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SOME RESULTS OF EXTENDED τ-GAUSS HYPERGEOMETRIC FUNCTION AND τ-KUMMER HYPERGEOMETRIC FUNCTION BY USING WIMAN'S FUNCTION

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Abstract. The main aim of this research paper is to introduce a new extension of τ-Gauss hypergeometric function and τ-Kummer hypergeometric function by using an extended beta function. Some functional relations, summation relations, integral representations, linear transformation formulas, and derivative formulas for these extended functions are derived.

1. Introduction and Preliminaries

In the past years, various extensions, properties and applications of special functions have been discussed by many researchers and authors. Here, we aim to study some results of beta function and τ-Gauss hypergeometric function by using Wiman's function.

Classical Euler Beta function and Gamma function are defined as [10]

$$B(y_1, y_2) = \int_0^1 t^{y_1-1}(1-t)^{y_2-1}dt, \tag{1.1}$$

where $\Re(y_1) > 0$ and $\Re(y_2) > 0$.

$$\Gamma(y_1) = \int_0^\infty t^{y_1-1}e^{-t}dt, \tag{1.2}$$

where $\Re(y_1) > 0$.

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Gauss hypergeoemtric function ${}_2F_1(z)$ is defined as [9]:

$$ {}_2F_1(\delta_1,\delta_2;\delta_3;z) = F(\delta_1,\delta_2;\delta_3;z) = \sum_{n=0}^{\infty} \frac{B(\delta_2+n,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}.(\delta_1)_n \frac{z^n}{n!}. \tag{1.3} $$

$$ (\Re(\delta_3) > \Re(\delta_2) > 0 \text{ and } |z| < 1) $$

The integral representation of the Gauss hypergeometric function is defined as:

$$ F(\delta_1,\delta_2;\delta_3;z) = \frac{1}{B(\delta_2,\delta_3-\delta_2)} \int_0^1 t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}(1-zt)^{-\delta_1}dt. \tag{1.4} $$

Confluent hypergeometric function is defined as [9]:

$$ {}_1F_1(\delta_2,\delta_3;z) = \Phi(\delta_2,\delta_3;z) = \sum_{k=0}^{\infty} \frac{B(\delta_2+n,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)} \frac{z^n}{n!}. \tag{1.5} $$

$$ (\Re(\delta_3) > \Re(\delta_2) > 0) $$

The integral representation of the confluent hypergeometric function is defined as :

$$ \Phi(\delta_2,\delta_3;z) = \frac{1}{B(\delta_2,\delta_3-\delta_2)} \int_0^1 t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}e^{zt}dt. \tag{1.6} $$

τ-Gauss hypergeometric function ${}_2R_1(z)$ is defined as [12]:

$$ {}_2R_1(\delta_1,\delta_2,\delta_3;\tau;z) = \frac{\Gamma(\delta_3)}{\Gamma(\delta_2)} \sum_{n=0}^{\infty} \frac{(\delta_1)_n\Gamma(\delta_2+n\tau)}{\Gamma(\delta_3+n\tau)} \frac{z^n}{n!}. \tag{1.7} $$

$$ (\tau > 0; |z| < 1; \Re(\delta_3) > \Re(\delta_2) > 0 \text{ when } |z| = 1) $$

The integral representation of the τ-Gauss hypergeometric function is defined as:

$$ {}_2R_1(\delta_1,\delta_2,\delta_3;\tau;z) = \frac{1}{B(\delta_2,\delta_3-\delta_2)} \int_0^1 t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}(1-zt^{\tau})^{-\delta_1}dt. \tag{1.8} $$

$$ (\tau > 0, |\arg(1-z)| < \pi; \Re(\delta_3) > \Re(\delta_2) > 0) $$

The τ-Kummer hypergeometric function is defined as [7]:

$$ {}_1\Phi_1(\delta_2,\delta_3;\tau,z) = \frac{\Gamma(\delta_3)}{\Gamma(\delta_2)} \sum_{n=0}^{\infty} \frac{\Gamma(\delta_2+n\tau)}{\Gamma(\delta_3+n\tau)} \frac{z^n}{n!}, \tag{1.9} $$

$$ (\tau > 0, \delta_2 \in \mathbb{C}; \delta_3 \in \mathbb{C} \setminus \mathbb{Z}_0^-, \Re(\delta_3) > \Re(\delta_2) > 0) $$

whereas the integral representation of the τ-Kummer hypergeometric function is defined as:

$$ {}_1\Phi_1(\delta_2,\delta_3;\tau,z) = \frac{1}{B(\delta_2,\delta_3-\delta_2)} \int_0^1 t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}e^{zt^{\tau}}dt. \tag{1.10} $$

One parameter Mittag-Leffler function and two parameter Mittag-Leffler function (Wiman's function) is defined as [5, 8, 13]:

$$ E_{y_1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ky_1+1)'} \tag{1.11} $$

where $\Re(y_1) \geqslant 0$ *and* $z \in \mathbb{C}$.

$$E_{y_1,y_2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ky_1 + y_2)'} \tag{1.12}$$

where $\Re(y_1) \geqslant 0$, $\Re(y_2) \geqslant 0$ and $z \in \mathbb{C}$.

In 1997, Chaudhary *et al.* extended the classical Euler beta function defined in [2] and then, in 2004, he used this extended beta function to generalize the Gauss hypergeometric function and confluent hypergeometric function as [3]:

$$B(y_1, y_2; y) = \int_0^1 t^{y_1-1}(1-t)^{y_2-1} e^{\frac{-y}{t(1-t)}}\, dt, \tag{1.13}$$

where $\Re(y_1) > 0$, $\Re(y_2) > 0$, $\Re(y) > 0$.

Later, in 2014, Choi *et al.* [4] defined a generalization of extended beta function, using it to broaden the definition of extended hypergeometric Gauss and extended Confluent hypergeometric function as follow:

$$B(y_1, y_2; u, v) = \int_0^1 t^{y_1-1}(1-t)^{y_2-1} e^{(-\frac{u}{t} - \frac{v}{1-t})} dt \tag{1.14}$$

where $\Re(y_1) > 0$, $\Re(y_2) > 0$, $\Re(u) > 0$ and $\Re(v) > 0$.

$$F_{r,s}(r_0, r_1, r_2; z) = \sum_{k=0}^{\infty} \frac{B(r_1 + k, r_2 - r_1; r, s)}{B(r_1, r_2 - r_1)} (r_0)_k \frac{z^k}{k!} \tag{1.15}$$

where $\Re(r_1) > \Re(r_1) > 0$, $r \geqslant 0$, $s \geqslant 0$ and $|z| < 1$.

$$\Phi_{r,s}(r_1, r_2; z) = \sum_{k=0}^{\infty} \frac{B(r_1 + k, r_2 - r_1; r, s)}{B(r_1, r_2 - r_1)} \frac{z^k}{k!}. \tag{1.16}$$

Remark 1.1. After putting $r = 0$, and $s = 0$, equations (1.15) and (1.16) gives Gauss hypergeometric function and Confluent hypergeometric function respectively:

$$F_{0,0}(r_0, r_1, r_2; z) = F(r_0, r_1, r_2; z) = \sum_{k=0}^{\infty} \frac{B(r_1 + k, r_2 - r_1)}{B(r_1, r_2 - r_1)} .(r_0)_k \frac{z^k}{k!} \tag{1.17}$$

and

$$\Phi_{0,0}(r_1, r_2; z) = \Phi(r_1, r_2; z) = \sum_{k=0}^{\infty} \frac{B(r_1 + k, r_2 - r_1)}{B(r_1, r_2 - r_1)} \frac{z^k}{k!}. \tag{1.18}$$

In 2018, Shabad *et al.* [11] extended classic Euler beta function using one parameter Mittag-Leffler function as:

$$B_u^y(y_1, y_2) = \int_0^1 t^{y_1-1}(1-t)^{y_2-1} E_u(\frac{-y}{t(1-t)}) dt \tag{1.19}$$

where $\Re(y_1)$, $\Re(y_2) > 0$, $\Re(u) > 0$ and $\Re(y) > 0$.

$$F_{r,\alpha}(r_0, r_1, r_2; z) = \sum_{k=0}^{\infty} \frac{B_\alpha^r(r_1 + k, r_2 - r_1)}{B(r_1, r_2 - r_1)} (r_0)_k \frac{z^k}{k!} \tag{1.20}$$

where $\Re(r_2) > \Re(r_1) > 0$, $\alpha \geqslant 0$, $r \geqslant 0$ and $|z| < 1$.

$$\Phi_{r,\alpha}(r_1, r_2; z) = \sum_{k=0}^{\infty} \frac{B_{\alpha}^{r}(r_1 + k, r_2 - r_1)}{B(r_1, r_2 - r_1)} \frac{z^k}{k!} \tag{1.21}$$

where $\Re(r_2) > \Re(r_1) > 0$, $\alpha \geqslant 0$, $r \geqslant 0$.

Remark 1.2. After setting $\alpha = 1$ and $r = 0$, equations (1.20) and (1.21) gives Gauss hypergeometric and Confluent hypergeometric function respectively:

$$F_{0,1}(r_0, r_1, r_2; z) = F(r_0, r_1, r_2; z)\,, \tag{1.22}$$

$$\Phi_{0,1}(r_1, r_2; z) = \Phi(r_1, r_2; z)\,. \tag{1.23}$$

Very recently, Goyal *et al.* [6] introduced an extension of the beta function using the Wiman function:

$$B_{(u_1,u_2)}^{(u)}(y_1, y_2) = \int_0^1 t^{y_1-1}(1-t)^{y_2-1} E_{u_1,u_2}\left(\frac{-u}{t(1-t)}\right) dt, \tag{1.24}$$

where $\min\{\Re(y_1),\ \Re(y_2)\} > 0$, $\Re(u_1) > 0$, $\Re(u_2) > 0$, $u \geqslant 0$ and E_{u_1,u_2} is 2-parameter Mittag Leffler function given by (1.12).

2. Extension of τ-Gauss hypergeometric and τ-Kummer hypergeometric function

In this section, we introduce a new extension of τ-Gauss hypergeometric function and τ-Kummer hypergeometric function by using extended beta function given in (1.24).

Definition 2.1. A new extended τ-Gauss hypergeometric function $R_{(r_1,r_2)}^{(r)}(\delta_1, \delta_2; \delta_3; \tau; z)$ is defined as:

$$R_{(r_1,r_2)}^{(r)}(\delta_1, \delta_2; \delta_3; \tau; z) = \sum_{n=0}^{\infty} \frac{B_{(r_1,r_2)}^{(r)}(\delta_2 + n\tau, \delta_3 - \delta_2)}{B(\delta_2, \delta_3 - \delta_2)} (\delta_1)_n \frac{z^n}{n!} \tag{2.1}$$

where $\Re(\delta_3) > \Re(\delta_2) > 0$, $\Re(r_1) > 0$, $\Re(r_2) > 0$, $r \geqslant 0$, $|z| < 1$ and $B_{(r_1,r_2)}^{(r)}(y_1, y_2)$ is extended beta function.

We have another representation of a new extended τ-Gauss hypergeometric function after substituting beta function value in terms of gamma function, i.e.,

$$R_{(r_1,r_2)}^{(r)}(\delta_1, \delta_2; \delta_3; \tau; z) = \frac{\Gamma(\delta_3)}{\Gamma(\delta_2)\Gamma(\delta_3 - \delta_2)} \sum_{n=0}^{\infty} (\delta_1)_n B_{(r_1,r_2)}^{(r)}(\delta_2 + n\tau, \delta_3 - \delta_2) \frac{z^n}{n!}. \tag{2.2}$$

Definition 2.2. A new extended τ-Kummer hypergeometric function $\Phi_{(r_1,r_2)}^{(r)}(\delta_2; \delta_3; \tau; z)$ is defined as:

$$\Phi_{(r_1,r_2)}^{(r)}(\delta_2; \delta_3; \tau; z) = \sum_{n=0}^{\infty} \frac{B_{(r_1,r_2)}^{(r)}(\delta_2 + n\tau, \delta_3 - \delta_2)}{B(\delta_2, \delta_3 - \delta_2)} \frac{z^n}{n!} \tag{2.3}$$

where $\Re(\delta_3) > \Re(\delta_2) > 0$, $\Re(r_1) > 0$, $\Re(r_2) > 0$, $r \geqslant 0$ and $B_{(r_1,r_2)}^{(r)}(y_1, y_2)$ is extended beta function.

After substituting beta function value in terms of gamma function, we have another representation of a new τ-kummer hypergeometric function as follows:

$$\Phi^{(r)}_{(r_1,r_2)}(\delta_2;\delta_3;\tau;z) = \frac{\Gamma(\delta_3)}{\Gamma(\delta_2)\Gamma(\delta_3-\delta_2)}\sum_{n=0}^{\infty} B^{(r)}_{(r_1,r_2)}(\delta_2+n\tau,\delta_3-\delta_2)\frac{z^n}{n!} \tag{2.4}$$

where $\Re(\delta_3) > \Re(\delta_2) > 0$, $\Re(r_1) > 0$, $\Re(r_2) > 0$, $r \geqslant 0$.

Remark 2.3. We know that τ-Gauss hypergeometric function does not change if the δ_1 and δ_2 parameters are swapped while keeping δ_3 fixed. This symmetric property with respect to parameters δ_1 and δ_2 can also be deducted from the new extended τ-Gauss hypergeometric function (2.1):

$$R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z) = R^{(r)}_{(r_1,r_2)}(\delta_2,\delta_1;\delta_3;\tau;z). \tag{2.5}$$

Then, after replacement of δ_1 with δ_2 from above property (2.5), we have

$$\begin{aligned} R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z) &= \sum_{n=0}^{\infty} \frac{B^{(r)}_{(r_1,r_2)}(\delta_2+n\tau,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}(\delta_1)_n\frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{B^{(r)}_{(r_1,r_2)}(\delta_1+n\tau,\delta_3-\delta_1)}{B(\delta_1,\delta_3-\delta_1)}(\delta_2)_n\frac{z^n}{n!}. \end{aligned} \tag{2.6}$$

Remark 2.4. (i) If we set $\tau = 1$ and $r_2 = 1$ in (2.1) and (2.3), then we get a new extension of Gauss hypergeometric function and confluent hypergeometric function given by (1.20) and (1.21), respectively.

$$F^{(r)}_{(r_1,1)}(\delta_1,\delta_2;\delta_3;z) = F_{(r,r_1)}(\delta_1,\delta_2;\delta_3;z) \tag{2.7}$$

and

$$\Phi^{(r)}_{(r_1,1)}(\delta_2;\delta_3;z) = \Phi_{(r,r_1)}(\delta_2;\delta_3;z). \tag{2.8}$$

(ii) If we set $r = 0$, and $r_1 = r_2 = 1$ in (2.1) and (2.3), then we get τ-Gauss hypergeometric function and τ-Kummer hypergeometric function given by (1.7) and (1.9), respectively.

$$R^{(0)}_{(1,1)}(\delta_1,\delta_2;\delta_3;\tau;z) = R(\delta_1,\delta_2;\delta_3;\tau;z) \tag{2.9}$$

and

$$\Phi^{(0)}_{(1,1)}(\delta_2;\delta_3;\tau;z) = \Phi(\delta_2;\delta_3;\tau;z). \tag{2.10}$$

(iii) If we set $r = 0$, $\tau = 1$ and $r_1 = r_2 = 1$ in (2.1) and (2.3), then we get Gauss hypergeometric function and confluent hypergeometric function given by (1.3) and (1.5), respectively.

$$F^{(0)}_{(1,1)}(\delta_1,\delta_2;\delta_3;z) = F(\delta_1,\delta_2;\delta_3;z) \tag{2.11}$$

and

$$\Phi^{(0)}_{(1,1)}(\delta_2;\delta_3;z) = \Phi(\delta_2;\delta_3;z). \tag{2.12}$$

Theorem 2.5. *Consider $R^{(r)}_{(r_1,r_2)}(\delta_1, \delta_2; \delta_3; \tau; z)$ and $\Phi^{(r)}_{(r_1,r_2)}(\delta_2; \delta_3; \tau; z)$ functions. Then the following functional relation holds:*

$$\begin{aligned} R^{(r)}_{(r_1,r_2)}(\delta_1, \delta_2; \delta_3; \tau; z) &= \frac{\delta_2}{\delta_3} R^{(r)}_{(r_1,r_2)}(\delta_1, \delta_2 + 1; \delta_3 + 1; \tau; z) \\ &\quad + \frac{\delta_3 - \delta_2}{\delta_3} R^{(r)}_{(r_1,r_2)}(\delta_1, \delta_2; \delta_3 + 1; \tau; z), \end{aligned} \tag{2.13}$$

where $\Re(\delta_3) > \Re(\delta_2) > 0$, $\Re(r_1) > 0$, $\Re(r_2) > 0$, $r \geqslant 0$ and $|z| < 1$.

$$\Phi^{(r)}_{(r_1,r_2)}(\delta_2; \delta_3; \tau; z) = \frac{\delta_2}{\delta_3} \Phi^{(r)}_{(r_1,r_2)}(\delta_2 + 1; \delta_3 + 1; \tau; z) + \Phi^{(r)}_{(r_1,r_2)}(\delta_2; \delta_3 + 1; \tau; z), \tag{2.14}$$

where $\Re(\delta_3) > 0$, $\Re(\delta_2) > 0$, $\Re(r_1) > 0$, $\Re(r_2) > 0$, and $r \geqslant 0$.

Proof. Using the following known relation from [6]

$$B^{(u)}_{(u_1,u_2)}(y_1, y_2) = B^{(u)}_{(u_1,u_2)}(y_1, y_2 + 1) + B^{(u)}_{(u_1,u_2)}(y_1 + 1, y_2) \tag{2.15}$$

in (2.1), we have

$$\begin{aligned} & R^{(r)}_{(r_1,r_2)}(\delta_1, \delta_2; \delta_3; \tau; z) \\ &= \sum_{n=0}^{\infty} (\delta_1)_n \frac{B^{(r)}_{(r_1,r_2)}(\delta_2 + n\tau + 1, \delta_3 - \delta_2) + B^{(r)}_{(r_1,r_2)}(\delta_2 + n\tau, \delta_3 - \delta_2 + 1)}{B(\delta_2, \delta_3 - \delta_2)} \frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} (\delta_1)_n \frac{B^{(r)}_{(r_1,r_2)}(\delta_2 + n\tau + 1, \delta_3 - \delta_2)}{B(\delta_2, \delta_3 - \delta_2)} \frac{z^n}{n!} \\ &\quad + \sum_{n=0}^{\infty} (\delta_1)_n \frac{B^{(r)}_{(r_1,r_2)}(\delta_2 + n\tau, \delta_3 - \delta_2 + 1)}{B(\delta_2, \delta_3 - \delta_2)} \frac{z^n}{n!} \\ &= \frac{B(\delta_2 + 1, \delta_3 - \delta_2)}{B(\delta_2, \delta_3 - \delta_2)} \sum_{n=0}^{\infty} (\delta_1)_n \frac{B^{(r)}_{(r_1,r_2)}(\delta_2 + n\tau + 1, \delta_3 - \delta_2)}{B(\delta_2 + 1, \delta_3 - \delta_2)} \frac{z^n}{n!} \\ &\quad + \frac{B(\delta_2, \delta_3 - \delta_2 + 1)}{B(\delta_2, \delta_3 - \delta_2)} \sum_{n=0}^{\infty} (\delta_1)_n \frac{B^{(r)}_{(r_1,r_2)}(\delta_2 + n\tau, \delta_3 - \delta_2 + 1)}{B(\delta_2, \delta_3 - \delta_2)} \frac{z^n}{n!}. \end{aligned}$$

Then, using the value of beta function in terms of gamma function together with (2.1), allow us to get the desired result. Similarly, using (2.15) in (2.3) and following the same rule that led to the result (2.13), we obtain the desired statement (2.14), and the theorem is fully proved. □

Remark 2.6. (i) From (2.5), we observe that for $R^{(r)}_{(r_1,r_2)}(\delta_1, \delta_2; \delta_3; \tau; z)$ another functional relation that also holds is:

$$\begin{aligned} R^{(r)}_{(r_1,r_2)}(\delta_1, \delta_2; \delta_3; \tau; z) &= \frac{\delta_1}{\delta_3} R^{(r)}_{(r_1,r_2)}(\delta_2, \delta_1 + 1; \delta_3 + 1; \tau; z) \\ &\quad + \frac{\delta_3 - \delta_1}{\delta_3} R^{(r)}_{(r_1,r_2)}(\delta_2, \delta_1; \delta_3 + 1; \tau; z). \end{aligned} \tag{2.16}$$

(ii) If we use (2.5) on the right side of (2.16), another functional relation obtains for $R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)$ is as follows:

$$R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)=\frac{\delta_1}{\delta_3}R^{(r)}_{(r_1,r_2)}(\delta_1+1,\delta_2;\delta_3+1;\tau;z)\\+\frac{(\delta_3-\delta_1)}{\delta_3}R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3+1;\tau;z). \tag{2.17}$$

Theorem 2.7. *Consider $R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)$ and $\Phi^{(r)}_{(r_1,r_2)}(\delta_2;\delta_3;\tau;z)$ functions. The following sum relation holds:*

$$R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)=(\delta_3-\delta_2)\sum_{k=0}^{\infty}\frac{(\delta_2)_k}{(\delta_3)_{k+1}}R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2+k;\delta_3+k+1;\tau;z), \tag{2.18}$$

where $\Re(\delta_3)>\Re(\delta_2)>0$, $\Re(r_1)>0$, $\Re(r_2)>0$, $r\geqslant 0$ and $|z|<1$.

$$\Phi^{(r)}_{(r_1,r_2)}(\delta_2;\delta_3;\tau;z)=(\delta_3-\delta_2)\sum_{k=0}^{\infty}\frac{(\delta_2)_k}{(\delta_3)_{k+1}}\Phi^{(r)}_{(r_1,r_2)}(\delta_2+k;\delta_3+k+1;\tau;z), \tag{2.19}$$

where $\Re(\delta_3)>0$, $\Re(\delta_2)>0$, $\Re(r_1)>0$, $\Re(r_2)>0$, and $r\geqslant 0$.

Proof. Using the following known relation from reference [6]:

$$B^{(u)}_{(u_1,u_2)}(y_1,y_2)=\sum_{k=0}^{\infty}B^{(u)}_{(u_1,u_2)}(y_1+k,y_2+1) \tag{2.20}$$

in (2.1), we have

$$\begin{aligned}R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)&=\sum_{n=0}^{\infty}\frac{\sum_{k=0}^{\infty}B^{(r)}_{(r_1,r_2)}(\delta_2+n\tau+k,\delta_3-\delta_2+1)}{B(\delta_2,\delta_3-\delta_2)}(\delta_1)_n\frac{z^n}{n!}\\&=\sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}\frac{B^{(r)}_{(r_1,r_2)}(\delta_2+n\tau+k,\delta_3-\delta_2+1)}{B(\delta_2,\delta_3-\delta_2)}(\delta_1)_n\frac{z^n}{n!}\right)\\&=\sum_{k=0}^{\infty}\frac{B(\delta_2+k,\delta_3-\delta_2+1)}{B(\delta_2,\delta_3-\delta_2)}\\&\quad\cdot\sum_{n=0}^{\infty}\frac{B^{(r)}_{(r_1,r_2)}(\delta_2+n\tau+k,\delta_3-\delta_2+1)}{B(\delta_2+k,\delta_3-\delta_2)}(\delta_1)_n\frac{z^n}{n!}.\end{aligned}$$

Moreover, from (2.1)

$$R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)=\sum_{k=0}^{\infty}\frac{B(\delta_2+k,\delta_3-\delta_2+1)}{B(\delta_2,\delta_3-\delta_2)}.R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2+k;\delta_3+k+1;\tau;z).$$

Using the value of beta function in terms of gamma function and value of gamma function in terms of pocchammer symbol in the above equation, we obtain the desired statement (2.18).

Similarly, using (2.20) in (2.3), and the following the same rule getting to result (2.18), we obtain the desired statement (2.19), and the theorem is fully proved. □

Remark 2.8. (i) From (2.5),we observe that for $R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)$ another summation relation that also holds is:

$$R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z) = (\delta_3-\delta_1)\sum_{k=0}^{\infty}\frac{(\delta_1)_k}{(\delta_3)_{k+1}}R^{(r)}_{(r_1,r_2)}(\delta_2,\delta_1+k;\delta_3+k+1;\tau;z). \tag{2.21}$$

(ii) If we use (2.5) on the right side of (2.21), we obtain another summation relation for $R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)$ as follows:

$$R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z) = (\delta_3-\delta_1)\sum_{k=0}^{\infty}\frac{(\delta_1)_k}{(\delta_3)_{k+1}}R^{(r)}_{(r_1,r_2)}(\delta_1+k,\delta_2;\delta_3+k+1;\tau;z). \tag{2.22}$$

Theorem 2.9. *Consider $R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)$ and $\Phi^{(r)}_{(r_1,r_2)}(\delta_2;\delta_3;\tau;z)$ functions. The following sum relation holds :*

$$R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z) = \sum_{k=0}^{\infty}\frac{(\delta_2-\delta_3+1)_k B(\delta_2+k,1)}{k!\,B(\delta_2,\delta_3-\delta_2)}R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2+k;\delta_2+k+1;\tau;z), \tag{2.23}$$

where $\Re(\delta_3)>0$, $\Re(\delta_2)>0$, $\Re(r_1)>0$, $\Re(r_2)>0$, $r\geqslant 0$ *and* $|z|<1$.

$$\Phi^{(r)}_{(r_1,r_2)}(\delta_2;\delta_3;\tau;z) = \sum_{k=0}^{\infty}\frac{(\delta_2-\delta_3+1)_k B(\delta_2+k,1)}{k!\,B(\delta_2,\delta_3-\delta_2)}\Phi^{(r)}_{(r_1,r_2)}(\delta_2+k;\delta_2+k+1;\tau;z), \tag{2.24}$$

where $\Re(\delta_3)>0$, $\Re(\delta_2)>0$, $\Re(r_1)>0$, $\Re(r_2)>0$, *and* $r\geqslant 0$.

Proof. Using the following known relation from reference [6]:

$$B^{(u)}_{(u_1,u_2)}(y_1,1-y_2) = \sum_{k=0}^{\infty}\frac{(y_2)_k}{k!}B^{(u)}_{(u_1,u_2)}(y_1+k,1) \tag{2.25}$$

in (2.1), we get

$$\begin{aligned}
&R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)\\
&\quad=\sum_{n=0}^{\infty}\left(\frac{\sum_{k=0}^{\infty}(1+\delta_2-\delta_3)_k B^{(r)}_{(r_1,r_2)}(\delta_2+n\tau+k,1)}{k!\,B(\delta_2,\delta_3-\delta_2)}\right)\frac{z^n}{n!}(\delta_1)_n\\
&\quad=\sum_{k=0}^{\infty}\frac{B(\delta_2+k,1)(1+\delta_2-\delta_3)_k}{k!\,B(\delta_2,\delta_3-\delta_2)}\sum_{n=0}^{\infty}\frac{B^{(r)}_{(r_1,r_2)}(\delta_2+n\tau+k,1)}{B(\delta_2+k,1)}(\delta_1)_n\frac{z^n}{n!}\\
&\quad=\sum_{k=0}^{\infty}\frac{B(\delta_2+k,1)(1+\delta_2-\delta_3)_k}{k!\,B(\delta_2,\delta_3-\delta_2)}R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2+k;\delta_2+k+1;\tau;z).
\end{aligned}$$

Similarly, using (2.25) in (2.3), and following the same rule getting to result (2.23), we obtain the desired statement (2.24), and the theorem is fully proved. ☐

Remark 2.10. (i) From (2.5), we observe that, for function $R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)$ another summation relation that also holds is:

$$R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z) = \sum_{k=0}^{\infty} \frac{B(\delta_1+k,1)(\delta_1-\delta_3+1)_k}{k!\, B(\delta_1,\delta_3-\delta_1)} R^{(r)}_{(r_1,r_2)}(\delta_2,\delta_1+k;\delta_1+k+1;\tau;z). \tag{2.26}$$

(ii) If we use (2.5) on the right hand side of (2.26), we have another summation relation for $R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)$ as follows:

$$R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z) = \sum_{k=0}^{\infty} \frac{B(\delta_1+k,1)(\delta_1-\delta_3+1)_k}{k!\, B(\delta_1,\delta_3-\delta_1)} R^{(r)}_{(r_1,r_2)}(\delta_1+k,\delta_2;\delta_1+k+1;\tau;z). \tag{2.27}$$

Theorem 2.11. *Consider $R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)$ and $\Phi^{(r)}_{(r_1,r_2)}(\delta_2;\delta_3;\tau;z)$ functions. The following results holds:*

$$R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z) = \sum_{l=0}^{k} \binom{k}{l} \frac{B(\delta_2+l,\delta_3-\delta_2+k-l)}{B(\delta_2,\delta_3-\delta_2)} R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2+l;\delta_3+k;\tau;z), \tag{2.28}$$

where $\Re(\delta_3)>0$, $\Re(\delta_2)>0$, $\Re(r_1)>0$, $\Re(r_2)>0$, $r\geqslant 0$, $\tau>0$, $k\in\mathbb{N}$ and $|z|<1$.

$$\Phi^{(r)}_{(r_1,r_2)}(\delta_2;\delta_3;\tau;z) = \sum_{l=0}^{k} \binom{k}{l} \frac{B(\delta_2+l,\delta_3-\delta_2+k-l)}{B(\delta_2,\delta_3-\delta_2)} \Phi^{(r)}_{(r_1,r_2)}(\delta_2+l;\delta_3+k;\tau;z), \tag{2.29}$$

where $\Re(\delta_3)>0$, $\Re(\delta_2)>0$, $\Re(r_1)>0$, $\Re(r_2)>0$, and $r\geqslant 0$, $\tau>0$ and $k\in\mathbb{N}$.

Proof. Using the following known relation from reference [6]:

$$B^{(u)}_{(u_1,u_2)}(y_1,y_2) = \sum_{l=0}^{k} \binom{k}{l} B^{(u)}_{(u_1,u_2)}(y_1+l,y_2+k-l) \tag{2.30}$$

in (2.1), we have

$$\begin{aligned}
R^{(r)}_{(r_1,r_2)}&(\delta_1,\delta_2;\delta_3;\tau;z)\\
&= \sum_{n=0}^{\infty} \frac{(\delta_1)_n}{B(\delta_2,\delta_3-\delta_2)} \left(\sum_{l=0}^{k} \binom{k}{l} B^{(r)}_{(r_1,r_2)}(\delta_2+n\tau+l,\delta_3-\delta_2+k-l) \right) \frac{z^n}{n!}\\
&= \sum_{l=0}^{k} \binom{k}{l} \frac{1}{B(\delta_2,\delta_3-\delta_2)} \left(\sum_{n=0}^{\infty} B^{(r)}_{(r_1,r_2)}(\delta_2+n\tau+l,\delta_3-\delta_2+k-l)(\delta_1)_n \frac{z^n}{n!} \right)\\
&= \sum_{l=0}^{k} \binom{k}{l} \frac{B(\delta_2+l,\delta_3-\delta_2+k-l)}{B(\delta_2,\delta_3-\delta_2)} \sum_{n=0}^{\infty} \frac{B^{(r)}_{(r_1,r_2)}(\delta_2+n\tau+l,\delta_3-\delta_2+k-l)}{B(\delta_2+l,\delta_3-\delta_2+k-l)} (\delta_1)_n \frac{z^n}{n!}
\end{aligned}$$

$$= \sum_{l=0}^{k} \binom{k}{l} \frac{B(\delta_2 + l, \delta_3 - \delta_2 + k - l)}{B(\delta_2, \delta_3 - \delta_2)} R^{(r)}_{(r_1,r_2)}(\delta_1, \delta_2 + l; \delta_3 + k; \tau; z).$$

Similarly, using (2.30) in (2.3), and following the same rule getting to result (2.28), we obtain the desired statement (2.29), and the theorem is fully proved. □

Remark 2.12. (i) From (2.5), we observe that, for function $R^{(r)}_{(r_1,r_2)}(\delta_1, \delta_2; \delta_3; \tau; z)$ the following result holds:

$$R^{(r)}_{(r_1,r_2)}(\delta_1, \delta_2; \delta_3; \tau; z) = \sum_{l=0}^{k} \binom{k}{l} \frac{B(\delta_1 + l, \delta_3 - \delta_1 + k - l)}{B(\delta_1, \delta_3 - \delta_1)} R^{(r)}_{(r_1,r_2)}(\delta_2, \delta_1 + l; \delta_3 + k; \tau; z). \tag{2.31}$$

(ii) If we use (2.5) on the right hand side of (2.31), we have another result for $R^{(r)}_{(r_1,r_2)}(\delta_1, \delta_2; \delta_3; \tau; z)$ as follows:

$$R^{(r)}_{(r_1,r_2)}(\delta_1, \delta_2; \delta_3; \tau; z) = \sum_{l=0}^{k} \binom{k}{l} \frac{B(\delta_1 + l, \delta_3 - \delta_1 + k - l)}{B(\delta_1, \delta_3 - \delta_1)} R^{(r)}_{(r_1,r_2)}(\delta_1 + l, \delta_2; \delta_3 + k; \tau; z). \tag{2.32}$$

Theorem 2.13. *Consider $R^{(r)}_{(r_1,r_2)}(\delta_1, \delta_2; \delta_3; \tau; z)$ and $\Phi^{(r)}_{(r_1,r_2)}(\delta_2; \delta_3; \tau; z)$ functions. The following integral representation holds :*

$$R^{(r)}_{(r_1,r_2)}(\delta_1, \delta_2; \delta_3; \tau; z) = \frac{1}{B(\delta_2, \delta_3 - \delta_2)} \int_0^1 t^{\delta_2 - 1}(1-t)^{\delta_3 - \delta_2 - 1}(1 - zt^{\tau})^{-\delta_1} \cdot E_{(r_1,r_2)}\left(\frac{-r}{t(1-t)}\right) dt, \tag{2.33}$$

where $\Re(\delta_3) > \Re(\delta_2) > 0$, $\Re(r_1) > 0$, $\Re(r_2) > 0$, $r \geqslant 0$, $\tau > 0$ and $|z| < 1$.

$$\Phi^{(r)}_{(r_1,r_2)}(\delta_2; \delta_3; \tau; z) = \frac{1}{B(\delta_2, \delta_3 - \delta_2)} \int_0^1 t^{\delta_2 - 1}(1-t)^{\delta_3 - \delta_2 - 1} e^{zt^{\tau}} \cdot E_{(r_1,r_2)}\left(\frac{-r}{t(1-t)}\right) dt, \tag{2.34}$$

where $\Re(\delta_3) > \Re(\delta_2) > 0$, $\Re(r_1) > 0$, $\Re(r_2) > 0$, $r \geqslant 0$ and $\tau > 0$.

Proof. Using the following known result from Reference [6]:

$$B^{(u)}_{(u_1,u_2)}(y_1, y_2) = \int_0^1 t^{y_1 - 1}(1-t)^{y_2 - 1} E_{(u_1,u_2)}\left(\frac{-u}{t(1-t)}\right) dt \tag{2.35}$$

in (2.1), we have

$$\begin{aligned} R^{(r)}_{(r_1,r_2)}(\delta_1, \delta_2; \delta_3; \tau; z) &= \frac{1}{B(\delta_2, \delta_3 - \delta_2)} \sum_{n=0}^{\infty} (\delta_1)_n \left(\int_0^1 t^{\delta_2 + n\tau - 1}(1-t)^{\delta_3 - \delta_2 - 1} E_{(r_1,r_2)}\left(\frac{-r}{t(1-t)}\right) dt \right) \frac{z^n}{n!} \\ &= \frac{1}{B(\delta_2, \delta_3 - \delta_2)} \sum_{n=0}^{\infty} (\delta_1)_n \left(\int_0^1 t^{\delta_2 - 1}(1-t)^{\delta_3 - \delta_2 - 1} E_{(r_1,r_2)}\left(\frac{-r}{t(1-t)}\right) t^{n\tau} dt \right) \frac{z^n}{n!} . \end{aligned}$$

Changing the order of integration and summation, we get:

$$R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)$$
$$=\frac{1}{B(\delta_2,\delta_3-\delta_2)}\int_0^1 t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}E_{(r_1,r_2)}\left(\frac{-r}{t(1-t)}\right)dt\sum_{n=0}^{\infty}(\delta_1)_n\frac{(zt^\tau)^n}{n!}\,.$$

Since,

$$\sum_{n=0}^{\infty}(\delta_1)_n\frac{(zt^\tau)^n}{n!}=(1-zt^\tau)^{-\delta_1};\qquad |t|<1\,.$$

The last expression becomes the required result:

$$R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)$$
$$=\frac{1}{B(\delta_2,\delta_3-\delta_2)}\int_0^1 t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}(1-zt^\tau)^{-\delta_1}\cdot E_{(r_1,r_2)}\left(\frac{-r}{t(1-t)}\right)dt\,.$$

Similarly, using (2.35) in (2.3), and following the same rule getting to result (2.33), we obtain the desired statement (2.34), and the theorem is fully proved. □

Remark 2.14. From (2.5), we observe that, for $R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)$, the following integral representation holds:

$$R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)$$
$$=\frac{1}{B(\delta_1,\delta_3-\delta_1)}\int_0^1 t^{\delta_1-1}(1-t)^{\delta_3-\delta_1-1}(1-zt^\tau)^{-\delta_2}\cdot E_{(r_1,r_2)}\left(\frac{-r}{t(1-t)}\right)dt. \quad (2.36)$$

Corollary 2.15. Consider $R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)$ and $\Phi^{(r)}_{(r_1,r_2)}(\delta_2;\delta_3;\tau;z)$ functions. The following integral representations hold:

$$R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)=\frac{2}{B(\delta_2,\delta_3-\delta_2)}\int_0^{\frac{\pi}{2}}(\cos x)^{2\delta_2-1}(\sin x)^{2\delta_3-2\delta_2-1}$$
$$\cdot E_{(r_1,r_2)}\left(\frac{-r}{\cos^2 x\sin^2 x}\right)[1-z(\cos x)^{2\tau}]^{-\delta_1}dx \quad (2.37)$$

and

$$\Phi^{(r)}_{(r_1,r_2)}(\delta_2;\delta_3;\tau;z)=\frac{2}{B(\delta_2,\delta_3-\delta_2)}\int_0^{\frac{\pi}{2}}(\cos x)^{2\delta_2-1}(\sin x)^{2\delta_3-2\delta_2-1}$$
$$\cdot E_{(r_1,r_2)}\left(\frac{-r}{\cos^2 x\sin^2 x}\right)e^{z(\cos x)^{2\tau}}dx. \quad (2.38)$$

Proof. The corollary is easily proved by choosing $t=\cos^2 x$ in (2.33) and (2.34). □

Corollary 2.16. Consider $R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)$ and $\Phi^{(r)}_{(r_1,r_2)}(\delta_2;\delta_3;\tau;z)$ functions. The following integral representations hold:

$$R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)=\frac{2}{B(\delta_2,\delta_3-\delta_2)}\int_0^{\frac{\pi}{2}}(\sin x)^{2\delta_2-1}(\cos x)^{2\delta_3-2\delta_2-1}$$
$$\cdot E_{(r_1,r_2)}\left(\frac{-r}{\cos^2 x\sin^2 x}\right)[1-z(\sin x)^{2\tau}]^{-\delta_1}dx \quad (2.39)$$

and

$$\Phi^{(r)}_{(r_1,r_2)}(\delta_2;\delta_3;\tau;z) = \frac{2}{B(\delta_2,\delta_3-\delta_2)}\int_0^{\frac{\pi}{2}}(\sin x)^{2\delta_2-1}(\cos x)^{2\delta_3-2\delta_2-1} \cdot E_{(r_1,r_2)}\left(\frac{-r}{\cos^2 x\sin^2 x}\right)e^{z(\sin x)^{2\tau}}dx. \tag{2.40}$$

Proof. The corollary is easily proved by choosing $t=\sin^2 x$ in (2.33) and (2.34). □

Theorem 2.17. *The Mellin transformation for $R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)$ and $\Phi^{(r)}_{(r_1,r_2)}(\delta_2;\delta_3;\tau;z)$ are given by:*

$$M\{R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z);s\} = \frac{\Gamma_0^{(r_1,r_2)}(s)B(\delta_2+s,\delta_3-\delta_2+s)}{B(\delta_2,\delta_3-\delta_2)}R(\delta_1,\delta_2+s;\delta_3+2s;\tau;z) \tag{2.41}$$

where $\Re(\delta_3)>\Re(\delta_2)>0$, $\Re(r_1)>0$, $\Re(r_2)>0$, $s\geqslant 0$, $r\geqslant 0$, $\tau>0$ *and* $|z|<1$ *and*

$$M\{\Phi^{(r)}_{(r_1,r_2)}(\delta_2;\delta_3;\tau;z);s\} = \frac{\Gamma_0^{(r_1,r_2)}(s)B(\delta_2+s,\delta_3-\delta_2+s)}{B(\delta_2,\delta_3-\delta_2)}\Phi(\delta_2+s;\delta_3+2s;\tau;z) \tag{2.42}$$

where $\Re(\delta_3)>\Re(\delta_2)>0$, $\Re(r_1)>0,\Re(r_2)>0$, $s\geqslant 0$, $r\geqslant 0$ *and* $\tau>0$.

Proof.

$$M\{R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z);s\} = \int_0^\infty\left\{\frac{1}{B(\delta_2,\delta_3-\delta_2)}\int_0^1 t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}(1-zt^\tau)^{-\delta_1} \cdot E_{(r_1,r_2)}\left(\frac{-r}{t(1-t)}\right)dt\right\}r^{s-1}dr\,.$$

Interchanging the order of integrations, the last expression read as:

$$M\{R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z);s\} = \frac{1}{B(\delta_2,\delta_3-\delta_2)}\int_0^1 t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}(1-zt^\tau)^{-\delta_1}\left[\int_0^\infty r^{s-1}E_{(r_1,r_2)}\left(\frac{-r}{t(1-t)}\right)dr\right]dt\,.$$

Let $v=\frac{r}{t(1-t)}$, then:

$$= \frac{1}{B(\delta_2,\delta_3-\delta_2)}\int_0^1 t^{\delta_2+s-1}(1-t)^{\delta_3-\delta_2+s-1}(1-zt^\tau)^{-\delta_1}\left[\int_0^\infty v^{s-1}E_{(r_1,r_2)}(-v)dv\right]dt\,.$$

From the known result in reference [1]:

$$\Gamma_r^{(r_1,r_2)}(x_1) = \int_0^\infty t^{x_1-1}E_{(r_1,r_2)}\left(-t-\frac{r}{t}\right)dt$$

and, using it with $r=0$ in the above equation, we get

$$M\{R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z);s\} = \frac{1}{B(\delta_2,\delta_3-\delta_2)}\int_0^1 t^{\delta_2+s-1}(1-t)^{\delta_3-\delta_2+s-1}(1-zt^\tau)^{-\delta_1}\left[\Gamma_0^{(r_1,r_2)}(s)\right]dt$$

$$= \frac{B(\delta_2+s,\delta_3-\delta_2+s)}{B(\delta_2,\delta_3-\delta_2)} \int_0^1 \frac{t^{\delta_2+s-1}(1-t)^{\delta_3-\delta_2+s-1}(1-zt^{\tau})^{-\delta_1}}{B(\delta_2+s,\delta_3-\delta_2+s)} \left[\Gamma_0^{(r_1,r_2)}(s)\right] dt$$
$$= \frac{B(\delta_2+s,\delta_3-\delta_2+s)\Gamma_0^{(r_1,r_2)}(s)}{B(\delta_2,\delta_3-\delta_2)} \int_0^1 \frac{t^{\delta_2+s-1}(1-t)^{\delta_3-\delta_2+s-1}(1-zt^{\tau})^{-\delta_1}}{B(\delta_2+s,\delta_3-\delta_2+s)} dt\,.$$

Finally, from integral representation of Gauss hypergeometric function (1.4), we get our desired result:

$$M\{R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z);s\}$$
$$= \frac{B(\delta_2+s,\delta_3-\delta_2+s)\Gamma_0^{(r_1,r_2)}(s)}{B(\delta_2,\delta_3-\delta_2)} R(\delta_1,\delta_2+s;\delta_3+2s;\tau;z).$$

Similarly, using Mellin transformation equation (2.34), and following the same rule getting to result (2.41), we obtain the desired statement (2.42), and the theorem is fully proved. □

Remark 2.18. (i) From (2.5), we observe that, for function $R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)$ the following result holds:

$$M\{R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z);s\}$$
$$= \frac{B(\delta_1+s,\delta_3-\delta_1+s)\Gamma_0^{(r_1,r_2)}(s)}{B(\delta_1,\delta_3-\delta_1)} R(\delta_2,\delta_1+s;\delta_3+2s;\tau;z). \tag{2.43}$$

(ii) Using symmetric property (2.5) on the right side of (2.43), we have another Mellin representation for $R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)$ as follows:

$$M\{R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z);s\}$$
$$= \frac{B(\delta_1+s,\delta_3-\delta_1+s)\Gamma_0^{(r_1,r_2)}(s)}{B(\delta_1,\delta_3-\delta_1)} R(\delta_1+s,\delta_2;\delta_3+2s;\tau;z). \tag{2.44}$$

Theorem 2.19. *Consider $R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)$ and $\Phi^{(r)}_{(r_1,r_2)}(\delta_2;\delta_3;\tau;z)$ functions. The following differentiation formulas hold:*

$$\frac{d^n}{dz^n}\{R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)\}$$
$$= \frac{(\delta_1)_n(\delta_2)_{n\tau}}{(\delta_3)_{n\tau}} R^{(r)}_{(r_1,r_2)}(\delta_1+n,\delta_2+n\tau;\delta_3+n\tau;\tau;z) \tag{2.45}$$

where $\Re(\delta_3)>\Re(\delta_2)>0$, $\Re(r_1)>0$, $\Re(r_2)>0$, $n\geqslant 0$, $r\geqslant 0$, $\tau>0$ and $|z|<1$.

$$\frac{d^n}{dz^n}\{\Phi^{(r)}_{(r_1,r_2)}(\delta_2;\delta_3;\tau;z)\} = \frac{(\delta_2)_{n\tau}}{(\delta_3)_{n\tau}} \Phi^{(r)}_{(r_1,r_2)}(\delta_2+n\tau;\delta_3+n\tau;\tau;z) \tag{2.46}$$

where $\Re(\delta_3)>0$, $\Re(\delta_2)>0$, $\Re(r_1)>0$, $\Re(r_2)>0$, $n\geqslant 0$, $r\geqslant 0$ and $\tau>0$.

Proof.

$$R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z) = \sum_{n=0}^{\infty}(\delta_1)_n \frac{B^{(r)}_{(r_1,r_2)}(\delta_2+n\tau,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)} \frac{z^n}{n!}\,,$$

$$\frac{\partial}{\partial z}\left\{R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)\right\} = \sum_{n=0}^{\infty}(\delta_1)_n \frac{B^{(r)}_{(r_1,r_2)}(\delta_2+n\tau,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}\frac{z^{n-1}}{(n-1)!}\,.$$

Replacing $n \to n+1$ in the above equation and after simplification, we get

$$\begin{aligned}
&\frac{\partial}{\partial z}\left\{R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)\right\}\\
&= \sum_{n=0}^{\infty}(\delta_1)_{n+1}\frac{B^{(r)}_{(r_1,r_2)}(\delta_2+n\tau+\tau,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}\frac{z^n}{n!}\\
&= \frac{B(\delta_2+\tau,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}\sum_{n=0}^{\infty}(\delta_1)(\delta_1+1)_n\frac{B^{(r)}_{(r_1,r_2)}(\delta_2+n\tau+\tau,\delta_3-\delta_2)}{B(\delta_2+\tau,\delta_3-\delta_2)}\frac{z^n}{n!}\\
&= \frac{(\delta_1)\Gamma(\delta_2+\tau)\Gamma(\delta_3)}{\Gamma(\delta_3+\tau)\Gamma(\delta_2)}\sum_{n=0}^{\infty}(\delta_1+1)_n\frac{B^{(r)}_{(r_1,r_2)}(\delta_2+n\tau+\tau,\delta_3-\delta_2)}{B(\delta_2+\tau,\delta_3-\delta_2)}\frac{z^n}{n!}\\
&= \frac{(\delta_1)\Gamma(\delta_2+\tau)\Gamma(\delta_3)}{\Gamma(\delta_3+\tau)\Gamma(\delta_2)}R^{(r)}_{(r_1,r_2)}(\delta_1+1,\delta_2+\tau;\delta_3+\tau;\tau;z)\,.
\end{aligned}$$

So, iterating this differential equation n-times, we get

$$\frac{d^n}{dz^n}\{R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)\} = \frac{(\delta_1)_n(\delta_2)_{n\tau}}{(\delta_3)_{n\tau}}R^{(r)}_{(r_1,r_2)}(\delta_1+n,\delta_2+n\tau;\delta_3+n\tau;\tau;z).$$

Similarly, following the same rule getting to result (2.45) applied in (2.3), we obtain the desired statement (2.46), and the theorem is fully proved. □

Remark 2.20. Using (2.5) on the right hand side of (2.45), we can write another derivative formula for $R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)$:

$$\begin{aligned}
&\frac{d^n}{dz^n}\{R^{(r)}_{(r_1,r_2)}(\delta_1,\delta_2;\delta_3;\tau;z)\}\\
&= \frac{(\delta_1)_n(\delta_2)_{n\tau}}{(\delta_3)_{n\tau}}R^{(r)}_{(r_1,r_2)}(\delta_2+n\tau,\delta_1+n;\delta_3+n\tau;\tau;z).
\end{aligned}\tag{2.47}$$

3. Conclusion

In our present investigation, we have first introduced a new extension of τ-Gauss hypergeometric and τ-Kummer hypergeometric function and investigated some properties of these extended functions. We are also trying to find certain applications of the results obtained here in some significant research areas such as statistics, physics, engineering, applied mathematics, and computer algebra.

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