such that $S_{e_1} \cap S_{e_2} \in \lambda'$.
Clearly, $p^{-1}(e_1),\ p^{-1}(e_2) \in K$. Now, $p^{-1}(S_{e_1}),\ p^{-1}(S_{e_2})$ are open sets in (K, V) containing $p^{-1}(e_1)$ and $p^{-1}(e_2)$ respectively (as p is continuous). Also, $p^{-1}(S_{e_1}) \cap p^{-1}(S_{e_2}) = p^{-1}(S_{e_1} \cap S_{e_2}) \in p^{-1}(\lambda')$. Thus $(K, V, p^{-1}(\lambda'))$ is an ideal Hausdorff space. $\square$

Theorem 2.6. *Let (K, V, λ) be an ideal Hausdorff space. If L is a compact subspace of K and k_0 is not in L, then there exist open sets $S,\ T$ in (K, V, λ) such that $k_0 \in S$, $L \subseteq T$ and $S \cap T \in \lambda$.*

Proof. Since k_0 is not in L, so $k_0 \in K \setminus L$. Given that (K, V, λ) is an ideal Hausdorff space. Then, for each point e of L, there exist open sets S_e containing k_0 and T_e containing e such that $S_e \cap T_e \in \lambda$. The collection $\{T_e \mid \ e \in L\}$ is a covering of L by sets open in K. As L is compact, then finitely many of them $T_{e_1},\ T_{e_2}, ..., T_{e_l}$ cover L. Thus the open set $T = T_{e_1} \cup T_{e_2} \cup ... \cup T_{e_l}$ contains L that is $L \subseteq T$. Further, the space is an ideal Hausdorff space, so we have following open sets $S_{e_1},\ S_{e_2},\ \dots$ and S_{e_l} containing k_0 such that $T_{e_1} \cap S_{e_1} \in \lambda$, $T_{e_2} \cap S_{e_2} \in \lambda$,..., and $T_{e_l} \cap S_{e_l} \in \lambda$. Put $S = S_{e_1} \cap S_{e_2} \cap ... \cap S_{e_l}$ and it is an open set containing k_0 that is $k_0 \in S$.

Now,

$$\begin{aligned} S \cap T \quad &= [S_{e_1} \cap S_{e_2} \cap \cdots \cap S_{e_l}] \cap [T_{e_1} \cup T_{e_2} \cup \cdots \cup T_{e_l}] \\ &= [(S_{e_1} \cap S_{e_2} \cap \cdots \cap S_{e_l}) \cap T_{e_1}] \cup [(S_{e_1} \cap S_{e_2} \cap \\ &\quad \cdots \cap S_{e_l}) \cap T_{e_2}] \cup \cdots \cup [(S_{e_1} \cap S_{e_2} \cap \cdots \cap S_{e_l}) \cap T_{e_l}] \\ &\subseteq (S_{e_1} \cap T_{e_1}) \cup (S_{e_2} \cap T_{e_2}) \cup \cdots \cup (S_{e_l} \cap T_{e_l}) \in \lambda. \end{aligned}$$

This implies that $S \cap T \in \lambda$. □

Corollary 2.7. Let (K, V, λ) be a compact ideal Hausdorff space. If L_1 and L are two disjoint closed subspaces of K, then there exist open sets S, T in (K, V, λ) such that $L_1 \subseteq S$, $L \subseteq T$ and $S \cap T \in \lambda$.

Theorem 2.8. *Let (K, V, λ) be an ideal Hausdorff space and (K', V') be an another topological space. Suppose $p : K \to K'$ be a homeomorphism, then (K', V') is an ideal Hausdorff space with respect to the ideal $p(\lambda)$.*

Proof. Let e_1 and e_2 be two distinct points of K. Then, there exist $v_1, v_2 \in V$ such that $e_1 \in v_1, e_2 \in v_2$ and $v_1 \cap v_2 \in \lambda$. Since p is a homeomorphism, so $p(v_1)$ and $p(v_2)$ are open in (K', V') containing $p(e_1)$ and $p(e_2)$ respectively. Also, $v_1 \cap v_2 \in \lambda$. Thus, $p(v_1 \cap v_2) \in p(\lambda)$. This implies that $p(v_1) \cap p(v_2) \in p(\lambda)$ as p is injective. Hence, (K', V') is an ideal Hausdorff space. □

Generalized version of the result
"Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism" [8] is as follows:

Theorem 2.9. *Let (K, V, λ) be an ideal Hausdorff space and (K', V') be another topological space. If (K', V') is compact, then every continuous bijective map p is a homeomorphism from (K', V') onto (K, V^g).*

Proof. To prove that p is a homeomorphism we need to show that p^{-1} is continuous.
For this purpose we shall show that images of closed sets of (K', V') under p are closed in V^g
Let A be a closed set in (K', V'), then A is compact. Since p is continuous, $p(A)$ is compact.
Since $p(A)$ is a compact subspace of an ideal Hausdorff space (K, V^g), so $p(A)$ is closed in (K, V^g). Hence, p^{-1} is continuous. Therefore, $p : (K', V') \to (K, V^g)$ is a homeomorphism. □

Followings are another methods for preserving the ideal Hausdorffness
Let $\int : K \to K'$ be a map and λ' be an ideal on K'. Then
$\int^{\leftarrow}(\lambda') = \{A : A \subseteq \int^{-1}(J) \in \int^{-1}(\lambda')\}$ is an ideal on K ([3]).
Let $(K',\ V')$ be a topological space and K be a non empty set, and $\int : K \to K'$ be a map. Then $\int^{-1}(V') = \{\int^{-1}(v) : v \in V'\}$ gives a topology on K. This topology is called the weak topology induced by $\int$ and V'.

Theorem 2.10. *Let $(K',\ V',\ \lambda')$ be an ideal Hausdorff space and K be any non empty set. Suppose $\int : K \to K'$ be a map, then $(K,\ \int^{-1}(V'),\ \int^{\leftarrow}(\lambda'))$ is an ideal Hausdorff space.*

Proof. Let $e_1,\ e_2 \in K$. Then $\int(e_1),\ \int(e_2) \in K'$. Given that $(K',\ V',\ \lambda')$ is an ideal Hausdorff space, then there exist open sets $S_{\int(e_1)}$ containing $\int(e_1)$ and $S_{\int(e_2)}$ containing $\int(e_2)$ such that $S_{\int(e_1)} \cap S_{\int(e_2)} \in \lambda'$. Then $\int^{-1}(S_{\int(e_1)})$ and $\int^{-1}(S_{\int(e_2)})$ are open sets containing e_1 and e_2 respectively. Also, $\int^{-1}(S_{\int(e_1)}) \cap \int^{-1}(S_{\int(e_1)}) = \int^{-1}(S_{\int(e_1)} \cap S_{\int(e_2)}) \in \int^{-1}(\lambda')$. This implies $\int^{-1}(S_{\int(e_1)}) \cap \int^{-1}(S_{\int(e_1)}) \in \int^{\leftarrow}(\lambda')$. Thus $(K,\ \int^{-1}(V'),\ \int^{\leftarrow}(\lambda'))$ is an ideal Hausdorff space. □

Lemma 2.11. *If (K, V, λ) is an ideal Hausdorff space then (K, V^g) is an ideal Hausdorff space.*

Following theorem is the generalization of the result
"Let A and B be disjoint compact subspaces of the Hausdorff space X. Show that there exist disjoint open sets U and V containing A and B respectively"[8].

Theorem 2.12. *Let L_1 and L_2 be disjoint compact subspaces of an ideal Hausdorff space (K, V, λ). Then there exist open sets S and T such that $L_1 \subseteq S$, $L_2 \subseteq T$ and $S \cap T \in \lambda$.*

Proof. Given that (K, V, λ) is an ideal Hausdorff space, then for each point e of L_1 and g of L_2, there exist open sets $S_e \in V$ containing e and $T_g \in V$ containing g such that $S_e \cap T_g \in \lambda$. The collection $\{S_e \mid\ e \in L_1\}$ is a covering of L_1 by sets open in K. As L_1 is compact, then finitely many of them $S_{e_1},\ S_{e_2}, ..., S_{e_l}$ cover L_1. Thus the open set $S = S_{e_1} \cup S_{e_2} \cup .... \cup S_{e_l}$ contains L_1.
Also, the collection $\{T_g \mid\ g \in L_2\}$ is a covering of L_2 by sets open in K. As L_2 is compact, then finitely many of them $T_{g_1},\ T_{g_2}, ..., T_{g_l}$ cover L_2. Thus the open set $T = T_{g_1} \cup T_{g_2} \cup .... \cup T_{g_l}$ contains L_2. Since L_1 and L_2 are disjoint, so $e_i \neq g_i$ $\forall i = 1, 2, \ldots, l$. Further, the space is an ideal Hausdorff space, so $S_{e_i} \cap T_{g_i} \in \lambda$ $\forall i = 1, 2, \ldots, l$.
Now, $U \cap V$
$= [S_{e_1} \cup S_{e_2} \cup .... \cup S_{e_l}] \cap [T_{g_1} \cup T_{g_2} \cup .... \cup T_{g_l}]$
$= [S_{e_1} \cap (T_{g_1} \cup T_{g_2} \cup .... \cup T_{g_l})] \cup [S_{e_2} \cap (T_{g_1} \cup T_{g_2} \cup .... \cup T_{g_n})] \cup \ldots \cup [S_{e_l} \cap (T_{g_1} \cup T_{g_2} \cup .... \cup T_{g_l})]$
$= (S_{e_1} \cap T_{g_1}) \cup (S_{e_1} \cap T_{g_2}) \cup \ldots \cup (S_{e_1} \cap T_{g_l}) \cup (S_{e_2} \cap T_{g_1}) \cup (S_{e_2} \cap T_{g_2}) \cup \ldots \cup (S_{e_2} \cap T_{g_l}) \cup \ldots \cup (S_{e_l} \cap T_{g_1}) \cup (S_{e_l} \cap T_{g_2}) \cup \ldots \cup (S_{e_l} \cap T_{g_l}) \in \lambda$. That is $S \cap T \in \lambda$. □

For further study of ideal Hausdorff space we are considering following theorem from [10].

Theorem 2.13. *Let $(K,\ V)$ be a topological space and λ be an ideal on K. $(K,\ V)$ is an ideal Hausdorff space if and only if $(K,\ V^g)$ is Hausdorff.*

We will be reserved for any finer topology.

Example 2.14. Let $K = \mathbb{R}$, $V = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$, $\lambda = \wp(\mathbb{R})$ and $V^f = \{\emptyset, \mathbb{R}, \mathbb{Q}, \{i\}, \mathbb{Q} \cup \{i\}\}$ where $i \in \mathbb{R} \setminus \mathbb{Q}$. Clearly $V \subseteq V^f$. Here, (K, V, λ) is an ideal Hausdorff space but (K, V^f) is not a Hausdorff space.

Theorem 2.15. *Let (K, V) be a locally compact space and λ be an ideal on K. Then for locally compact ideal Hausdorff space (K, V, λ), there exists a space (L, V_L) such that*
1) K is a subspace (with respect to V^g topology) of L,
2) The set $L \setminus K$ contains a single point,

3) (L, V_L) is locally compact.

Proof. Given that (K, V, λ) is a locally compact ideal Hausdorff space. Suppose $L = K \cup \{\gamma\}$, where $\gamma \notin K$.
Let us consider the collection $V_L = V^g \bigcup \{L \setminus P : \ P \textit{ is a compact subspace of } K \textit{ with respect to the } V^g \textit{ topology}\}$. Here we call the elements of V^g and $\{L \setminus P : \ P \textit{ is a compact subspace of } K \textit{ with respectto the } V^g \textit{ topology}\}$ as **first type** and **second type** respectively.
Now our assertion is V_L is a topology on L. As, the empty set is an element of V^g, and the space L is a set of second type. Further the intersection of two elements of V_L is always an element of V_L.
Next we take arbitrary collection of elements of first type, then it is obvious that its union is always a member of V_L.
Let $\{L \setminus P_\beta : \beta \in \delta \ (\textit{index set})\}$ be a collection of second type. Now $\bigcap\limits_\beta P_\beta$ is closed in K with respect to the V^g topology (it is hold throughout the theorem 2.13 and the result "every compact subspace of a Hausdorff space is closed"). Further $\bigcap\limits_\beta P_\beta$ is a closed subspace of a compact space P_β, hence $\bigcap\limits_\beta P_\beta$ is a compact subspace of K with respect to the V^g topology. Thus $L \setminus \bigcap\limits_\beta P_\beta = \bigcup\limits_\beta (L \setminus P_\beta)$ is an element of second type.
Mixed types: For first type $\{S_\alpha\}$ and second type $\{L \setminus P_\beta\}$, $(\bigcup\limits_\alpha S_\alpha) \bigcup (\bigcup\limits_\beta (L \setminus P_\beta)) = S \cup (L \setminus P) = L \setminus (P \setminus S)$, which is a second type as $P \setminus S$ is a closed subspace of P (with respect to the V^g topology), so it is compact (due to V^g topology), where $S = \bigcup\limits_\alpha S_\alpha$ and $\bigcap\limits_\beta P_\beta = P$.
Subspace: If S is of first type, then $S \cap K = S$; if $L \setminus P$ is of second type, then $(L \setminus P) \cap K = K \setminus P$; both of these sets are open (due to V^g topology) in K. Also, any set open in K is a set of first type and therefore open in L by definition.
Finally, we shall show that (L, V_L) is locally compact. Let $\mathcal{C}$ be an open covering of L. Then $\mathcal{C}$ must contain an open set of second type, say $L \setminus P$, since none of the open sets of first type contain the point γ. Take all the members of $\mathcal{C}$ different from $L \setminus P$ and intersect them with K; they form a collection of open sets (due to V^g topology) of K covering P. As P is compact, finitely many of them cover P; the corresponding finite collection of elements of $\mathcal{C}$ will, along with the element $L \setminus P$, cover all of L. Hence (L, V_L) is locally compact. □

Theorem 2.16. *Let λ be an ideal on a topological space (K, V). Suppose (L, V_L) is a space which satisfy :*
1) (K, V) is a subspace of (L, V_L),
2) The set $L \setminus K$ consists of a single point,
3) (L, V_L) is a compact Hausdorff space.
Then (K, V^g, λ) is a locally compact Hausdorff space.

Proof. Suppose a space L satisfying conditions (i)-(iii) exists. Then (K, V^g) is Hausdorff space because it is a subspace of the Hausdorff space (L, V_L).
Given any $e \in K$, we show that (K, V^g) is locally compact at e. We choose open sets S

and T of (L, V_L) containing e and the single point of $L \setminus K$, respectively. Then the set $P = L \setminus T$ is closed in L, so it is a compact subspace of L. Since P lies in K, it is also compact as a subspace of (K, V^g); it contains the neighborhood S of e. □

Let (K, V) be a locally compact space and λ be an ideal on K. Then for locally compact ideal Hausdorff space (K, V, λ), (L, V_L) is an one-point compactification of (K, V^g), when (L, V_L) is a Hausdorff space. It is happened, because $Cl_{V_L}(K) = L$ (Cl_{V_L} denotes the closure operator with respect to the V_L topology).

Theorem 2.17. *Let λ be an ideal on a topological space (K, V) and λ' be an ideal on a topological space (K', V'). If the product space $K \times K'$ is ideal Hausdorff space with respect to the ideal $\lambda\lambda'$, then both (K, V, λ) and (K', V', λ') are ideal Hausdorff spaces.*

Proof. Let $e_1 \neq e_2$ in K. Then $(e_1, k_0) \neq (e_2, k_0)$ for some $k_0 \in K'$. Since $K \times K'$ is ideal Hausdorff space with respect to the ideal $\lambda\lambda'$, so there exist open sets $S_1 \times T_1$ and $S_2 \times T_2$ in $K \times K'$ such that $(e_1, k_0) \in S_1 \times T_1$ and $(e_2, k_0) \in S_2 \times T_2$ and $(S_1 \times T_1) \cap (S_2 \times T_2) \in \lambda\lambda'$. This implies $(S_1 \cap S_2) \times (T_1 \cap T_2) \in \lambda\lambda'$.
Thus $(S_1 \cap S_2) \times (T_1 \cap T_2) \subseteq A \times B$, where $A \in \lambda, B \in \lambda'$. Therefore $S_1 \cap S_2 \subseteq A$ and $T_1 \cap T_2 \subseteq B$. Hence $S_1 \cap S_2 \in \lambda$ and $T_1 \cap T_2 \in \lambda'$ as λ and λ' are ideals. Now S_1, S_2 are open sets in K and $e_1 \in S_1, e_2 \in S_2$ such that $S_1 \cap S_2 \in \lambda$. Thus (K, V, λ) is an ideal Hausdorff space. Hence (K', V', λ') is an ideal Hausdorff space. □

Converse statement of the above theorem is not true in general. That is, if (K, V, λ) and (K', V', λ') are ideal Hausdorff spaces, then the product space $K \times K'$ need not be an ideal Hausdorff space with respect to the ideal $\lambda\lambda'$.

Example 2.18. Let $K = \{\tan a, \tan b\},\ K' = \{\tan a, \tan c\},\ V = \{\emptyset, K, \{\tan a\}\}$, $V' = \{\emptyset, K', \{\tan a\}\},\ \lambda = \{\emptyset, \{\tan a\}\},\ \lambda' = \{\emptyset, \{\tan a\}\}$. Then the product topology $V \times V' = \{\emptyset, K \times K', \{(\tan a, \tan a)\}, \{(\tan a, \tan a), (\tan a, \tan c)\}$, $\{(\tan a, \tan a), (\tan b, \tan a)\}, \{(\tan a, \tan a), (\tan a, \tan c), (\tan b, \tan a)\}\}$ and $\lambda\lambda' = \{\emptyset, \{(\tan a, \tan a)\}\}$.
In this example $K \times K'$ is not an ideal Hausdorff space with respect to the ideal $\lambda\lambda'$.

$\mathbb{R}$ with usual topology is an ideal Hausdorff space, where the ideal is $\wp(\mathbb{R})$. For n-manifold $\mathbb{R}^n$, $(\prod_{i=1}^{n} A_i)^g(\wp(\mathbb{R})\wp(\mathbb{R})\ldots\wp(\mathbb{R})) = \prod_{i=1}^{n} A_i^g(\wp(\mathbb{R}))$.
Consequently we have, a n-manifold $\mathbb{R}^n$ is an ideal Hausdorff space, where the ideal is $\wp(\mathbb{R})\wp(\mathbb{R})\ldots\wp(\mathbb{R})$.

3. Completely ideal Hausdorff spaces

In the previous section we have discussed a generalized space of Hausdorff space. At this present section we shall give an another space which is generalization of Hausdorff space but weaker than the ideal Hausdorff space. That is, this space split the Hausdorff space and ideal Hausdorff space.

Definition 3.1. A space $\mathcal{P}$ is called completely ideal Hausdorff space if for each pair p_1, p_2 of distinct points of K, there are two open sets $v_1,\ v_2$ in V such that $p_1 \in v_1,\ p_2 \in v_2$ and $\overline{v_1} \cap \overline{v_2} \in \lambda$, where $\overline{v_1}$ denotes the closure of v_1.

Every completely ideal Hausdorff space is an ideal Hausdorff space but converse is not true in general. For this we shall give the following example:

Example 3.2. Let $K = \{\tan a, \tan b, \tan c\}$, $V = \{\emptyset, \{\tan b\}, \{\tan a, \tan b\}, \{\tan b, \tan c\}, K\}$, $\lambda = \{\emptyset, \{\tan b\}\}$. Here, (K, V, λ) is an ideal Hausdorff space but not a completely ideal Hausdorff space.

Completely ideal Hausdorff space obeys hereditary property.

Theorem 3.3. *Every subspace of a completely ideal Hausdorff space is completely ideal Hausdorff.*

Proof. Let L be a subspace of the completely ideal Hausdorff space (K, V, λ) and let e, g be two distinct points in L. Then there exist two open sets S and T in K such that $e \in S$, $g \in T$ and $\overline{S} \cap \overline{T} \in \lambda$. Hence, $e \in L \cap S$, $g \in L \cap T$ and
$(\overline{L \cap S}) \cap (\overline{L \cap T}) \subseteq (\overline{L} \cap \overline{S}) \cap (\overline{L} \cap \overline{T}) = (\overline{S} \cap \overline{T}) \cap \overline{L} \subseteq \overline{S} \cap \overline{T} \in \lambda$. This implies
$(\overline{L \cap S}) \cap (\overline{L \cap T}) \in \lambda$. Hence L is completely ideal Hausdorff as $L \cap S$ and $L \cap T$ are open in L. □

Theorem 3.4. *Let λ be an ideal on a topological space (K, V) and λ' be an ideal on a topological space (K', V'). If the product space $K \times K'$ is completely ideal Hausdorff space with respect to the ideal $\lambda\lambda'$, then both (K, V, λ) and (K', V', λ') are completely ideal Hausdorff spaces.*

Proof. Let $e_1 \neq e_2$ in K. Then $(e_1, k_0) \neq (e_2, k_0)$ for some $k_0 \in K'$. Since $K \times K'$ is completely ideal Hausdorff space with respect to the ideal $\lambda\lambda'$, so there exist open sets $S_1 \times T_1$ and $S_2 \times T_2$ in $K \times K'$ such that $(e_1, k_0) \in S_1 \times T_1$ and $(e_2, k_0) \in S_2 \times T_2$ and $\overline{(S_1 \times T_1)} \cap \overline{(S_2 \times T_2)} \in \lambda\lambda'$. This implies $(\overline{S_1} \cap \overline{S_2}) \times (\overline{T_1} \cap \overline{T_2}) \in \lambda\lambda'$.
Thus $(\overline{S_1} \cap \overline{S_2}) \times (\overline{T_1} \cap \overline{T_2}) \subseteq A \times B$, where $A \in \lambda, B \in \lambda'$. Therefore $\overline{S_1} \cap \overline{S_2} \subseteq A$ and $\overline{T_1} \cap \overline{T_2} \subseteq B$. Hence $\overline{S_1} \cap \overline{S_2} \in \lambda$ and $\overline{T_1} \cap \overline{T_2} \in \lambda'$ as λ and λ' are ideals. Now S_1, S_2 are open sets in K and $e_1 \in S_1$, $e_2 \in S_2$ respectively such that $\overline{S_1} \cap \overline{S_2} \in \lambda$. Thus (K, V, λ) is completely ideal Hausdorff space. Hence (K', V', λ') is completely ideal Hausdorff space. □

Converse of the above theorem is not true in general as similar result is not true in the ideal Hausdorff space.

Theorem 3.5. *Let (K', V', λ') be a completely ideal Hausdorff space and (K, V) be another topological space. Let $p : K \to K'$ be a homeomorphism, then $(K, V, p^{-1}(\lambda'))$ is a completely ideal Hausdorff space.*

Proof. Let e_1, $e_2 \in K'$ and $e_1 \neq e_2$. Given that (K', V', λ') is a completely ideal Hausdorff space. So, there exist two open sets S_{e_1} containing e_1 and S_{e_2} containing e_2 in (K', V') such that $\overline{S_{e_1}} \cap \overline{S_{e_2}} \in \lambda'$.
Clearly, $p^{-1}(e_1)$, $p^{-1}(e_2) \in K$. Now, $p^{-1}(S_{e_1})$, $p^{-1}(S_{e_2})$ are open sets in (K, V) containing $p^{-1}(e_1)$ and $p^{-1}(e_2)$ respectively (as p is continuous). Also, $\overline{p^{-1}(S_{e_1})} \cap \overline{p^{-1}(S_{e_2})} = p^{-1}(\overline{S_{e_1}}) \cap p^{-1}(\overline{S_{e_2}})$ (as p is homeomorphism) $= p^{-1}(\overline{S_{e_1}} \cap \overline{S_{e_2}}) \in p^{-1}(\lambda')$. Thus $(K, V, p^{-1}(\lambda'))$ is a completely ideal Hausdorff space. □

Theorem 3.6. *Let (K, V, λ) be a completely ideal Hausdorff space and (K', V') be an another topological space. Suppose $p : K \to K'$ be a homeomorphism, then (K', V') is a completely ideal Hausdorff space with respect to the ideal $p(\lambda)$.*

Proof. Let e_1 and e_2 be two distinct points of K. Given that (K, V, λ) is a completely ideal Hausdorff space. Then, there exist two open sets S_{e_1} containing e_1 and S_{e_2} containing e_2 in (K, V) such that $\overline{S_{e_1}} \cap \overline{S_{e_2}} \in \lambda$.
Since p is a homeomorphism, so $p(S_{e_1})$ and $p(S_{e_2})$ are open in (K', V') containing $p(e_1)$ and $p(e_2)$ respectively. Also, $\overline{S_{e_1}} \cap \overline{S_{e_2}} \in \lambda$. Thus, $p(\overline{S_{e_1}} \cap \overline{S_{e_2}}) \in p(\lambda)$. This implies that $p(\overline{S_{e_1}}) \cap p(\overline{S_{e_2}}) \in p(\lambda)$ as p is injective. Also, $p(\overline{S_{e_1}}) \cap p(\overline{S_{e_2}}) = \overline{p(S_{e_1})} \cap \overline{p(S_{e_2})}$ as p is a homeomorphism. Thus $\overline{p(S_{e_1})} \cap \overline{p(S_{e_2})} \in p(\lambda)$. Hence, (K', V') is a completely ideal Hausdorff space. □

Theorem 2.10 does not hold for a completely ideal Hausdorff space as $\int$ is not a homeomorphism.

Theorem 3.7. *Let λ be an ideal on a topological space (K, V). If (K, V, λ) is a completely ideal Hausdorff space then (K, V^g, λ) is a completely ideal Hausdorff space.*

Proof. Let e_1 and e_2 be two distinct points of K. Given that (K, V, λ) is a completely ideal Hausdorff space. Then, there exist two open sets S_{e_1} containing e_1 and S_{e_2} containing e_2 in (K, V) such that $\overline{S_{e_1}} \cap \overline{S_{e_2}} \in \lambda$.
As $V \subseteq V^g$, S_{e_1} containing e_1 and S_{e_2} containing e_2 are also open sets in (K, V^g). Thus (K, V^g, λ) is a completely ideal Hausdorff space. □

Converse of this theorem need not hold in general.

Example 3.8. Let $K = \{a,\ b,\ c\}$, $V = \{\emptyset,\ \{b\},\ \{a, b\},\ \{b, c\},\ K\}$ and $\lambda = \{\emptyset,\ \{b\}\}$. Then $V^g = \{\emptyset,\ \{b\},\ \{a, b\},\ \{b, c\},\ K,\ \{a\},\ \{c\},\ \{a, c\}\} = \wp(K)$. In this example (K, V^g, λ) is a completely ideal Hausdorff space but (K, V, λ) is not a completely ideal Hausdorff space.

Conclusion

In the conclusion part, we may note the role of ideals in ideal topological spaces for studying compactification spaces.

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(Kulchhum Khatun) Department of Mathematics, University of Gour Banga, Malda 732103, West Bengal, India
E-mail address: kulchhumkhatun123@gmail.com

(Ahmad Al-Omari) Department of Mathematics, Faculty of Sciences, Al al- Bayt University, P.O. Box 130095, Mafraq 25113, Jordan
E-mail address: omarimutah1@yahoo.com

(Shyamapada Modak) Department of Mathematics, University of Gour Banga, Malda 732103, West Bengal, India
E-mail address: spmodak2000@yahoo.co.in