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A NOTE ON THE COMPUTABILITY OF HILBERT-SCHMIDT OPERATOR

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Abstract. In this paper,we study the computability of Paley-Weiner space. Also, a computability result based on the Classical Sampling Theorem is proved. Finally, we discuss the computability of the Hilbert-Schmidt operator.

1. Introduction

The Turing machine related accession to computability in the broad area of analysis is known as computable analysis. Banach and Mazur [1], Grzegorczyk [9], Kreitz and Weihrauch [11], Lacombe [14], Pour-El and Richards [18], Turing [19], and many others have made significant contributions to this field.For the representation-oriented accession to computable analysis' the standard idea is to represent infinite objects, such as functions, sets or real numbers by infinite strings over an alphabet (which at the very least comprises the symbols 0 and 1). As a result, a surjective function $\delta :\subseteq \Sigma^\omega \to X$ can be used to represent a set X, where where Σ^ω stands for the set of infinite sequences over Σ, and the inclusion symbol signifies the possibility of a partial mapping. Here, the pair (X, δ) is referred to as a represented space.

In case a machine (Turing) is able to computes endlessly long sequences and converts every sequence p on the (input) tape into the appropriate sequence $F(p)$ on the output tape, then the function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ is called computable.
We refer Brattka and Dillhage [6] definition for the concept of a computable function between two represented spaces. Also for various other definitions and terminology related to the contents of this paper, one may refer to the references [3, 9, 13, 16, 17]

The admission of evaluation and type conversion is a distinguishing feature of the function space model.

Definition 1.1. Let δ_i be a representation of X_i, $i = 1, 2, 3$. Then

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(i) **Evaluation.** $ev : C(X_1, X_2) \times X_1 \to X_2$ defined by $(f, x) \to f(x)$ is $([[\delta_1 \to \delta_2], \delta_1], \delta_2)$-computable.

(ii) **Type Conversion.** For any total function $f : X_1 \times X_2 \to X_3$, define a function $T(f) : X_1 \to X_3^{X_2}$ by $T(f)(x)(y) = f(x, y)$.Then f is $((\delta_1, \delta_2), \delta_3)$ -computable iff T(f) is $(\delta_1, [\delta_2 \to \delta_3])$ - computable.

Here, the set of all functions from X_2 to X_3 is denoted by $X_3^{X_2}$.

As shown by evaluation and type conversion in [20], the computable points in $(C(X, Y), [\delta \to \delta'])$, where (X, δ) and (Y, δ') are topological sequential spaces that are represented admissibly, are the (δ, δ')-computable functions from X to Y.

Suppose that the representation $\delta^{\mathbb{N}} \equiv [\delta_{\mathbb{N}} \to \delta]$ is for the set of sequences $X^{\mathbb{N}}$ for the space (X, δ). Write the computed points in (X, δ) as the computed sequences in (X, δ). Lacombe [15] first proposed the idea of computable metric space (computable MS). But we offer Brattka's definition, which is as follows.

Definition 1.2. [5] The triplet (X, d, α) is refered to as a computable MS if
(a) The pair (X, d) is a MS.
(b) $\alpha : \mathbb{N} \to X$ is dense in X.
(c) The double sequence $d \circ (\alpha \times \alpha) : \mathbb{N}^2 \to \mathbb{R}$ is a computable in $\mathbb{R}$.

Let (X, d, α) be a computable MS. Then its Cauchy Representation (CR) $\delta_X :\subseteq \Sigma^\omega \to X$ is given as

$$\delta_X(01^{n_0+1}01^{n_1+1}01^{n_2+1}...) := \lim_{i\to\infty} \alpha(n_i),$$

for all $n_i \in \mathbb{N}$ such that $(\alpha(n_i))_{i\in\mathbb{N}}$ converges and

$$d(\alpha(n_i), \alpha(n_j)) < 2^{-i}, \text{ for all } j > i.$$

The CR of computable MS are assumed to be used in the following sections. Regarding the relevant metric topology, all CR's are admissible.
For the terminology, definitions and other details related to computable metric spaces (CMS) and computable normed linear space (computable n.l.s) we refer the reader to consult [5].

One may observe that a computable n.l.s. is intutively a computable linear space. The triplet $(X, \|\cdot\|, e)$ is referred to as a computable Banach space (Computable BS) if the underlying space $(X, \|\cdot\|)$ is a Banach space.

Throughout the paper, it is assumed that computable n.l.s. are represented by their CRs,that are admissible with respect to their norm topology. If the CRs of two computable Banach spaces (computable BS) with the same underlying set are computably equivalent, then the two computable BS are called computably equivalent.

For the definition of computable Hilbert spaces one may read [4].

The standard representation of the dual space $H^{'}$ is stated by

$$\delta_{H'}\langle p, q\rangle = f \Leftrightarrow [\delta_H \to \delta_{\mathbb{F}}](p) = f \text{ and } \delta_{\mathbb{R}}(q) = \|f\|,$$

for all $f \in H^{'}$. A computable isomorphism $T : H \to H$ is an isomorphism such that T as well as T^{-1} are computable.

We denote the set of step functions on $\mathbb{R}$ with rational values by $RSF = \{\sum_{j=0}^{n} c_j \chi_{(a_j,b_j)}, n \in \mathbb{N}, a_j, b_j \in \mathbb{Q}, c_j \in \mathbb{Q}^2, a_j \leq b_j, 0 \leq j \leq n\}$. Here, we identify the set $\mathbb{Q} + \iota\mathbb{Q}$ with $\mathbb{Q}^2$. The set RSF is a countable dense subset of L_p for $1 \leq p \leq \infty$. Let $\alpha_{RSF} :\subseteq \Sigma^* \to RSF$ denote some notation of the set RSF with recursive enumerable domain such that the mappings which assign to a step function $\sum_{j=0}^{n} c_j \chi_{(a_j,b_j)}$, the end points a_j, b_j respectively, the values c_j are computable. The space $(L_p, \|.\|, RSF, \alpha_{RSF})$ is a computable metric space for any computable p with $1 \leq p \leq \infty$. See [12] for details.

Given separable Hilbert spaces $\mathcal{H}$ *and* $\mathcal{K}$, we say that a bounded linear operator $T : \mathcal{H} \to \mathcal{K}$ is Hilbert-Schmidt if there exists an orthonormal basis (e_n) for $\mathcal{H}$ such that $\sum \|Te_n\|^2 < \infty$. The space of Hilbert-Schmidt operators mapping $\mathcal{H}$ into $\mathcal{H}$ is denoted by

$$B_2(\mathcal{H}, \mathcal{K}) = \{T \in B(\mathcal{H}, \mathcal{K}) : T \text{ is Hilbert-Schmidt}\}.$$

$B_2(\mathcal{H}, \mathcal{K})$ forms a Hilbert space with respect to the inner product

$$\langle T, U\rangle_{HS} = \sum_n \langle Te_n, Ue_n\rangle,\ T, U \in B_2(\mathcal{H}, \mathcal{K}),$$

where (e_n) is any orthonormal basis for $\mathcal{H}$ and

$$\|T\|_{HS}^2 = \sum_n \|Te_n\|^2.$$

$\|T\|_{HS}$ is called the Hilbert-Schmidt norm of T. The tensor product of $x \in \mathcal{H}$ and $y \in \mathcal{K}$ is the operator $x \otimes y \in B(\mathcal{H}, \mathcal{K})$ defined by $(x \otimes y)(z) = \langle z, x\rangle y,\ z \in \mathcal{H}$. $x \otimes y$ is a bounded linear operator with finite rank and so is Hilbert-Schmidt. If (e_n) is an orthonormal basis for $\mathcal{H}$ and (f_n) is an orthonormal basis for $\mathcal{K}$, then $(e_m \otimes f_n)_{m,n\in\mathbb{N}}$ is an orthonormal basis for $B_2(\mathcal{H}, \mathcal{K})$. Since $(\phi_{n,k})_{n,k\in\mathbb{Z}} = (e^{2\pi inx}\ \chi_{[k,k+1]}(x))_{n,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$, $(\phi_{n,k} \otimes \phi_{n'k'})_{n,k,n',k'\in\mathbb{Z}}$ is an orthonormal basis for $B_2(L^2(\mathbb{R}), L^2(\mathbb{R}))$.

1.1. Overview. In this paper, we consider the space $L^2_{[-\frac{1}{2},\frac{1}{2}]}(\mathbb{R})$ and consequently constructed another space $P_W(\mathbb{R})$, called the Paley-Weiner space, and prove that it is computable. Also, we prove a result based on the Classical Sampling Theorem [10]. Further, the computability of the pre-image of an integral operator is discussed. Finally, we prove that the space $B_2(L^2(\mathbb{R}), L^2(\mathbb{R}))$ is isometrically isomorphic to the space $L^2(\mathbb{R})$.

2. Main Results

Consider the space $L^2_{[-\frac{1}{2},\frac{1}{2}]}(\mathbb{R})$ defined as

$$L^2_{[-\frac{1}{2},\frac{1}{2}]}(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : f(x) = 0 \text{ for a.e. } |x| > \frac{1}{2}\}.$$

$L^2_{[-\frac{1}{2},\frac{1}{2}]}(\mathbb{R})$ is the subspace of $L^2(\mathbb{R})$ consisting of functions that are time-limited to the interval $[-\frac{1}{2},\frac{1}{2}]$. The sequence $(\varepsilon_n)_{n\in\mathbb{Z}} = (e^{2\pi nx}\chi_{[-\frac{1}{2},\frac{1}{2}]}(x))_{n\in\mathbb{Z}} = (M_n\ \chi_{[-\frac{1}{2},\frac{1}{2}]})_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2_{[-\frac{1}{2},\frac{1}{2}]}(\mathbb{R})$ and an orthonormal sequence in $L^2(\mathbb{R})$. Since the Fourier transform is unitary, the sequence $(\hat{\varepsilon}_n)_{n\in\mathbb{Z}}$ is also orthonormal in $L^2(\mathbb{R})$. In terms of the sinc function given by $d_\pi(\xi) = \frac{sin\pi\xi}{\pi\xi}$, we can write

$$\hat{\varepsilon}_n(\xi) = T_n d_\pi(\xi).$$

The functions $T_n d_\pi$ has a Fourier transform that is non-zero only within $[-\frac{1}{2},\frac{1}{2}]$. In fact,

$$(T_n d_\pi)^\wedge(\xi) = e^{-2\pi in\xi}\chi_{[-\frac{1}{2},\frac{1}{2}]}(\xi).$$

The closed span of the orthonormal sequence is given by $(\hat{\varepsilon}_n)_{n\in\mathbb{Z}} = \{T_n d_\pi\}_{n\in\mathbb{Z}}$ is

$$P_W(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \hat{f}(\xi) = 0 \text{ for a.e. } |\xi| > \frac{1}{2}\}.$$

The space $P_W(\mathbb{R})$ is called the **Paley-Weiner space** of functions bandlimited to the interval $[-\frac{1}{2},\frac{1}{2}]$. Here, a function $f \in L^2(\mathbb{R})$ is said to be bandlimited to $[-\Omega,\Omega]$ if $supp(\hat{f}) \subseteq [-\Omega,\Omega]$. The space $P_W(\mathbb{R})$ is a closed subspace of $L^2(\mathbb{R})$.

In the following result, we show that $P_W(\mathbb{R})$ is a computable Hilbert space.

Theorem 2.1. *The space $P_W(\mathbb{R})$ is a computable Hilbert space.*

Proof. The sequence $(T_n d_\pi)_{n\in\mathbb{Z}}$ is an orthonormal basis for $P_W(\mathbb{R})$. Since the Fourier transform $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a computable operator and $d_\pi = (\chi_{[-\frac{1}{2},\frac{1}{2}]})^\wedge$, we get that d_π is a computable element in $L^2(\mathbb{R})$. In view of Theorem 2.1 in [16], we get that $(T_n d_\pi)_{n\in\mathbb{Z}}$ is a computable sequence in $L^2(\mathbb{R})$. Thus, $P_W(\mathbb{R})$ is a computable subspace of $L^2(\mathbb{R})$ and hence a computable Hilbert space. □

Recall that a sequence $(f_i)_{i\in\mathbb{N}}$ of elements in a Hilbert space H is called a *frame* for H if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i\in\mathbb{N}} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \text{ for all } f \in H.$$

If only the upper inequality holds in above, then $(f_i)_{i\in\mathbb{N}}$ is said to be a *Bessel sequence* in H. The numbers A and B are called a lower and upper frame bound for the frame $(f_i)_{i\in\mathbb{N}}$. In case $A = B$, then the frame $(f_i)_{i\in\mathbb{N}}$ is called a tight frame. For further details on frames, one may read [7]. Also, for litrature related to computability of frame systems one may refer [13, 16, 17].

The Classical Sampling Theorem discussed in [10] is an important result in signal processing that gives an algorithm for reconstructing a bandlimited function f from the countably many sample values $f(bn), n \in \mathbb{Z}$, where $0 < b < 1$ is fixed. Based on the Classical sampling theorem, we obtain the following result:

Theorem 2.2. *If b is a computable real number such that $0 < b \leq 1$, then the map*

$$\phi :\subseteq l^2(\mathbb{Z}) \to L^2(\mathbb{R})$$
$$(f(bn))_{n\in\mathbb{Z}} \to f,$$

where $dom\phi = \{(f(bn))_{n\in\mathbb{Z}} : f \in P_W(\mathbb{R})\}$ *is a computable map.*

Proof. The sequence $(T_{bn}d_\pi)_{n\in\mathbb{Z}}$ is a $\frac{1}{b}$-tight computable frame for $P_W(\mathbb{R})$. By Theorem 2.1, the map $T : l^2(\mathbb{Z}) \to P_W(\mathbb{R})$ given by $T((c_n)) = \sum_{n\in\mathbb{Z}} c_n T_{bn}d_\pi$ is computable. Since $\langle f, T_{bn}d_\pi\rangle = f(bn)$, given $\delta_{l^2(\mathbb{Z})}$ name of the sequence $(f(bn))_{n\in\mathbb{Z}}$, we can get $\delta_{P_W(\mathbb{R})}$ name of $\sum_{n\in\mathbb{Z}} \langle f, T_{bn}d_\pi\rangle T_{bn}d_\pi$. In view of Theorem 10.7 in [10], we may write $f = b\sum_{n\in\mathbb{Z}} \langle f, T_{bn}d_\pi\rangle T_{bn}d_\pi$ with unconditional convergence of the series in L^2-norm. So we get $\delta_{P_W(\mathbb{R})}$ name of f and since $P_W(\mathbb{R})$ is a computable subspace of $L^2(\mathbb{R})$, we can get $\delta_{L^2(\mathbb{R})}$ name of f. □

The **Integral operator** L_k with kernel $k \in L^2(\mathbb{R}^2)$ is defined by $L_k : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, where $L_k f : \mathbb{R} \to \mathbb{C}$ is given by $L_k f(y) = \int_{\mathbb{R}} k(x,y)f(x)dx$. L_k is a bounded map of $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$. In fact, L_k is a Hilbert-Schmidt operator and $\|L_k\|_{HS} = \|k\|_{L^2}$. The sequence $G(\chi_{[0,1]}, 1, 1) = (e^{2\pi inx}\chi_{[k,k+1]}(x))_{k,n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$. We use the notation $\phi_{n,k}(x) = e^{2\pi inx}\chi_{[k,k+1]}(x)$, $x \in \mathbb{R}$. The sequence $(\phi_{n,k})_{n,k\in\mathbb{Z}}$ is a computable sequence in $L^2(\mathbb{R})$. Thus, the computable Hilbert space $(L^2(\mathbb{R}), \langle\cdot\rangle, (\phi_{n,k})_{n,k\in\mathbb{Z}})$ with representation induced by the orthonormal basis $(\phi_{n,k})_{n,k\in\mathbb{Z}}$, can be considered computably equivalent to $(L^2(\mathbb{R}), \langle\cdot\rangle, RSF, \alpha_{RSF})$. The sequence $(\phi_{n,k} \otimes \phi_{n'k'})_{n,k,n',k'\in\mathbb{Z}}$ forms an orthonormal basis for $L^2(\mathbb{R}^2)$, where $\phi_{n,k} \otimes \phi_{n'k'}$ is the tensor product of $\phi_{n,k}$ and $\phi_{n'k'}$, given by

$$(\phi_{n,k} \otimes \phi_{n'k'})(x,y) = \overline{\phi_{n,k}(x)}\ \phi_{n'k'}(y),\ (x,y) \in \mathbb{R}^2.$$

The space $(L^2(\mathbb{R}^2), \langle\cdot\rangle, \phi_{n,k} \otimes \phi_{n'k'})$ also forms a computable Hilbert space if the representation induced by the orthonormal basis $(\phi_{n,k} \otimes \phi_{n'k'})$ is considered. Here, the inner product $\langle\cdot\rangle$ is given by

$$\langle f, g\rangle = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)\ \overline{g(x,y)}\ dxdy,\ \ f, g \in L^2(\mathbb{R}^2)$$

and the norm is given by $\|f\|^2 = \sum |\langle f, \phi_{n,k} \otimes \phi_{n',k'}\rangle|^2$. In the following result, we prove that the mapping which takes k in $L^2(\mathbb{R})$ to the Hilbert-Schmidt operator L_k is computable.

Theorem 2.3. *The map* $\phi : L^2(\mathbb{R}^2) \to C(L^2(\mathbb{R}), L^2(\mathbb{R}))$ *given by* $k \mapsto L_k$ *is* $(\delta_{L^2(\mathbb{R}^2)}, [\delta_{L^2(\mathbb{R})} \to \delta_{L^2(\mathbb{R})}])$ *computable.*

Proof. Given $\delta_{L^2(\mathbb{R}^2)}$ name of $k \in L^2(\mathbb{R}^2)$, which contains a coded list of names of elements $\psi_{p,q} = \sum_{finite} c_{n,k,n',k'}(\phi_{n,k} \otimes \phi_{n'k'})$, $p = \langle n, k\rangle$, $q = \langle n', k'\rangle$ such that $\|k - \psi_{p,q}\| < 2^{-n}$ and for $\delta_{L^2(\mathbb{R})}$ name of $f \in L^2(\mathbb{R})$, which contains a coded list of names of elements $\tau_n = \alpha_{RSF}(u_n)$ such that $\|f - \tau_n\| < 2^{-n}$, we have

$$\begin{aligned}\|L_k f - L_{\psi_{p,q}}\tau_n\| &\leq \|L_k(f - \tau_n)\| + \|(L_k - L_{\psi_{p,q}})\tau_n\| \\ &\leq \|k\|\|f - \tau_n\| + \|L_k\tau_n - L_{\psi_{p,q}}\tau_n\|.\end{aligned}$$

Also, we compute

$$\begin{aligned}\|L_k\tau_n - L_{\psi_{p,q}}\tau_n\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \Big| \int_{\mathbb{R}} k(x,y)\ \tau_n(x)\ dx - \int_{\mathbb{R}} \psi_{p,q}(x,y)\ \tau_n(x)\ dx\Big|^2\ dy \\ &\le \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |k(x,y) - \psi_{p,q}(x,y)|\ |\tau_n(x)|\ dx\Big)^2\ dy \\ &\le \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |k(x,y) - \psi_{p,q}(x,y)|^2\ dx\Big)\Big(\int_{\mathbb{R}} |\tau_n(x)|^2\ dx\Big)\ dy \\ &= \|\tau_n\|^2_{L^2(\mathbb{R})}\ \|k - \psi_{p,q}\|^2_{L^2(\mathbb{R}^2)}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\|L_k f - L_{\psi_{p,q}}\tau_n\| &\le \|k\|\|f - \tau_n\| + \|\tau_n\|\|k - \psi_{p,q}\| \\ &\le 2\|f - \tau_n\| + (1 + \|\tau_1\|)\|k - \psi_{p,q}\|.\end{aligned}$$

Compute $n \in \mathbb{N}$ such that $2^{-n} \le \min\{\frac{2^{-(k+1)}}{(1+\|\tau_1\|)}, \frac{2^{-(k+1)}}{2}\}$. Then

$$\begin{aligned}\|L_k f - L_{\psi_{p,q}}\tau_n\| &\le 2 \cdot 2^{-n} + 2^{-n}(1 + \|\tau_1\|) \\ &\le 2^{-(k+1)} + 2^{-(k+1)} \\ &= 2^{-k}.\end{aligned}$$

It suffices to compute $\delta_{L^2(\mathbb{R})}$ name of $L_{\psi_{p,q}}\tau_n$ for the case when $\psi_{p,q} = \phi_{n,k} \otimes \phi_{n'k'}$ and $\tau_n = \chi_{[a,b]}$ for $a, b \in \mathbb{Q}$. Note that

$$\begin{aligned}L_{\psi_{p,q}}\tau_n(y) &= \int \psi_{p,q}(x,y)\ \tau_n(x)\ dx \\ &= \overline{\phi_{n',k'}(y)}\ \int e^{2\pi i n x}\ \chi_{[k,k+1]}(x)\ \chi_{[a,b]}(x)\ dx \\ &= C\ \overline{\phi_{n',k'}(y)},\end{aligned}$$

where C is a computable number. The result follows in view of fact that $(\phi_{n,k})_{n,k\in\mathbb{Z}}$ is a computable sequence in $L^2(\mathbb{R})$. □

The space $(B_2(L^2(\mathbb{R}), L^2(\mathbb{R})), \langle\cdot\rangle_{HS}, \phi_{n,k} \otimes \phi_{n'k'})$ forms a computable Hilbert space if we consider the representation induced by the sequence $(\phi_{n,k} \otimes \phi_{n'k'})_{n,k,n',k'\in\mathbb{Z}}$. Here, $\phi_{n,k} \otimes \phi_{n'k'} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is given by $f \mapsto \langle f, \phi_{n,k}\rangle\phi_{n',k'},\ f \in L^2(\mathbb{R})$. Also, the space $(L^2(\mathbb{R}^2), \langle\cdot\rangle_{HS}, \phi_{n,k} \otimes \phi_{n'k'})$ forms a computable Hilbert space, where

$$(\phi_{n,k} \otimes \phi_{n'k'})(x,y) = \overline{\phi_{n,k}(x)}\ \phi_{n'k'}(y), \quad (x,y) \in \mathbb{R}^2.$$

In the following result, we prove that the two spaces $L^2(\mathbb{R}^2)$ and $(B_2(L^2(\mathbb{R}), L^2(\mathbb{R}))$ are computably isometrically isomorphic. Here, we use the standard representation δ_H of a computable Hilbert space $(H, \langle\cdot\rangle, (e_n))$, where (e_n) is an orthonormal basis, given by

$$\delta_H(\langle p, q\rangle) = f \Leftrightarrow [\delta_{\mathbb{N}} \to \delta_{\mathbb{F}}](p) = \langle f, e_n\rangle \text{ and } \delta_{\mathbb{R}}(q) = \|f\|$$

for all $f \in H$. This representation is computably equivalent to the Cauchy Representation of H.

Theorem 2.4. *The space $L^2(\mathbb{R}^2)$ is computably isometrically isomorphic to $B_2(L^2(\mathbb{R}), L^2(\mathbb{R}))$.*

Proof. The map $\phi : L^2(\mathbb{R}^2) \to B_2(L^2(\mathbb{R}), L^2(\mathbb{R}))$ given by $k \mapsto L_k$ is an isometric isomorphism by Theorem B.13 [10]. Denote the standard representation of $L^2(\mathbb{R}^2)$ as $\delta_{L^2(\mathbb{R}^2)}$ and of $B_2(L^2(\mathbb{R}), L^2(\mathbb{R}))$ as δ_{B_2}. Since

$$\begin{aligned}
\langle k, \phi_{n,k} \otimes \phi_{n'k'} \rangle_{L^2(\mathbb{R}^2)} &= \int_{\mathbb{R}} \int_{\mathbb{R}} k(x,y)\phi_{n,k}(x)\overline{\phi_{n',k'}(y)}dxdy \\
&= \int_{\mathbb{R}} L_k\phi_{n,k}(y)\overline{\phi_{n',k'}(y)}dy \\
&= \langle L_k\phi_{n,k}, \phi_{n',k'} \rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle L_k, \phi_{n',k'} \otimes \phi_{nk} \rangle_{HS} &= \sum_{n,k} \langle L_k\phi_{n,k}, (\phi_{n',k'} \otimes \phi_{nk})\phi_{n,k} \rangle \\
&= \sum_{n,k} \langle L_k\phi_{n,k}, \langle \phi_{n',k'}, \phi_{nk} \rangle \phi_{n,k} \rangle \\
&= \langle L_k\phi_{n',k'}, \phi_{n,k} \rangle
\end{aligned}$$

and $\|L_k\| = \|k\|$, we obtain that the map ϕ is $(\delta_{L^2(\mathbb{R}^2)}, \delta_{B_2})$ computable.

□

2.1. Conclusion. The space$L^2_{[-\frac{1}{2},\frac{1}{2}]}(\mathbb{R})$ is considered and using it we construct another space $P_W(\mathbb{R})$, called the Paley-Weiner space, and prove that it is computable. An algorithm for reconstructing a bandlimited function f from the countably numerous sample values $f(bn), n \in \mathbb{Z}$, where $0 < b < 1$ is constant, is provided by the Classical Sampling Theorem [10]. We prove a result based on the Classical Sampling Theorem. Also, the computability of the pre-image of an integral operator is discussed. Finally, we consider the space $B_2(L^2(\mathbb{R}), L^2(\mathbb{R}))$ and prove that it is isometrically isomorphic to the space $L^2(\mathbb{R})$.

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