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WEAK (I, J)-CONTINUOUS MULTIFUNCTIONS IN IDEAL BITOPOLOGICAL SPACES

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Abstract. In this paper, we introduce a new type of weak (I, J)-continuous multifunction between bitopological spaces. Also we study several characterizations of this type of multifunctions and some illustrative examples are given.

1. Introduction

It is well known today that the notions of multifunctions play a very important role in General topology. Upper and lower continuity have been extensively studied on multifunctions $F : (X, \tau) \to (Y, \sigma)$. Currently using the notions of ideal topological spaces, different types of upper and lower continuity in a multifunction $F : (X, \tau, I) \to (Y, \sigma)$ has been studied and characterized [1, 5, 12, 13]. On the other hand, generalized open sets are basic and important in many topological notions and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity by utilizing generalized open sets. The concept of ideal topological spaces has been introduced and studied by Kuratowski [9] and the local function of a subset A of a topological space (X, τ) was introduced by Vaidyanathaswamy [14] as follows: given a topological space (X, τ) with an ideal I on X and if $P(X)$ is the set of all subsets of X, a set operator $(.)^* : P(X) \to P(X)$, called the local function of A with respect to τ and I, is defined as follows: for $A \subseteq X$, $A^*(\tau, I) = \{x \in X \ / \ U \cap A \notin I \text{ for every } U \in \tau_x\}$, where $\tau_x = \{U \in \tau : x \in U\}$. A Kuratowski closure operator $cl^*(,)$ for a topology $\tau^*(\tau, I)$

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called the *-topology, finer than τ is defined by $cl^*(A) = A \cup A^*(\tau, I)$. We will denote $A^*(\tau, I)$ by A^*. In this paper, given a bitopological space (X, τ_1, τ_2) and two ideals I, J, the ideals topological spaces (X, τ_1, I) and (X, τ_2, J), forms the ideal bitopological space $(X, \tau_1, \tau_2, I, J)$ in order to introduce and study upper (lower) (i, j)-weakly (I, J)-continuous multifunctions on bitopological spaces.

2. Preliminaries

Throughout this paper, (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces in which no separation axioms are assumed, unless explicitly stated and if I is an ideal on X, (X, τ, I) mean an ideal topological space. For a subset A of (X, τ), $\mathrm{Cl}(A)$ (resp. $\mathrm{Int}(A)$) denote the closure (resp. interior) of A with respect to τ. A subset A is said to be a regular open [4] (resp. semiopen [10], preopen [11] , semi preopen [2]) if $A = \mathrm{Int}(\mathrm{Cl}(A))$ (resp. $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$, $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$, $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$). The complement of a regular open (resp. semiopen, semi-preopen) set is called regular closed (resp. semiclosed, semi-preclosed) set. A subset S of (X, τ, I) is an I-open [6], if $S \subseteq \mathrm{Int}(S^*)$. The complement of an I-open set is called I-closed set. The I-closure (resp. I-interior), can be defined in the same way as $\mathrm{Cl}(A)$ (resp. $\mathrm{Int}(A)$) and will be denoted by I-$\mathrm{Cl}(A)$ (resp. I-$\mathrm{Int}(A)$). The family of all I-open (resp. I-closed, regular open, regular closed, semiopen, semi closed, preopen, semi-preclosed) subsets of a (X, τ, I), denoted by $IO(X, \tau)(resp. IC(X, \tau), RO(X, \tau), RC(X, \tau), SO(X, \tau), SC(X, \tau),$ $PO(X, \tau), SPO(X, \tau), SPC(X, \tau))$. We set $IO(X, x) = \{A : A \in IO(X)$ and $x \in A\}$. By a multifunction $F : (X, \tau) \to (Y, \sigma)$, following [3], we shall denote the upper (resp. lower) inverse of a set B of Y by $F^+(B)$ (resp. $F^-(B)$), that is $F^+(B) = \{x \in X : F(x) \subset B\}$ (resp. $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$). In particular, $F^-(y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$ and for each $A \subset X$, $F(A) = \cup_{x \in A} F(x)$. Then F is said to be surjective if $F(X) = Y$ and injective if $x \neq y$ implies $F(x) \cap F(y) = \emptyset$. A triple (X, τ_1, τ_2) consisting of a nonempty set X with two topologies τ_1 and τ_2 on X is called bitopological space. Throughout this paper, the indices i and j take values in $\{1, 2\}$ and $i \neq j$. For a A of (X, τ_1, τ_2), $i \,\mathrm{Int}\, A$ (resp. $i \,\mathrm{Cl}\, A$) means, the interior (resp. closure) of A with respect to the topology τ_i. In the same order of idea. An ideal bitopological space $(X, \tau_1, \tau_2, I, J)$ consist of two ideal topological spaces (X, τ_1, I) and (X, τ_2, J)

Definition 2.1. [7] *Let (X, τ_1, τ_2) be a bitopological space and A a subset of X. A point $x \in X$ is said to be in the (i, j)-θ-closure of A, denoted by (i, j)-$\mathrm{Cl}_\theta(A)$ if $A \cap j \,\mathrm{Cl}(U) \neq \emptyset$ for every τ_i-open set U containing x, where $i, j = 1, 2$ and $i \neq j$.*

Definition 2.2. [7] *A subset A of X is said to be (i, j)-θ-closed if $A = (i, j)$-$\mathrm{Cl}_\theta(A)$. A subset A of X is said to be (i, j)-θ-open of $X \backslash A$ is (i, j)-θ-closed. The (i, j)-θ-interior of A, denoted by (i, j)-$\mathrm{Int}_\theta(A)$, is defined as the union of all (i, j)-θ-open sets contained in A. Hence $x \in (i, j)$-$\mathrm{Int}_\theta(A)$ if, and only if there exists a τ_i-open set U containing x such that $x \in U \subset j \,\mathrm{Cl}(U) \subset A$.*

Lemma 2.3. [7] *For a subset of a bitopological space (X, τ_1, τ_2), we have the following*

(1) (i, j)-$\mathrm{Cl}_\theta(X \backslash A) = X \backslash (i, j)$-$\mathrm{Int}_\theta(A)$.

(2) (i, j)-$\mathrm{Int}_\theta(X \backslash A) = X \backslash (i, j)$-$\mathrm{Cl}_\theta(A)$.

Lemma 2.4. [7] *Let (X, τ_1, τ_2) be a bitopological space. If U is a τ_j-open subset of X, then (i, j)-$\mathrm{Cl}_\theta(U) = i\,\mathrm{Cl}(U)$.*

Definition 2.5. *A subset A of a bitopological space (X, τ_1, τ_2) is said to be:*

(1) (i, j)*-preopen* [8] *if* $A \subset i\,\mathrm{Int}(j\,\mathrm{Cl}(A))$;
(2) (i, j)*-regular open* [4] *if* $A = i\,\mathrm{Int}(j\,\mathrm{Cl}(A))$;

On each definition above, $i, j = 1, 2$ and $i \neq j$.

The complement of an (i, j)-preopen (resp. (i, j)-regular open) set is called an (i, j)-preclosed (resp. (i, j)-regular closed) set.

Definition 2.6. *Let $(X, \tau_1, \tau_2, I, J)$ be an ideal bitopological space and let $A \subset X$. Then*

(1) A is said to be b-(I, J)-open in $(X, \tau_1, \tau_2, I, J)$ if $A \in IO(X, \tau_1) \cup JO(X, \tau_2)$,
(2) A is said to be b-(I, J)-closed in $(X, \tau_1, \tau_2, I, J)$ if $X - A$ is b-(I, J)-open in (X, τ_1, τ_2).

The family of all b-(I, J)-open (resp. closed) sets in $(X, \tau_1, \tau_2, I, J)$ is denoted by b-$(I, J)O(X, \tau_1, \tau_2)$ (resp. b-$(I, J)C(X, \tau_1, \tau_2)$).

Definition 2.7. *Let $(X, \tau_1, \tau_2, I, J)$ be an ideal bitopological space and $A \subset X$. Then*

(1) The b-(I, J)-closure of A in $(X, \tau_1, \tau_2, I, J)$ denoted by (τ_1, τ_2)-$(I, J)\,\mathrm{Cl}(A)$ and defined (τ_1, τ_2)-$(I, J)\,\mathrm{Cl}(A) = I\,\mathrm{Cl}(A) \cap J\,\mathrm{Cl}(A)$.
(2) The b-(τ_1, τ_2)-(I, J)-interior of A in $(X, \tau_1, \tau_2, I, J)$ denoted by (τ_1, τ_2)-$(I, J)\,\mathrm{Int}(A)$ and defined (τ_1, τ_2)-$(I, J)\,\mathrm{Int}(A) = I\,\mathrm{Int}(A) \cup J\,\mathrm{Int}(A)$.

Example 2.8. *Let $(X, \tau_1, \tau_2, I, J)$ be an ideal bitopological space, where $X = \mathbb{R}$, $\tau_1 = \{\emptyset, \mathbb{R}, \mathbb{R} \setminus \mathbb{Q}\}$, $\tau_2 = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$ and $I = J = \{\emptyset\}$. The I-open sets in (X, τ_1) (resp. J-open sets in (X, τ_2)) are the preopen sets in (X, τ_1) (resp. (X, τ_2)). Take $A = \mathbb{Q}$, then I-$\mathrm{Cl}_{\tau_1}(\mathbb{Q}) = \mathbb{Q}$ and J-$\mathrm{Cl}_{\tau_2}(\mathbb{Q}) = \mathbb{R}$; in consequence, (τ_1, τ_2)-(I, J)-$\mathrm{Cl}(\mathbb{Q}) = \mathbb{Q} \cap \mathbb{R} = \mathbb{Q}$ and (τ_1, τ_2)-(I, J)-$\mathrm{Int}(\mathbb{Q}) = \emptyset \cup \mathbb{Q} = \mathbb{Q}$.*

3. On (i, j)-upper (lower) (I, J)-continuous multifunctions

Definition 3.1. *A multifunction $F : (X, \tau_1, \tau_2, I, J) \to (Y, \sigma_1, \sigma_2)$ is said to be*

(1) (i, j)-upper weakly (I, J)-continuous if for each $x \in X$ and each σ_i-open set V containing $F(x)$, there exists a b-(I, J)-open set U containing x such that $F(U) \subset j\,\mathrm{Cl}(V)$,
(2) (i, j)-lower weakly (I, J)-continuous if for each $x \in X$ and each σ_i-open set V such that $F(x) \cap V \neq \emptyset$, there exists a b-$(I, J)$-open set U containing x such that $F(u) \cap j\,\mathrm{Cl}(V) \neq \emptyset$ for each $u \in U$.

Example 3.2. *In Example 2.8, take $X = Y$ and $\sigma_1 = \tau_1$ and $\sigma_2 = \tau_2$ and define $F : (X, \tau_1, \tau_2, I, J) \to (Y, \sigma_1, \sigma_2)$ as*

$$F(x) = \begin{cases} \mathbb{Q} & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \\ \mathbb{R} \setminus \mathbb{Q} & \text{if } x \in \mathbb{Q} \end{cases}$$

It is easy to that F is (i, j)-upper weakly (I, J)-continuous and (i, j)-lower weakly (I, J)-continuous.

Theorem 3.3. *Let $(X, \tau_1, \tau_2, I, J)$ be an ideal bitopological space and (Y, σ_1, σ_2) a bitopological space. For a multifunction $F : (X, \tau_1, \tau_2, I, J) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is an (i, j)-upper weakly (I, J)-continuous;
(2) $F^+(V) \subset (\tau_1, \tau_2)$-$(I, J)$-$\mathrm{Int}(F^+(j\,\mathrm{Cl}(V)))$ for every σ_i-open set V of Y,
(3) (τ_1, τ_2)-(I, J)-$\mathrm{Cl}(F^-(j\,\mathrm{Int}(K))) \subset F^-(K)$ for every σ_i-closed set K of Y,
(4) (τ_1, τ_2)-(I, J)-$\mathrm{Cl}(F^-(j\,\mathrm{Int}(i\,\mathrm{Cl}(B)))) \subset F^-(i\,\mathrm{Cl}(B))$ for every subset B of Y,
(5) $F^+(i\,\mathrm{Int}(B)) \subset (\tau_1, \tau_2)$-$(I, J)$-$\mathrm{Int}(F^+(j\,\mathrm{Cl}(i\,\mathrm{Int}(B))))$ for every subset B of Y,
(6) (τ_1, τ_2)-(I, J)-$\mathrm{Cl}(F^-(j\,\mathrm{Int}(K))) \subset F^-(K)$ for every (i, j)-regular closed set K of Y,
(7) (τ_1, τ_2)-(I, J)-$\mathrm{Cl}(F^-(V)) \subset F^-(i\,\mathrm{Cl}(V))$ for every σ_j-open set V of Y.

Proof. (1) $\Rightarrow$ (2): Let V be any σ_i-open set and $x \in F^+(V)$. Then we have $F(x) \subset V$. By hypothesis, there exists an b-(I, J)-open set U containing x such that $F(U) \subset j\,\mathrm{Cl}(V)$. Therefore, we have $x \in U \subset F^+(j\,\mathrm{Cl}(V))$. Since $U \in b$-$(I, J)O(X, \tau_1, \tau_2)$, we obtain $x \in (\tau_1, \tau_2)$-$(I, J)\,\mathrm{Int}(F^+(j\,\mathrm{Cl}(V)))$. Hence $F^+(V) \subset (\tau_1, \tau_2)$-$(I, J)\,\mathrm{Int}(F^+(j\,\mathrm{Cl}(V)))$.
(2) $\Rightarrow$ (3): Let K be any σ_i-closed set of Y. Then $Y - K$ is σ_i-open in Y and we have $X - F^-(K) = F^+(Y - K) \subset (\tau_1, \tau_2)$-$(I, J)\,\mathrm{Int}(F^+(j\,\mathrm{Cl}(Y - K))) = (\tau_1, \tau_2)$-$(I, J)\,\mathrm{Int}(F^+(Y - j\,\mathrm{Int}(K))) = (\tau_1, \tau_2)$-$(I, J)\,\mathrm{Int}(X - F^-(j\,\mathrm{Int}(K))) = X - (\tau_1, \tau_2)$-$(I, J)\,\mathrm{Cl}(F^-(j\,\mathrm{Int}(K)))$. Therefore, (τ_1, τ_2)-$(I, J)\,\mathrm{Cl}(F^-(j\,\mathrm{Int}(K))) \subset F^-(K)$.
(3)$\Rightarrow$ (4): Let B be any subset of Y. Then $i\,\mathrm{Cl}(B)$ is σ_i-closed in Y and by (3), we have (τ_1, τ_2)-$(I, J)\,\mathrm{Cl}(F^-(j\,\mathrm{Int}(i\,\mathrm{Cl}(B))) \subset F^-(i\,\mathrm{Cl}(B))$.
(4) $\Rightarrow$ (5): Let B be any subset of Y. By (4), we have $X - (\tau_1, \tau_2)$-$(I, J)\,\mathrm{Int}(F^+(j\,\mathrm{Cl}(i\,\mathrm{Int}(B)))) = (\tau_1, \tau_2)$-$(I, J)\,\mathrm{Cl}(X - F^+(j\,\mathrm{Cl}(i\,\mathrm{Int}(B)))) = (\tau_1, \tau_2)$-$(I, J)\,\mathrm{Cl}(F^-(Y - j\,\mathrm{Cl}(i\,\mathrm{Int}(B))) = (\tau_1, \tau_2)$-$(I, J)\,\mathrm{Cl}(F^-(j\,\mathrm{Int}(i\,\mathrm{Cl}(Y - B))) \subset F^-(i\,\mathrm{Cl}(Y - B)) = X - F^+(i\,\mathrm{Int}(B))$. Therefore, we obtain $F^+(i\,\mathrm{Int}(B)) \subset (\tau_1, \tau_2)$-$(I, J)\,\mathrm{Int}(F^+(j\,\mathrm{Cl}(i\,\mathrm{Int}(B))))$.
(5) $\Rightarrow$ (6): Let K be any (i, j)-regular closed set of Y. Then by (5), we have $X - F^-(K) = X - F^-(i\,\mathrm{Cl}(j\,\mathrm{Int}(K))) = F^+(Y - i\,\mathrm{Cl}(j\,\mathrm{Int}(K))) = F^+(i\,\mathrm{Int}(Y - j\,\mathrm{Int}(K))) \subset (\tau_1, \tau_2)$-$(I, J)\,\mathrm{Int}(F^+(j\,\mathrm{Cl}(i\,\mathrm{Int}(Y - j\,\mathrm{Int}(K))))) = (\tau_1, \tau_2)$-$(I, J)\,\mathrm{Int}(F^+(Y - j\,\mathrm{Int}(i\,\mathrm{Cl}(j\,\mathrm{Int}(K))))) \subset (\tau_1, \tau_2)$-$(I, J)\,\mathrm{Int}(F^+(Y - j\,\mathrm{Int}(K))) = (\tau_1, \tau_2)$-$(I, J)\,\mathrm{Int}(X - F^-(j\,\mathrm{Int}(K))) = X - (\tau_1, \tau_2)$-
$(I, J)\,\mathrm{Cl}(F^-(j\,\mathrm{Int}(K))$. Therefore, we obtain (τ_1, τ_2)-$(I, J)\,\mathrm{Cl}(F^-(j\,\mathrm{Int}(K))) \subset F^-(K)$.
(6) $\Rightarrow$ (7): Let V be any σ_j-open set of Y. Then $i\,\mathrm{Cl}(V)$ is (i, j)-regular closed. By (6), we have (τ_1, τ_2)-$(I, J)\,\mathrm{Cl}(F^-(V)) \subset (\tau_1, \tau_2)$-$(I, J)\,\mathrm{Cl}(F^-(j\,\mathrm{Int}(i\,\mathrm{Cl}(V)))) \subset F^-(i\,\mathrm{Cl}(V))$.

(7) $\Rightarrow$ (1): Let $x \in X$ and V be a σ_i-open set containing $F(x)$. Then $Y - j\operatorname{Cl}(V)$ is σ_j-open in Y and we have $X - (\tau_1, \tau_2)$-$(I, J)\operatorname{Int}(F^+(j\operatorname{Cl}(V))) = (\tau_1, \tau_2)$-$(I, J)\operatorname{Cl}(F^-(Y - j\operatorname{Cl}(V))) \subset F^-(i\operatorname{Cl}(Y - j\operatorname{Cl}(V))) = F^-(Y - i\operatorname{Int}(j\operatorname{Cl}(V))) = X - F^+(i\operatorname{Int}(j(Cl(V))) \subset X - F^+(V)$. Therefore, $x \in F^+(V) \subset (\tau_1, \tau_2)$-$(I, J)\operatorname{Int}(F^+ j\operatorname{Cl}(V)))$. Follows that, there exists $U \in (I, J)O(X, \tau_1, \tau_2)$ such that $x \in U \subset F^+(j\operatorname{Cl}(V))$, hence $F(U) \subset j\operatorname{Cl}(V)$. This shows that F is an (i, j)-upper weakly (I, J)-continuous. □

Theorem 3.4. *For a multifunction $F : (X, \tau_1, \tau_2, I, J) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is an (i, j)-lower weakly (I, J)-continuous;
(2) $F^-(V) \subset (\tau_1, \tau_2)$-$(I, J)\operatorname{Int}(F^-(j\operatorname{Cl}(V)))$ for every σ_i-open set V of Y,
(3) (τ_1, τ_2)-$(I, J)\operatorname{Cl}(F^+(j\operatorname{Int}(K))) \subset F^+(K)$ for every σ_i-closed set K of Y,
(4) (τ_1, τ_2)-$(I, J)\operatorname{Cl}(F^+(j\operatorname{Int}(i\operatorname{Cl}(B)))) \subset F^+(i\operatorname{Cl}(B))$ for every subset B of Y,
(5) $F^-(i\operatorname{Int}(B)) \subset (\tau_1, \tau_2)$-$(I, J)\operatorname{Int}(F^-(j\operatorname{Cl}(i\operatorname{Int}(B))))$ for every subset B of Y,
(6) (τ_1, τ_2)-$(I, J)\operatorname{Cl}(F^+(j\operatorname{Int}(K))) \subset F^+(K)$ for every (i, j)-regular closed set K of Y,
(7) (τ_1, τ_2)-$(I, J)\operatorname{Cl}(F^+(V)) \subset F^+(i\operatorname{Cl}(V))$ for every σ_j-open set V of Y.

Proof. The proof is similar to that of Theorem 3.3. □

Theorem 3.5. *For a multifunction $F : (X, \tau_1, \tau_2, I, J) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is an (i, j)-upper weakly (I, J)-continuous,
(2) (τ_1, τ_2)-$(I, J)\operatorname{Cl}(F^-(j\operatorname{Int}(i\operatorname{Cl}(V)))) \subset F^-(i\operatorname{Cl}(V))$ for every (j, i)-preopen set V of Y,
(3) (τ_1, τ_2)-$(I, J)\operatorname{Cl}(F^-(V)) \subset F^-(i\operatorname{Cl}(V))$ for every (j, i)-preopen set V of Y,
(4) $F^+(V) \subset (\tau_1, \tau_2)$-$(I, J)\operatorname{Int}(F^+(j\operatorname{Cl}(V)))$ for every (i, j)-preopen set V of Y.

Proof. (1) $\Rightarrow$ (2): Let V be any (j, i)-preopen set of Y. Since $j\operatorname{Int}(i\operatorname{Cl}(V))$ is σ_j-open, by Theorem 3.5, we obtain (τ_1, τ_2)-$(I, J)\operatorname{Cl}(F^-(j\operatorname{Int}(i\operatorname{Cl}(V))))$
$\subset F^-(i\operatorname{Cl}(j\operatorname{Int}(i\operatorname{Cl}(V)))) \subset F^-(i\operatorname{Cl}(V))$.
(2) $\Rightarrow$ (3): Let V be any (j, i)-preopen set of Y. Then we have (τ_1, τ_2)-$(I, J)\operatorname{Cl}(F^-(V)) \subset (\tau_1, \tau_2)$-$(I, J)\operatorname{Cl}(F^-(j\operatorname{Int}(i\operatorname{Cl}(V)))) \subset F^-(i\operatorname{Cl}(V))$.
(3) $\Rightarrow$ (4): Let V be any (i, j)-preopen set of Y. By (3), we have $X - (\tau_1, \tau_2)$-$(I, J)\operatorname{Int}(F^+(j\operatorname{Cl}(V))) = (\tau_1, \tau_2)$-$(I, J)\operatorname{Cl}(X - F^+(j\operatorname{Cl}(V))) = (\tau_1, \tau_2)$-$(I, J)\operatorname{Cl}(F^-(Y - j\operatorname{Cl}(V))) \subset F^-(i\operatorname{Cl}(Y - j\operatorname{Cl}(V))) = X - F^+(i\operatorname{Int}(j\operatorname{Cl}(V))) \subset X - F^+(V)$. Therefore, we obtain $F^+(V) \subset (\tau_1, \tau_2)$-$(I, J)\operatorname{Int}(F^+(j\operatorname{Cl}(V)))$.
(4) $\Rightarrow$ (1): Let V be any σ_i-open set of Y. Then V is (i, j)-preopen in Y and $F^+(V) \subset (\tau_1, \tau_2)$-$(I, J)\operatorname{Int}(F^+(j\operatorname{Cl}(V)))$. By Theorem 3.5, F is (i, j)-upper weakly (I, J)-continuous. □

Theorem 3.6. *For a multifunction $F : (X, \tau_1, \tau_2, I, J) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is an (i, j)-lower weakly (I, J)-continuous,

(2) (τ_1,τ_2)-$(I,J)\,\mathrm{Cl}(F^+(j\,\mathrm{Int}(i\,\mathrm{Cl}(V)))) \subset F^+(i\,\mathrm{Cl}(V))$ *for every* (j,i)*-preopen set* V *of* Y*,*
(3) (τ_1,τ_2)-$(I,J)\,\mathrm{Cl}(F^+(V)) \subset F^+(i\,\mathrm{Cl}(V))$ *for every* (j,i)*-preopen set* V *of* Y*,*
(4) $F^-(V) \subset (\tau_1,\tau_2)$-$(I,J)\,\mathrm{Int}(F^-(j\,\mathrm{Cl}(V)))$ *for every* (i,j)*-preopen set* V *of* Y*.*

Proof. The proof is similar to that of Theorem 3.5. □

Lemma 3.7. *If* $F:(X,\tau_1,\tau_2,I,J)\to(Y,\sigma_1,\sigma_2)$ *is an* (i,j)*-lower weakly* (I,J)*-continuous, then for each* $x\in X$ *and each subset* B *of* Y *with* $F(x)\cap(i,j)$-$\mathrm{Int}_\theta(B)\neq\emptyset$*, there exists a* b*-*(I,J)*-open set* U *containing* x *such that* $U\subset F^-(B)$*.*

Proof. Since $F(x)\cap(i,j)$-$\mathrm{Int}_\theta(B)\neq\emptyset$, there exists a σ_i-open set V of Y such that $V\subset j\,\mathrm{Cl}(V)\subset B$ and $F(x)\cap V\neq\emptyset$. Since F is an (i,j)-lower weakly (I,J)-continuous, there exists a b-(I,J)-open set U containing x such that $F(u)\cap j\,\mathrm{Cl}(V)\neq\emptyset$ for every $u\in U$ and hence, $U\subset F^-(B)$. □

Theorem 3.8. *For a multifunction* $F:(X,\tau_1,\tau_2,I,J)\to(Y,\sigma_1,\sigma_2)$*, the following properties are equivalent:*

(1) F *is an* (i,j)*-lower weakly* (I,J)*-continuous,*
(2) (τ_1,τ_2)-$(I,J)\,\mathrm{Cl}(F^+(B))\subset F^+((i,j)$-$\mathrm{Cl}_\theta(B))$ *for every subset* B *of* Y*,*
(3) $F((\tau_1,\tau_2)$-$(I,J)\,\mathrm{Cl}(A))\subset(i,j)$-$\mathrm{Cl}_\theta(F(A))$ *for every subset* A *of* X*.*

Proof. (1) ⇒ (2): Let B be any subset of Y. Suppose that $x\notin F^+((i,j)$-$\mathrm{Cl}_\theta(B))$. Then $x\in F^-(Y\setminus a^(i,j)$-$\mathrm{Cl}_\theta(B))=F^-((i,j)$-$\mathrm{Int}_\theta(Y-B))$. Then, by Lemma 3.7, there exists a b-(I,J)-open set U containing x such that $U\subset F^-(Y-B)=X-F^+(B)$. Thus we have $U\cap F^+(B)=\emptyset$. Then $x\in X-(\tau_1,\tau_2)$-$(I,J)\,\mathrm{Cl}(F^+(B))$ and hence, (τ_1,τ_2)-$(I,J)\,\mathrm{Cl}(F^+(B))\subset F^+((i,j)$-$\mathrm{Cl}_\theta(B))$.
(2) ⇒ (3): Let A be a subset of X. By (2), we have (τ_1,τ_2)-$(I,J)\,\mathrm{Cl}(A)\subset(\tau_1,\tau_2)$-$(I,J)\,\mathrm{Cl}(F^+(F(A)))\subset F^+((i,j)$-$\mathrm{Cl}_\theta(F(A)))$. Hence
$F((\tau_1,\tau_2)$-$(I,J)\,\mathrm{Cl}(A))\subset(i,j)$-$\mathrm{Cl}_\theta(F(A))$.
(3) ⇒ (1): Let V be any σ_j-open set of Y. Then (i,j)-$\mathrm{Cl}_\theta(V)=i\,\mathrm{Cl}(V)$ and by (3) we have $F((\tau_1,\tau_2)$-$(I,J)\,\mathrm{Cl}(F^+(V)))\subset(i,j)$-$\mathrm{Cl}_\theta(F(F^+(V)))$
$\subset(i,j)$-$\mathrm{Cl}_\theta(V)=i\,\mathrm{Cl}(V)$. Therefore, we obtain (τ_1,τ_2)-$(I,J)\,\mathrm{Cl}(F^+(V))$
$\subset F^+(i\,\mathrm{Cl}(V))$. By Theorem 3.4, F is an (i,j)-lower weakly (I,J)-continuous. □

Theorem 3.9. *For a multifunction* $F:(X,\tau_1,\tau_2,I,J)\to(Y,\sigma_1,\sigma_2)$*, the following properties are equivalent:*

(1) F *is an* (i,j)*-upper weakly* (I,J)*-continuous,*
(2) (τ_1,τ_2)-$(I,J)\,\mathrm{Cl}(F^-(j\,\mathrm{Int}((i,j)$-$\mathrm{Cl}_\theta(B))))\subset F^-((i,j)$-$\mathrm{Cl}_\theta(B))$ *for every subset* B *of* Y*,*
(3) (τ_1,τ_2)-$(I,J)\,\mathrm{Cl}(F^-(j\,\mathrm{Int}(i\,\mathrm{Cl}(B))))\subset F^-((i,j)$-$\mathrm{Cl}_\theta(B))$ *for every subset* B *of* Y*.*

Proof. (1) ⇒ (2): Let B be any subset of Y. Then the (i,j)-$\mathrm{Cl}_\theta(B)$ is σ_i-closed in Y. Therefore, by Theorem 3.3, (τ_1,τ_2)-

$(I, J)\operatorname{Cl}(F^-(j\operatorname{Int}((i, j)\text{-}\operatorname{Cl}_\theta(B)))) \subset F^-((i, j)\text{-}\operatorname{Cl}_\theta(B))$.
(2) $\Rightarrow$ (3): This is obvious, since $i\operatorname{Cl}(B) \subset (i, j)\text{-}\operatorname{Cl}_\theta(B)$.
(3) $\Rightarrow$ (1): Let K be any (i, j)-regular closed set of Y. Then, by Lemma 2.4, $(i, j)\text{-}\operatorname{Cl}_\theta(j\operatorname{Int}(K)) = i\operatorname{Cl}(j\operatorname{Int}(K))$ and we have $(\tau_1, \tau_2)\text{-}(I, J)\operatorname{Cl}(F^-(j\operatorname{Int}(K))) = (\tau_1, \tau_2)\text{-}(I, J)\operatorname{Cl}(F^-(j\operatorname{Int}(i\operatorname{Cl}(j\operatorname{Int}(K))))) \subset F^-((i, j)\text{-}\operatorname{Cl}_\theta(j\operatorname{Int}(K))) = F^-(K)$. Therefore, $(\tau_1, \tau_2)\text{-}(I, J)\operatorname{Cl}(F^-(j\operatorname{Int}(K))) \subset F^-(K)$ and by Theorem 3.3, F is an (i, j)-upper weakly (I, J)-continuous. □

Theorem 3.10. *For a multifunction $F : (X, \tau_1, \tau_2, I, J) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is an (i, j)-lower weakly (I, J)-continuous,
(2) $(\tau_1, \tau_2)\text{-}(I, J)\operatorname{Cl}(F^+(j\operatorname{Int}((i, j)\text{-}\operatorname{Cl}_\theta(B)))) \subset F^+((i, j)\text{-}\operatorname{Cl}_\theta(B))$ for every subset B of Y,
(3) $(\tau_1, \tau_2)\text{-}(I, J)\operatorname{Cl}(F^+(j\operatorname{Int}(i\operatorname{Cl}(B)))) \subset F^+((i, j)\text{-}\operatorname{Cl}_\theta(B))$ for every subset B of Y.

Proof. The proof is similar to that of Theorem 3.9. □

Conclusion

(1) We define and characterize the (i, j)-upper weakly (I, J)-continuous.
(2) We define and characterize the (i, j)-lower weakly (I, J)-continuous.
(3) We give some illustrative examples.

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