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CHARACTERISTICS OF VARIOUS MAPS VIA OPERATION APPROACH ON αg-OPEN SETS

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Abstract. The objective of this paper is to introduce new maps namely αg-(γ,δ)-continuous, αg-(γ,δ)-homeomorphism, αg-δ-continuous and αg_γ-δ-continuous maps and examine their properties in topological spaces. We also derive Urysohn's lemma for αg-γ-normal space using αg_γ-closed sets.

1. Introduction

Njastad [12] developed the concept of α-open sets in 1965. In 1970, Levine [6] proposed the notion of generalised closed sets in topological spaces. Later in 1994, Maki et al. [7] presented αg-closed sets in topological spaces. Kasahara [4] put forth the idea of an operation on topological spaces as well as the notion of α-closed graphs of maps in topological spaces. After this, the maps with α-closed graphs was analyzed by Jankovic [3]. Following his work, Ogata [13] instigated γ-open sets using the operation γ on open sets in topological spaces and derived the characteristics of (γ,β)-continuous, (γ,β)-homeomorphism and (γ,β)-open maps. The generalizations of (γ,β)-continuous maps in topological spaces were studied extensively by many authors [1, 2, 5]. Recently, Mershia Rabuni and Balamani [8, 9] put forth αg_γ-open sets concept using the operation γ on $\varsigma_{\alpha g}$ and analyzed its properties in topological spaces. Mershia Rabuni and Balamani [10, 11] developed various spaces like αg_γ-T_i $(i = 0, 1, 2, 1/2)$, αg_γ-$T_i^{'}$ $(i = 0, 1, 2)$, αg-γ-regular, αg-γ-normal via operation on αg-open sets and investigated their characterizations.

In this paper, we propose the definitions of some new continuous maps such as αg_γ-δ-continuous, αg-δ-continuous, αg_γ-δ-continuous. Also, we establish the inter continuity connections amongst these continuities. Further we introduce αg-(γ,δ)-homeomorphism, αg-(γ,δ)-open maps and derive their properties. Finally, we obtain the Urysohn's lemma for αg-γ-normal space.

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2. Preliminaries

Throughout this paper, $(\mathfrak{L}, \varsigma)$ and $(\mathfrak{K}, \xi)$ signifies topological spaces.

Definition 2.1. Let $(\mathfrak{L}, \varsigma)$ be a topological space. A subset J of $(\mathfrak{L}, \varsigma)$ is called

(1) α-open [12] if $J \subseteq int(cl(int(J)))$. The collection of all α-open sets in $\mathfrak{L}$ is denoted by ς_α. The complement of an α-open set is called α-closed. The intersection of all α-closed sets containing J is called α-closure of J and is denoted by $\alpha cl(J)$.

(2) αg-closed [7] if $\alpha cl(J) \subseteq U$ whenever $J \subseteq U$ and U is open in $(\mathfrak{L}, \varsigma)$. The complement of an αg-closed set is called αg-open. The collection of all αg-open sets in $\mathfrak{L}$ is denoted by $\varsigma_{\alpha g}$.

Definition 2.2 ([8])**.** A map γ from $\varsigma_{\alpha g}$ into the power set $P(\mathfrak{L})$ of $\mathfrak{L}$ is called an operation if $H \subseteq \gamma(H)$ $\forall$ $H \in \varsigma_{\alpha g}$, where $\gamma(H)$ is the value of H under the operation γ.

Definition 2.3 ([8])**.** A non-empty subset J of $(\mathfrak{L}, \varsigma)$ with an operation γ on $\varsigma_{\alpha g}$ is called an αg_γ-open set if $\forall$ $l \in J$, $\exists$ an αg-open set $H \ni l \in H$ and $\gamma(H) \subseteq J$. The collection of all αg_γ-open sets in $(\mathfrak{L}, \varsigma)$ is denoted by $\varsigma_{\alpha g_\gamma}$. The complement of an αg_γ-open set is called αg_γ-closed.

Definition 2.4 ([8])**.** A topological space $(\mathfrak{L}, \varsigma)$ is αg_γ-regular if $\forall$ $l \in \mathfrak{L}$ and $\forall$ αg-open set H_1 containing l, $\exists$ an αg-open set H_2 containing l such that $\gamma(H_2) \subseteq H_1$.

Definition 2.5 ([8])**.** An operation γ on $\varsigma_{\alpha g}$ is an αg-open operation if $\forall$ αg-open set H_1 containing $l \in \mathfrak{L}$, $\exists$ an αg_γ-open set H_2 containing l such that $H_2 \subseteq \gamma(H_1)$.

Definition 2.6 ([8])**.** A point $l \in \mathfrak{L}$ is an αg_γ-closure point of a set J if $\gamma(H) \cap J \neq \phi$ $\forall$ αg-open set H containing l.
$\alpha gcl_\gamma(J) = \{l \in \mathfrak{L} : \gamma(H) \cap J \neq \phi, \forall\ H, \alpha g \text{ - open set containing } l\ \}$

Definition 2.7 ([8])**.** $\alpha g_\gamma cl(J) = \cap\{K \subseteq \mathfrak{L} : J \subseteq K \text{ where } \mathfrak{L} \setminus K \in \varsigma_{\alpha g_\gamma}\}$

Definition 2.8. [9] In a topological space $(\mathfrak{L}, \varsigma)$, $J \subseteq \mathfrak{L}$ is an αg_γ-generalized closed (concisely αg_γ-g.closed) set if $\alpha gcl_\gamma(J) \subseteq U$ whenever $J \subseteq U$ and U is αg_γ-open in $(\mathfrak{L}, \varsigma)$.

Definition 2.9 ([10])**.** A topological space $(\mathfrak{L}, \varsigma)$ is

(1) αg_γ-T_0 if for $l \neq n$ in $\mathfrak{L}$, $\exists$ an αg-open set $H \ni$ either $l \in H$; $n \notin \gamma(H)$ or $n \in H$; $l \notin \gamma(H)$.

(2) αg_γ-T_1 if for $l \neq n$ in $\mathfrak{L}$, $\exists$ αg-open sets H_1 and $H_2 \ni l \in H_1; n \notin \gamma(H_1)$ and $n \in H_2; l \notin \gamma(H_2)$.

(3) αg_γ-T_2 if for $l \neq n$ in $\mathfrak{L}$, $\exists$ αg -open set H_1 and $H_2 \ni l \in H_1; n \in H_2$ and $\gamma(H_1) \cap \gamma(H_2) = \phi$.

(4) αg_γ-$T_0{}'$ if for $l \neq n$ in $\mathfrak{L}$, $\exists$ an αg_γ-open set $H \ni$ either $l \in H$; $n \notin H$ or $n \in H$; $l \notin H$.

(5) αg_γ-$T_1^{'}$ if for $l \neq n$ in $\mathfrak{L}$, $\exists$ αg_γ-open sets H_1 and $H_2 \ni l \in H_1; n \notin H_1$ and $n \in H_2; l \notin H_2$.

(6) αg_γ-$T_2^{'}$ if for $l \neq n$ in $\mathfrak{L}$, $\exists$ αg_γ-open sets H_1 and $H_2 \ni l \in H_1; n \in H_2$ and $H_1 \cap H_2 = \phi$.

(7) αg_γ-$T_{1/2}$ space if $\forall$ αg_γ-g.closed set of $(\mathfrak{L}, \varsigma)$ is αg_γ-closed.

Definition 2.10 ([11])**.** A topological space $(\mathfrak{L}, \varsigma)$ is αg-γ-normal if for any pair of disjoint αg_γ-closed sets J and G of $\mathfrak{L}$, $\exists$ disjoint αg_γ-open sets H_1 and H_2 containing J and G respectively.

Definition 2.11 ([3])**.** A point $l \in \mathfrak{L}$ is in γ-closure of a set $J \subseteq \mathfrak{L}$, if $\gamma(H) \cap J \neq \phi$ $\forall$ open set H containing l, where γ is an operation on ς. The γ-closure of J is denoted by $Cl_\gamma(J)$.

Definition 2.12 ([13])**.** An operation γ on ς is called open if $\forall$ open set U containing $x \in \mathfrak{L}$, $\exists$ a γ-open set $H_2 \ni l \in H_2$ and $H_2 \subseteq \gamma(H_1)$.

Definition 2.13 ([4])**.** A topological space $(\mathfrak{L}, \varsigma)$ is γ-regular if $\forall$ $l \in \mathfrak{L}$ and $\forall$ open set H_1 containing l, $\exists$ an open set H_2 containing $l \ni \gamma(H_2) \subseteq H_1$.

Theorem 2.14. [10] *A topological space with an αg-open operation is αg_γ - $T_{1/2}$ iff $\{l\}$ is αg_γ-closed or αg_γ-open $\forall$ $l \in \mathfrak{L}$*

Theorem 2.15. [11] *The following are equivalent for a topological space $(\mathfrak{L}, \varsigma)$ with an operation γ on $\varsigma_{\alpha g}$:*

(1) $\mathfrak{L}$ is αg-γ- normal.

(2) For each αg_γ-closed set J and each αg_γ-open set U containing J, there exists an αg_γ-open set V containing J such that $\alpha g_\gamma cl(V) \subseteq U$.

(3) For each pair of disjoint αg_γ-closed sets J and K of $\mathfrak{L}$, there exists an αg_γ-open set V containing J such that $\alpha g_\gamma cl\,(V) \cap K = \phi$.

3. αg-(γ, δ) Continuous Maps

Definition 3.1. A map $\hbar$:$(\mathfrak{L}, \varsigma, \gamma) \to (\mathfrak{K}, \xi, \delta)$ is said to be αg-(γ, δ) continuous if $\forall$ $l \in \mathfrak{L}$ and every αg-open set H_2 in $(\mathfrak{K}, \xi, \delta)$ containing $\hbar(l)$, $\exists$ an αg-open set H_1 in $(\mathfrak{L}, \varsigma, \gamma) \ni l \in H_1$ and $\hbar(\gamma(H_1)) \subseteq \delta(H_2)$.

Example 3.2. Consider $\mathfrak{L} = \mathfrak{K} = \{\ell, \mathscr{o}, \mathscr{p}, \mathscr{b}\}$ with $\varsigma = \{\phi, \{\ell, \mathscr{o}\}, \mathfrak{L}\}$ and $\xi = \{\phi, \{\ell, \mathscr{o}, \mathscr{p}\}, \mathfrak{K}\}$ Then $\varsigma_{\alpha g} = P(\mathfrak{L}) \setminus [\{\mathscr{p}, \mathscr{b}\}, \{\ell, \mathscr{p}, \mathscr{b}\}, \{\mathscr{o}, \mathscr{p}, \mathscr{b}\}]$ and $\xi_{\alpha g} = \{\phi, \{\ell\}, \{\mathscr{o}\}, \{\mathscr{p}\}, \{\ell, \mathscr{o}\}, \{\ell, \mathscr{p}\}, \{\mathscr{o}, \mathscr{p}\}, \{\ell, \mathscr{o}, \mathscr{p}\}, \mathfrak{K}\}$. Let γ and δ be two operations on $\varsigma_{\alpha g}$ and $\xi_{\alpha g}$ defined as

$$\gamma(J) = J \quad \forall \quad J \in \varsigma_{\alpha g} \text{ and } \delta(G) = \begin{cases} G & \text{if } \mathscr{p} \in G \\ \mathfrak{K} & \text{otherwise} \end{cases} \quad \forall \quad G \in \xi_{\alpha g}$$

Then the map $\hbar$:$(\mathfrak{L}, \varsigma, \gamma) \to (\mathfrak{K}, \xi, \delta)$ defined as $\hbar(\ell) = \ell, \hbar(\mathscr{o}) = \mathscr{p}, \hbar(\mathscr{p}) = \mathscr{b}$, $\hbar(\mathscr{b}) = \mathscr{o}$ is αg-(γ, δ)-continuous.

Theorem 3.3. *Consider the topological spaces $(\mathfrak{L}, \varsigma, \gamma), (\mathfrak{K}, \xi, \delta)$ and $(\mathfrak{P}, \sigma, \mu)$ with operations γ, δ and μ on $\varsigma_{\alpha g}, \xi_{\alpha g}$ and $\sigma_{\alpha g}$ respectively. If $\hbar$:$(\mathfrak{L}, \varsigma, \gamma) \to (\mathfrak{K}, \xi, \delta)$*

is αg-(γ,δ)-continuous and $\hbar : (\mathfrak{K},\xi,\delta) \to (\mathfrak{P},\sigma,\mu)$ is αg-(δ,μ)-continuous, then $\hbar \circ \hbar : (\mathfrak{L},\varsigma,\gamma) \to (\mathfrak{P},\sigma,\mu)$ is αg-(γ,μ)-continuous.

Proof. Consider any point l in $\mathfrak{L}$ and any αg-open set H_3 of $\mathfrak{P}$ containing $\hbar(\hbar(\ell))$. Since $\hbar$ is αg-(δ,μ)-continuous, $\exists$ an αg-open set H_2 of $\mathfrak{K} \ni \hbar(l) \in H_2$ and $\hbar(\delta(H_2)) \subseteq \mu(H_3)$. Also since $\hbar$ is αg-(γ,δ)-continuous, $\exists$ an αg-open set H_1 of $\mathfrak{L}$ containing $l \ni \hbar(\gamma(H_1)) \subseteq \delta(H_2)$. Consequently, $\Rightarrow \hbar(\hbar(\gamma(H_1))) \subseteq \hbar(\delta(H_2)) \subseteq \mu(H_3)$. Therefore $\hbar \circ \hbar$ is αg-(γ,μ)-continuous. □

Theorem 3.4. *Consider $\hbar$:$(\mathfrak{L},\varsigma,\gamma) \to (\mathfrak{K},\xi,\delta)$ to be a map.*

(a) $\hbar$ is αg-(γ,δ)-continuous.
(b) $\hbar$ $(\alpha gcl_\gamma(J)) \subseteq \alpha gcl_\delta(\hbar(J))$, $\forall$ $J \subseteq \mathfrak{L}$.
(c) $(\alpha gcl_\gamma(\hbar^{-1}(J)) \subseteq \hbar^{-1}(\alpha gcl_\delta(J))$, $\forall J \subseteq \mathfrak{K}$.
(d) for any αg_δ-closed set J of $(\mathfrak{K},\xi,\delta)$, $\hbar^{-1}(J)$ is αg_γ-closed in $(\mathfrak{L},\varsigma,\gamma)$
(e) for any αg_δ-open set J of $(\mathfrak{K},\xi,\delta)$, $\hbar^{-1}(J)$ is αg_γ-open in $(\mathfrak{L},\varsigma,\gamma)$

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e)

Proof. (a) $\Rightarrow$ (b) Suppose $n \in \hbar(\alpha gcl_\gamma(J))$ and H_2 is any αg-open set in $(\mathfrak{K},\xi,\delta)$ containing n. Then by assumption, $\exists$ a point $l \in \alpha gcl_\gamma(J)$ and an αg-open set $H_1 \ni \hbar(l) = n, l \in H_1$ and $\hbar(\gamma(H_1)) \subseteq \delta(H_2)$. Since $l \in \alpha gcl_\gamma(J), \gamma(H_1) \cap J \neq \phi$. Then $\hbar(\gamma(H_1) \cap J) \subseteq \hbar(\gamma(H_1)) \cap \hbar(J) \subseteq \delta(H_2) \cap \hbar(J) \neq \phi$. $\because H_2$ is an αg-open set containing n, $n \in \alpha gcl_\delta(\hbar(J))$. Therefore, $\hbar(\alpha gcl_\gamma(J)) \subseteq \alpha gcl_\delta(\hbar(J))$.

(b) $\Rightarrow$ (c) Let $J \subseteq \mathfrak{K}$. By (b), $\hbar(\alpha gcl_\gamma(\hbar^{-1}(J))) \subseteq \alpha gcl_\delta(\hbar(\hbar^{-1}(J))) \subseteq \alpha gcl_\delta(J)$. Hence $\alpha gcl_\gamma(\hbar^{-1}(J)) \subseteq \hbar^{-1}(\alpha gcl_\delta(J))$.

(c) $\Rightarrow$ (d) Consider J to be an αg_δ-closed set in $(\mathfrak{K},\xi,\delta)$. Then $J = \alpha gcl_\delta(J)$ [8]. By (c), $\hbar(\alpha gcl_\gamma(\hbar^{-1}(J))) \subseteq \alpha gcl_\delta(J) = J$. Hence $\alpha gcl_\gamma(\hbar^{-1}(J)) \subseteq \hbar^{-1}(J)$. And also, $\hbar^{-1}(J) \subseteq \alpha gcl_\gamma(\hbar^{-1}(J))$. Thus $\hbar^{-1}(J) = \alpha gcl_\gamma(\hbar^{-1}(J))$. This implies $\hbar^{-1}(J)$ is αg_γ-closed in $(\mathfrak{L},\varsigma,\gamma)$ [8].

(d) $\Rightarrow$ (e) Consider J to be an αg_δ-open set in $(\mathfrak{K},\xi,\delta)$. $\Rightarrow \mathfrak{K} \setminus J$ is an αg_δ-closed set in $(\mathfrak{K},\xi,\delta)$. By (d), $\hbar^{-1}(\mathfrak{K} \setminus J)$ is αg_γ-closed in $(\mathfrak{L},\varsigma,\gamma)$. Since $\hbar^{-1}(\mathfrak{K} \setminus J) = \mathfrak{K} \setminus \hbar^{-1}(J), \hbar^{-1}(J)$ is αg_γ-open in $(\mathfrak{L},\varsigma,\gamma)$.

□

Corollary 3.5. The statements of Theorem 3.4 are equivalent, if $(\mathfrak{K},\xi,\delta)$ is αg_δ-regular space.

Proof. From Theorem 3.4, it is enough to show that (e) implies (a) when $(\mathfrak{K},\xi,\delta)$ is an αg_δ-regular space.

Let l be any point in $\mathfrak{L}$ and H_2 be an αg-open set in $(\mathfrak{K},\xi,\delta) \ni \hbar(l) \in H_2$. Since $(\mathfrak{K},\xi,\delta)$ is an αg_δ-regular space, H_2 is αg_δ-open in $(\mathfrak{K},\xi,\delta)$. As $\hbar(l) \in H_2, l \in \hbar^{-1}(H_2)$ and by Theorem 3.4 (e), $\hbar^{-1}(H_2)$ is an αg_γ-open. As $\hbar^{-1}(H_2)$ is an αg_γ-open set, $\exists$ an αg-open set $H_1 \ni l \in H_1$ and $\gamma(H_1) \subseteq \hbar^{-1}(H_2)$. This implies $\hbar(\gamma(H_1)) \subseteq H_2 \subseteq \delta(H_2)$. Therefore $\hbar$ is αg-(γ,δ) continuous. □

Corollary 3.6. The statements of Theorem 3.4 are equivalent, if δ is an αg-open operation.

Proof. From Theorem 3.4, it is enough to prove that (e) implies (a) when δ is an αg-open operation.
Consider any point l in $\mathfrak{L}$ and an αg-open set H_2 containing $\hbar(l)$. Since δ is an αg-open operation, $\exists$ an αg_δ-open set $H_3 \ni \hbar(l) \in H_3$ and $H_3 \subseteq \delta(H_2)$. As $\hbar(l) \in H_3, l \in \hbar^{-1}(H_3)$ and by Theorem 3.4 (e), $\hbar^{-1}(H_3)$ is αg_γ-open. As $\hbar^{-1}(H_3)$ is αg_γ-open, $\exists$ an αg-open set H_1 containing $l \ni \gamma(H_1) \subseteq \hbar^{-1}(H_3) \subseteq \hbar^{-1}(\delta(H_2))$ which implies $\hbar(\gamma(H_1)) \subseteq \delta(H_2)$. Hence $\hbar$ is αg-(γ,δ) continuous. □

Definition 3.7. A map $\hbar$:$(\mathfrak{L},\varsigma,\gamma) \to (\mathfrak{K},\xi,\delta)$ is called

(i) αg-(γ,δ)- closed if $\hbar(J)$ is αg_δ-closed in $(\mathfrak{K},\xi,\delta)$ $\forall$ αg_γ-closed set J of $(\mathfrak{L},\varsigma,\gamma)$.

(ii) αg-(id,δ)- closed if $\hbar(J)$ is αg_δ-closed in $(\mathfrak{K},\xi,\delta)$ $\forall$ αg-closed set J of $(\mathfrak{L},\varsigma,\gamma)$.

Example 3.8. Consider $\mathfrak{L} = \mathfrak{K} = \{\hbar_1,\hbar_2,\hbar_3\}$ with $\varsigma = \{\phi,\{\hbar_3\},\mathfrak{L}\}$ and $\xi = \{\phi,\{\hbar_1\}$ $,\{\hbar_2\},\{\hbar_1,\hbar_2\},\{\hbar_1,\hbar_3\},\mathfrak{K}\}$. Then $\varsigma_{\alpha g} = P(\mathfrak{L})\backslash\{\hbar_1,\hbar_2\}$ and $\xi_{\alpha g} = P(\mathfrak{K})\backslash[\{\hbar_3\},\{\hbar_2,\hbar_3\}]$. Let $\hbar$:$(\mathfrak{L},\varsigma,\gamma) \to (\mathfrak{K},\xi,\delta)$ be a map defined by $\hbar(\hbar_1) = \hbar_2, \hbar(\hbar_2) = \hbar(\hbar_3) = \hbar_3 \ni$

$$\gamma(J) = \begin{cases} \{\hbar_2,\hbar_3\} & \text{if } J = \{\hbar_2\} \text{ or } \{\hbar_3\} \\ cl(J) & \text{otherwise} \end{cases} \quad \forall \quad J \in \varsigma_{\alpha g}$$

$$\delta(G) = \begin{cases} G & \text{if } G = \{\hbar_1,\hbar_2\} \\ cl(G) & \text{otherwise} \end{cases} \quad \forall \quad G \in \xi_{\alpha g}$$

Here αg_γ-closed sets are $\phi,\{\hbar_1\},\{\hbar_2\},\{\hbar_2,\hbar_3\},\mathfrak{L}$ and the αg_δ-closed sets are $\phi,\{\hbar_2\},\{\hbar_3\},\{\hbar_1,\hbar_3\},\mathfrak{K}$. Therefore, the map $\hbar$ is αg-(γ,δ)- closed.

Example 3.9. Consider $\mathfrak{L} = \mathfrak{K} = \{d_1,d_2,d_3\}$ with $\varsigma = \{\ \phi,\{d_1\},\ \{d_2\},\{d_1,d_2\},\ \mathfrak{L}\}$ and $\xi = \{\phi\ ,\ \{d_1\},\ \{d_1,d_2\},\ \mathfrak{K}\}$. Then $\varsigma_{\alpha g} = \varsigma$ and $\xi_{\alpha g} = P(\mathfrak{K})\backslash\ [\{d_3\},\ \{d_2,d_3\}]$. Let $\hbar$:$(\mathfrak{L},\varsigma,\gamma) \to (\mathfrak{K},\xi,\delta)$ be a map defined by $\hbar(d_1) = d_3$, $\hbar(d_2)= \hbar(d_3)= d_2 \ni$

$$id(J) = J \ \forall\ J \in \varsigma_{\alpha g} \text{ and } \delta(G) = \begin{cases} G & \text{if } d_1 \in G \\ cl(G) & \text{otherwise} \end{cases} \quad \forall \quad G \in \xi_{\alpha g}$$

Here αg_δ-closed sets are $\phi,\{d_2\},\{d_3\},\{d_2,d_3\},\mathfrak{K}$. Then the map $\hbar$ is αg-(id,δ)-closed.

Remark 3.10. If $\hbar$ is bijective and $\hbar^{-1} : (\mathfrak{K},\xi,\delta) \to (\mathfrak{L},\varsigma,\gamma)$ is αg-(δ,id)-continuous then $\hbar$ is αg-(id,δ)-closed.

Proof. As $\hbar$ is bijective and $\hbar^{-1}$ is αg-(δ,id)-continuous, by Theorem 3.4, for any αg-closed set J of $(\mathfrak{L},\varsigma,\gamma)$, $(\hbar^{-1})^{-1}(J)$ is αg_δ-closed in $(\mathfrak{K},\xi,\delta) \Rightarrow \hbar(J)$ is αg_δ-closed in $(\mathfrak{K},\xi,\delta)$. Therefore $\hbar$ is αg-(id,δ)-closed.

□

Definition 3.11. Let $\hbar$:$(\mathfrak{L},\varsigma,\gamma) \to (\mathfrak{K},\xi,\delta)$ be a map. Then $\hbar$ is called as αg-(γ,δ)-homeomorphism, if $\hbar$ is bijective and both $\hbar$ and $\hbar^{-1}$ are αg-(γ,δ)-continuous.

Theorem 3.12. *Let $(\mathfrak{L},\varsigma,\gamma)$ be an αg_γ-$T_{1/2}$ topological space with an αg-open operation and $\hbar$:$(\mathfrak{L},\varsigma,\gamma) \to (\mathfrak{K},\xi,\delta)$ be an αg-(γ,δ)-homeomorphism. Then $(\mathfrak{K},\xi,\delta)$ is αg_δ-$T_{1/2}$.*

Proof. Consider $\{n\}$ to be a singleton set of $(\mathfrak{K},\xi,\delta)$ and l be any point in $\mathfrak{L} \ni n = \hbar(l)$, which $\Rightarrow l = \hbar^{-1}(n)$. By Theorem 2.14, $\{l\}$ is either αg_γ-closed or αg_γ-open. As $\hbar^{-1}$ is αg-(γ,δ)-continuous, $\{n\} = \{\hbar(l)\}$ is αg_δ-closed or αg_δ-open. By Theorem 2.14, we get $(\mathfrak{K},\xi,\delta)$ is αg_δ-$T_{1/2}$. □

Theorem 3.13. *Let $\hbar$:$(\mathfrak{L}, \varsigma, \gamma) \rightarrow (\mathfrak{K}, \xi, \delta)$ be an injective and an αg-(γ, δ)-continuous map. If $(\mathfrak{K}, \xi, \delta)$ is an αg_δ-T_2 space, then $(\mathfrak{L}, \varsigma, \gamma)$ is an αg_γ-T_2 space.*

Proof. Assume that l and n are any two distinct points in $\mathfrak{L}$. Since $(\mathfrak{K}, \xi, \delta)$ is αg_δ-T_2 space, $\exists$ αg-open sets H_1 and H_2 in $(\mathfrak{K}, \xi, \delta)$ $\ni$ $\hbar(l) \in H_1, \hbar(n) \in H_2$ and $\delta(H_1) \cap \delta(H_2) = \phi$. Since $\hbar$ is αg-(γ, δ)-continuous, $\forall$ αg-open sets H_1 and H_2 $\exists \alpha g$-open sets P and Q in $(\mathfrak{L}, \varsigma, \gamma)$ containing l and n respectively $\ni \hbar(\gamma(P)) \subseteq \delta(H_1)$ and $\hbar(\gamma(Q)) \subseteq \delta(H_2)$. Hence $\hbar(\gamma(P)) \cap \hbar(\gamma(Q)) = \phi$ which implies $\hbar(\gamma(P) \cap \gamma(Q)) = \phi$. Since $\hbar$ is injective, $\hbar^{-1}(\hbar(\gamma(P) \cap \gamma(Q))) = \phi$. Thus $\gamma(P) \cap \gamma(Q) = \phi$. Therefore $(\mathfrak{L}, \varsigma, \gamma)$ is an αg_γ-T_2 space. $\square$

Theorem 3.14. *Let $\hbar$:$(X, \varsigma, \gamma) \rightarrow (\mathfrak{K}, \xi, \delta)$ be an injective and an αg-(γ, δ)-continuous map. If $(\mathfrak{K}, \xi, \delta)$ is an αg_δ-T_1 (resp. αg_δ-T_0) space, then $(\mathfrak{L}, \varsigma, \gamma)$ is an αg_γ-T_1 (resp. αg_δ-T_0) space.*

Proof. By Theorem 3.13. $\square$

Theorem 3.15. *Let $\hbar$:$(\mathfrak{L}, \varsigma, \gamma) \rightarrow (\mathfrak{K}, \xi, \delta)$ be a surjective and an αg-(γ, δ)-open map. If $(\mathfrak{L}, \varsigma, \gamma)$ is an αg_γ-$T_2^{'}$ (resp. αg_γ-$T_0^{'}$ and αg_γ-$T_1^{'}$) space, then $(\mathfrak{K}, \xi, \delta)$ is an αg_δ-$T_2^{'}$ (resp. αg_δ-$T_0^{'}$ and αg_δ-$T_1^{'}$) space.*

Proof. Consider an αg-(γ, δ)-open surjective map $\hbar$ and $l \neq m$ in $\mathfrak{K}$. Then $\exists$ $p \neq r$ in $\mathfrak{L}$ $\ni$ $\hbar(p) = l$ and $\hbar(r) = m$. Since $(\mathfrak{L}, \varsigma, \gamma)$ is αg_γ-$T_2^{'}$, $\exists$ αg_γ-open sets H_1 and H_2 containing p and r respectively $\ni H_1 \cap H_2 = \phi$. Since $\hbar$ is αg-(γ, δ)-open, $\hbar(H_1)$ and $\hbar(H_2)$ are αg_δ-open in $(\mathfrak{K}, \xi, \delta)$ $\ni l = \hbar(p) \in \hbar(H_1)$ and $m = \hbar(r) \in \hbar(H_2)$ and $\hbar(H_1) \cap \hbar(H_2) = \phi$. Hence $(\mathfrak{K}, \xi, \delta)$ is αg_δ-$T_2^{'}$. $\square$

4. αg-δ-Continuous Maps

Definition 4.1. A map $\hbar$:$(\mathfrak{L}, \varsigma, \gamma) \rightarrow (\mathfrak{K}, \xi, \delta)$ is called αg-δ-continuous if $\forall$ l in $\mathfrak{L}$ and $\forall$ αg-open set H_2 of $\mathfrak{K}$ containing $\hbar(l), \exists$ an αg-open set H_1 of $\mathfrak{L}$ containing $l \ni \hbar(H_1) \subseteq \delta(H_2)$.

Example 4.2. Consider $\mathfrak{L} = \mathfrak{K} = \{\ell_1, \ell_2, \ell_3\}$ with ς=$\{ \phi, \{\ell_1\}, \{\ell_1, \ell_2\}, \{\ell_1, \ell_3\}, \mathfrak{L}\}$ and $\xi = \{\phi, \{\ell_1\}, \{\ell_2\}, \{\ell_1, \ell_2\}, \mathfrak{K}\}$. Then $\varsigma_{\alpha g} = \varsigma$ and $\xi_{\alpha g} = \xi$. Let δ be an operations on $\xi_{\alpha g}$ defined as

$$\delta(G) = \begin{cases} \{\ell_1, \ell_2\} & \text{if } G \text{ is Singleton} \\ \mathfrak{K} & \text{otherwise} \end{cases} \quad \forall \quad G \in \xi_{\alpha g}$$

Then the map $\hbar$:$(\mathfrak{L}, \varsigma, \gamma) \rightarrow (\mathfrak{K}, \xi, \delta)$ defined as $\hbar(\ell_1) = \ell_2, \hbar(\ell_2) = \ell_1$, $\hbar(\ell_3) = \ell_3$ is αg-δ-continuous.

Theorem 4.3. *If $\hbar$:$(\mathfrak{L}, \varsigma, \gamma) \rightarrow (\mathfrak{K}, \xi, \delta)$ is αg-(γ, δ)-continuous, then $\hbar$ is αg-δ-continuous but not conversely.*

Proof. As $\hbar$ is αg-(γ, δ)-continuous, $\forall$ $l \in \mathfrak{L}$ and every αg-open set H_2 in $(\mathfrak{K}, \xi, \delta)$ containing $\hbar(l), \exists$ an αg-open set H_1 in $(\mathfrak{L}, \varsigma, \gamma)$ $\ni l \in H_1$ and $\hbar(\gamma(H_1)) \subseteq \delta(H_2)$. $\because \gamma$ is an operation, $H_1 \subseteq \gamma(H_1)$ implies that $\hbar(H_1) \subseteq \hbar(\gamma(H_1)) \subseteq \delta(H_2)$. Therefore $\hbar$ is αg-δ-continuous. $\square$

Example 4.4. Consider $\mathfrak{L} = \mathfrak{K} = \{\ell, p, q\}$ with $\varsigma = \xi = \{\phi, \{\ell\}, \mathfrak{K}\}$. Then $\varsigma_{\alpha g} = \varsigma$ and $\xi_{\alpha g} = P(\mathfrak{K}) \setminus \{p, q\}$. Let γ and δ be two operations on $\varsigma_{\alpha g}$ and $\xi_{\alpha g}$ defined as

$$\gamma(J) = \begin{cases} \mathfrak{L} & \text{if } q \in J \\ J & \text{otherwise} \end{cases} \quad \forall \quad J \in \varsigma_{\alpha g}$$

$$\delta(G) = \begin{cases} G \cup \{p\} & \text{if } G = \{\ell\} \text{ or} \{\ell, p\} \\ G & \text{otherwise} \end{cases} \quad \forall \quad G \in \xi_{\alpha g}$$

Then the map $\hbar{:}(\mathfrak{L}, \varsigma, \gamma) \to (\mathfrak{K}, \xi, \delta)$ defined as $\hbar(\ell) = p, \hbar(p) = q, \hbar(q) = \ell$ is αg-δ-continuous but not αg-(γ, δ)-continuous.

Remark 4.5. Composition of two αg-δ-continuous maps is αg-δ-continuous.

Proof. Analogous to the proof of Theorem 3.3. □

Theorem 4.6. *Let* $\hbar{:}(\mathfrak{L}, \varsigma, \gamma) \to (\mathfrak{K}, \xi, \delta)$ *be a map.*

(a) $\hbar$ *is* αg-δ*-continuous.*

(b) $\hbar(\alpha gcl(J)) \subseteq \alpha gcl_\delta(\hbar(J)), \forall J \subseteq \mathfrak{L}$.

(c) $\alpha gcl(\hbar^{-1}(J)) \subseteq \hbar^{-1}(\alpha gcl_\delta(J)), \forall J \subseteq \mathfrak{K}$.

Then (a) $\Rightarrow$ *(b)* $\Rightarrow$ *(c)*

Proof. Analogous to the proof of Theorem 3.4. □

Remark 4.7. If $\hbar{:}(\mathfrak{L}, \varsigma, \gamma) \to (\mathfrak{K}, \xi, \delta)$ is αg-δ-continuous, then $\hbar^{-1}(J)$ need not be αg-open in $(\mathfrak{L}, \varsigma, \gamma)$,$\forall$ αg_δ-open set J of $(\mathfrak{K}, \xi, \delta)$.

Example 4.8. Consider $\mathfrak{L} = \mathfrak{K} = \{u_1, u_2, u_3\}$ with $\varsigma = \{\phi, \{u_1\}, \mathfrak{L}\}$ and $\xi = \{\phi, \{u_1\}, \{u_3\}, \{u_1, u_3\}, \mathfrak{K}\}$. Then $\varsigma_{\alpha g} = P(\mathfrak{L}) \setminus \{u_2, u_3\}$ and $\xi_{\alpha g} = \xi$. Let δ be the operations on $\xi_{\alpha g}$ defined as

$$\delta(G) = \begin{cases} \{u_1, u_3\} & \text{if } G \text{ is Singleton} \\ \mathfrak{K} & \text{otherwise} \end{cases} \quad \forall \quad G \in \xi_{\alpha g}$$

Then $\xi_{\alpha g_\delta} = \{\phi, \{u_1, u_3\}, \mathfrak{K}\}$. Here the map $\hbar{:}(\mathfrak{L}, \varsigma, \gamma) \to (\mathfrak{K}, \xi, \delta)$ defined as $\hbar(u_1) = u_2, \hbar(u_2) = u_1, \hbar(u_3) = u_3$ is αg-δ-continuous, but for $J = \{u_1, u_3\}$, $\hbar^{-1}(J) = \{u_2, u_3\}$ is not αg-open in $(\mathfrak{L}, \varsigma, \gamma)$.

5. αg_γ-δ-Continuous Maps

Definition 5.1. A map $\hbar{:}(\mathfrak{L}, \varsigma, \gamma) \to (\mathfrak{K}, \xi, \delta)$ is said to be αg_γ-δ-continuous, if $\forall$ $l \in \mathfrak{L}$ and $\forall$ open set H_2 in $\mathfrak{K}$ containing $\hbar(l)$, $\exists$ an αg_γ-open set H_1 in $\mathfrak{L}$ containing $l \ni \hbar(H_1) \subseteq \delta(H_2)$.

Example 5.2. Consider $\mathfrak{L} = \mathfrak{K} = \{u_1, u_2, u_3\}$ with $\varsigma = \xi = \{\phi, \{u_1\}, \{u_2, u_3\}, \mathfrak{L}\}$. Then $\varsigma_{\alpha g} = \xi_{\alpha g} = P(\mathfrak{L})$. Let $\gamma : \varsigma_{\alpha g} \to P(\mathfrak{L})$ and $\delta : \xi_{\alpha g} \to P(\mathfrak{K})$ be two operations defined as

$$\gamma(J) = \begin{cases} J & \text{if } u_1 \in J \\ J \cup \{u_3\} & \text{if } u_1 \notin J \end{cases} \quad \forall \quad J \in \varsigma_{\alpha g}$$

$$\delta(G) = \begin{cases} G & \text{if } u_2 \in G \\ \{u_1, u_3\} & \text{if } u_2 \notin G \end{cases} \quad \forall \quad G \in \xi_{\alpha g}$$

Here $\varsigma_{\alpha g_\gamma} = P(\mathfrak{L}) \setminus \{u_2\}$. Then the map $\hbar:(\mathfrak{L}, \varsigma, \gamma) \to (\mathfrak{K}, \xi, \delta)$ defined as $\hbar(u_1) = u_2, \hbar(u_2) = u_3, \hbar(u_3) = u_1$ is αg_γ-δ-continuous.

Theorem 5.3. *Suppose $\hbar:(\mathfrak{L}, \varsigma, \gamma) \to (\mathfrak{K}, \xi, \delta)$ be a map.*

(i) $\hbar$ is αg_γ- δ-continuous.
(ii) $\hbar(\alpha g_\gamma cl(J)) \subseteq cl_\delta(\hbar(J)), \forall J \subseteq \mathfrak{L}$.
(iii) $\alpha g_\gamma cl(\hbar^{-1}(J)) \subseteq \hbar^{-1}(cl_\delta(J)), \forall J \subseteq \mathfrak{K}$.
(iv) for any δ-closed set J of $(\mathfrak{K}, \xi, \delta), \hbar^{-1}(J)$ is αg_γ-closed in $(\mathfrak{L}, \varsigma, \gamma)$.
(v) for any δ-open set J of $(\mathfrak{K}, \xi, \delta), \hbar^{-1}(J)$ is αg_γ-open in $(\mathfrak{L}, \varsigma, \gamma)$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v).

Proof. (i) $\Rightarrow$ (ii) Let $n \in \hbar(\alpha g_\gamma cl(J))$ and H_2 be any open set in $(\mathfrak{K}, \xi, \delta)$ containing n. By the definition of αg_γ-δ-continuous, $\forall$ open set H_2 in $\mathfrak{K}$ containing $\hbar(l), \exists$ an αg_γ-open set H_1 containing $l \ni \hbar(H_1) \subseteq \delta(H_2)$. Since $l \in \alpha g_\gamma cl(J)$ and H_1 is an αg_γ-open set containing $l, H_1 \cap J \neq \phi$. $\Rightarrow \hbar(H_1 \cap J) \subseteq \hbar(H_1) \cap \hbar(J) \subseteq \delta(H_2) \cap \hbar(J) \neq \phi$. By Definition 2.11, $n \in cl_\delta(\hbar(J))$. Therefore $\hbar(\alpha g_\gamma cl(J)) \subseteq cl_\delta(\hbar(J))$.

(ii) $\Rightarrow$ (iii) Assume $J \subseteq \mathfrak{K}$. By (ii), $\hbar(\alpha g_\gamma cl(\hbar^{-1}(J))) \subseteq cl_\delta(\hbar(\hbar^{-1}(J))) \subseteq cl_\delta(J)$. Hence $\alpha g_\gamma cl(\hbar^{-1}(J)) \subseteq \hbar^{-1}(cl_\delta(J))$.

(iii) $\Rightarrow$ (iv) Consider J to be a δ-closed set in $(\mathfrak{K}, \xi, \delta)$. Then $J = cl_\delta(J)$ [13]. By (iii), $\hbar(\alpha g_\gamma cl(\hbar^{-1}(J))) \subseteq \hbar(\hbar^{-1}(cl_\delta(J))) \subseteq cl_\delta(J) = J$. Hence $\alpha g_\gamma cl(\hbar^{-1}(J)) \subseteq \hbar^{-1}(J)$. Always, $\hbar^{-1}(J) \subseteq \alpha g_\gamma cl(\hbar^{-1}(J))$. Thus $\hbar^{-1}(J) = \alpha g_\gamma cl(\hbar^{-1}(J))$. Then $\hbar^{-1}(J)$ is αg_γ-closed in $(\mathfrak{L}, \varsigma, \gamma)$[8].

(iv) $\Rightarrow$ (v) Consider J to be a δ-open set in $(\mathfrak{K}, \xi, \delta)$. $\Rightarrow \mathfrak{K} \setminus J$ is a δ-closed set in $(\mathfrak{K}, \xi, \delta)$. By (iv), $\hbar^{-1}(\mathfrak{K} \setminus J)$ is αg_γ-closed in $(\mathfrak{L}, \varsigma, \gamma)$. Since $\hbar^{-1}(\mathfrak{K} \setminus J) = \mathfrak{K} \setminus \hbar^{-1}(J), \hbar^{-1}(J)$ is αg_γ-open in $(\mathfrak{L}, \varsigma, \gamma)$.

□

Corollary 5.4. If the space $(\mathfrak{L}, \varsigma, \gamma)$ is αg_γ-regular and the operation δ is open, then all statements of Theorem 5.3 are equivalent.

Proof. To show that all statements of Theorem 5.3 are equivalent, it is enough to prove that (v) $\Rightarrow$ (i) in Theorem 5.3.
Consider $l \in \mathfrak{L}$ and H_2 to be an open set in $(\mathfrak{K}, \xi, \delta)$ containing $\hbar(l)$. $\because$ the operation δ is open, $\exists$ an δ-open set H_1 in $\mathfrak{K} \ni \hbar(l) \in H_1$ and $H_1 \subseteq \delta(H_2)$. By (v), we get $l \in \hbar^{-1}(H_1) \in \varsigma_{\alpha g_\gamma}$. By using the definition of αg_γ-open set to $\hbar^{-1}(H_1), \exists$ an αg-open set H_3 containing $l \ni H_3 \subseteq \gamma(H_3) \subseteq \hbar^{-1}(H_1) \subseteq \hbar^{-1}(\delta(H_2))$. Since $(\mathfrak{L}, \varsigma, \gamma)$ is αg_γ regular, $\varsigma_{\alpha g} \subseteq \varsigma_{\alpha g_\gamma}$, H_3 is αg_γ-open in $(\mathfrak{L}, \varsigma, \gamma)$ [8]. Since $\xi_\delta \subseteq \xi$, H_1 is open in $(\mathfrak{K}, \xi, \delta) \ni \hbar(H_3) \subseteq (\delta(H_2))$. Therefore $\hbar$ is αg_γ-δ-continuous. □

Theorem 5.5. *Consider $\hbar:(\mathfrak{L}, \varsigma, \gamma) \to (\mathfrak{K}, \xi, \delta)$ to be a map and $(\mathfrak{K}, \xi, \delta)$ to be δ-regular*

(i) $\hbar$ is αg_γ-δ-continuous.
(ii) for any closed set J of $(\mathfrak{K}, \xi, \delta)$, $\hbar^{-1}(J)$ is αg_γ-closed in $(\mathfrak{L}, \varsigma, \gamma)$
(iii) for any open set J of $(\mathfrak{K}, \xi, \delta)$, $\hbar^{-1}(J)$ is αg_γ-open in $(\mathfrak{L}, \varsigma, \gamma)$
Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)

Proof. (i) $\Rightarrow$ (ii) Consider any closed set, J in $(\mathfrak{K}, \xi, \delta)$. $\Rightarrow J = cl(J) = cl_\delta(J)$ [13]. By Theorem 5.3, $\hslash(\alpha g_\gamma cl(J)) \subseteq cl_\delta(\hslash(J))$. Then $\hslash(\alpha g_\gamma cl(\hslash^{-1}(J))) \subseteq \hslash(\hslash^{-1}(cl_\delta(J))) \subseteq cl_\delta(J) = J$. Hence $\alpha g_\gamma cl(\hslash^{-1}(J)) \subseteq \hslash^{-1}(J)$. Always, $\hslash^{-1}(J) \subseteq \alpha g_\gamma cl(\hslash^{-1}(J))$. Thus $\hslash^{-1}(J) = \alpha g_\gamma cl(\hslash^{-1}(J))$. This implies $\hslash^{-1}(J)$ is αg_γ-closed in $(\mathfrak{L}, \varsigma, \gamma)$.

(ii) $\Rightarrow$ (iii) Assume that J is open in $(\mathfrak{K}, \xi, \delta)$. $\Rightarrow \mathfrak{K} \setminus J$ is a closed in $(\mathfrak{K}, \xi, \delta)$. By (ii), $\hslash^{-1}(\mathfrak{K} \setminus J)$ is αg_γ-closed in $(\mathfrak{L}, \varsigma, \gamma)$. Since $\hslash^{-1}(\mathfrak{K} \setminus J) = \mathfrak{K} \setminus \hslash^{-1}(J), \hslash^{-1}(J)$ is αg_γ-open in $(\mathfrak{L}, \varsigma, \gamma)$.

□

Corollary 5.6. If the space $(\mathfrak{L}, \varsigma, \gamma)$ is αg_γ regular and $(\mathfrak{K}, \xi, \delta)$ be δ-regular, then all statements of Theorem 5.5 are equivalent.

Proof. To show that all statements of Theorem 5.5 are equivalent, it is enough to show that (iii) $\Rightarrow$ (i) in Theorem 5.5.

Consider $l \in \mathfrak{L}$ and H_2 to be an open set in $(\mathfrak{K}, \xi, \delta)$ containing $\hslash(l)$. By (iii), $\hslash^{-1}(H_2)$ is αg_γ-open in $(\mathfrak{L}, \varsigma, \gamma)$. By using the definition of αg_γ-open set to $\hslash^{-1}(H_2)$, $\exists$ an αg-open set H_1 in $\mathfrak{L} \ni H_1 \subseteq \gamma(H_1) \subseteq \hslash^{-1}(H_2)$. This implies that $\hslash(H_1) \subseteq \hslash(\gamma(H_1)) \subseteq H_2 \subseteq \gamma(H_2)$. Since $(\mathfrak{L}, \varsigma, \gamma)$ is αg_γ-regular, H_1 is αg_γ-open in $(\mathfrak{L}, \varsigma, \gamma)$. Therefore $\hslash$ is αg_γ-δ-continuous. □

Remark 5.7. Composition of any two αg_γ-δ-continuous map need not be αg_γ-δ-continuous.

Proof. Since open sets and αg_γ-open sets are not relatable. □

Remark 5.8. αg_γ-δ-continuous is independent with αg-δ-continuous and αg-(γ, δ)-continuous.

Proof. Since αg-open sets and αg_γ-open sets are not relatable. □

6. Urysohn Lemma for αg-γ-Normal Space

Lemma 6.1. *An αg_γ regular space $(\mathfrak{L}, \varsigma, \gamma)$ is αg-γ-normal if and only if it is possible to define an αg_γ-δ-continuous map $\hslash$ from an αg_γ regular space $(\mathfrak{L}, \varsigma, \gamma)$ to a δ-regular space $([0, 1], \xi, \delta) \ni \hslash(J) = \{0\}$ and $\hslash(G) = \{1\}$.*

Proof. (Necessary Part): Consider $(\mathfrak{L}, \varsigma, \gamma)$ to be an αg-γ-normal space. Let J and G be any two αg_γ-closed sets $\ni J \cap G = \phi$. Then $\mathfrak{L} \setminus G$ is an αg_γ-open set and $J \subset (\mathfrak{L} \setminus G) = O_1$(say). By Theorem 2.15, $\exists$ an αg_γ-open set O_0 containing $J \ni \alpha g_\gamma cl(O_0) \subset O_1$. Now $\alpha g_\gamma cl(O_0)$ and G are disjoint αg_γ-closed subsets of $\mathfrak{L}$. Again, since $\mathfrak{L}$ is αg-γ-normal, by Theorem 2.15, $\exists$ an αg_γ-open set $O_{1/2}$ containing $\alpha g_\gamma cl(O_0) \ni \alpha g_\gamma cl(O_{1/2}) \subset O_1$.

Thus $J \subset O_0 \subset \alpha g_\gamma cl(O_0) \subset O_{1/2} \subset \alpha g_\gamma cl(O_{1/2}) \subset O_1 = (\mathfrak{L} \setminus G)$

Now $O_{1/2}$ and O_1 are αg_γ-open sets containing the αg_γ-closed sets J and $\alpha g_\gamma cl(O_{1/2})$ respectively, therefore in same manner $\exists$ αg_γ-open sets $O_{1/4}$ and $O_{3/4} \ni$

$J \subset O_{1/4} \subset \alpha g_\gamma cl(O_{1/4}) \subset O_{1/2} \subset O_{3/4} \subset \alpha g_\gamma cl(O_{3/4}) \subset O_1 = (\mathfrak{L} \setminus G)$

Proceeding in this way, for each rational number in [0,1] of the form

$$p = \frac{m}{2^n} \qquad (\text{ where } n = 1, 2, 3, \ldots..; m = 1, 3, 5, 7, \ldots.., 2^{n-1} \text{ })$$

we can find an αg_γ-open set $O_p \ni$ for $p < q$,

$$J \subset O_p \subset \alpha g_\gamma cl(O_p) \subset O_q \subset \alpha g_\gamma cl(O_q) \subset O_1 = (\mathfrak{L} \setminus G)$$

Let us denote the set of rational numbers of the form p in the interval $[0, 1]$ by Q. Define a map $\hbar$ from $\mathfrak{L}$ to the δ regular space $[0, 1]$ as

$$\hbar(l) = \begin{cases} 1 & \text{if } l \in G \\ glb\{p : p \in Q \text{ and } l \in O_p\} & \text{if } l \notin G \end{cases}$$

Now, if $l \in J$, then $l \in O_p\ \forall\ p \in Q$. So, $\hbar(l) = glbQ = 0$. Therefore $\hbar(l) = 0$ whenever $l \in J$. Hence $\hbar(J) = \{0\}$. And $\hbar(l) = 1$ whenever $l \in G$. Then $\hbar(G) = \{1\}$. Next to show that $\hbar$ is αg_γ-δ- continuous. The intervals of the form $(a, 1]$ and $[0, b)$, where $a, b \in (0, 1)$, forms the sub base for the closed interval $[0, 1]$. To show that $\hbar$ is αg_γ-δ-continuous, it is enough to show that $\hbar^{-1}((a, 1])$ and $\hbar^{-1}([0, b))$ are αg_γ-open in $\mathfrak{L}$.

$$\hbar^{-1}((a, 1]) = \{l \in \mathfrak{L} : \hbar(l) > a\}$$
$$\hbar^{-1}([0, b)) = \{l \in \mathfrak{L} : \hbar(l) < b\}$$

Let $l \in \hbar^{-1}((a, 1])$. Then $a < \hbar(l) \leq 1$. Since Q is dense in $[0, 1]$, $\exists\ p, q$ in $Q \ni a < p < q < \hbar(l)$. i.e., $\hbar(l) = glb\{s : s \in Q \text{ and } l \in O_s\} > q$. Then $l \notin O_q$. Now since $p < q$, $\alpha g_\gamma cl(O_p) \subset O_q$. Hence $l \notin \alpha g_\gamma cl(O_p)$. This implies $l \in \mathfrak{L} \setminus \alpha g_\gamma cl(O_p)$ where $p > a$. Then $l \in \cup\{\mathfrak{L} \setminus \alpha g_\gamma cl(O_p) : p > a\} = H_1$(say). Hence $\hbar^{-1}((a, 1]) \subset \cup\{\mathfrak{L} \setminus \alpha g_\gamma cl(O_p) : p > a\} = H_1$. Conversely, let $l \in \cup\{\mathfrak{L} \setminus \alpha g_\gamma cl(O_p) : p > a\} = H_1$, which is αg_γ-open in $\mathfrak{L}$. Then l belongs to some αg_γ-open set $\mathfrak{L} \setminus \alpha g_\gamma cl(O_s)$ for some $s \in Q$ with $s > a$. Implies that $l \notin \alpha g_\gamma cl(O_s)$. Now $O_t \subset O_s$ for $t < s$. Then $O_t \subset \alpha g_\gamma cl(O_t) \subset O_s \subset \alpha g_\gamma cl(O_s)$. As $l \notin \alpha g_\gamma cl(O_s), l \notin O_t$. Then $\hbar(l) = glb\{t : t \in Q \text{ and } l \in O_t\} \geq s > a$. Hence $l \in \hbar^{-1}((a, 1])$. Thus $H_1 \subset \hbar^{-1}((a, 1])$. Therefore $H_1 = \hbar^{-1}((a, 1])$.

Next let $l \in \hbar^{-1}([0, b))$. Then $0 \leq \hbar(l) < b$. By the definition of Q, $\exists$ some $p \in Q \ni \hbar(l) < p < b$. i.e., $\hbar(l) = glb\{t : t \in Q \text{ and } l \in O_t\} < p < b$. Then $l \in O_p$ for $p < b$. Implies that $l \in \cup\{O_p : p < b\} = H_2$(say). Thus $\hbar^{-1}([0, b)) \subset H_2$. Conversely, let $l \in H_2$. Then l belongs to some αg_γ-open set O_s for some $s \in Q$ with $s < b$. Therefore $\hbar(l) = glb\{t : t \in Q \text{ and } l \in O_t\} \leq s < b$. Thus $l \in \hbar^{-1}([0, b))$. This implies $H_2 \subset \hbar^{-1}([0, b))$. Therefore $H_2 = \hbar^{-1}([0, b))$. Hence $\hbar$ is αg_γ-δ-continuous.

(Sufficient Part:) Consider $(\mathfrak{L}, \varsigma, \gamma)$ to be a topological space. Consider two disjoint αg_γ-closed sets J and G. By the hypothesis, $\exists$ an αg_γ-δ-continuous map $\hbar : \mathfrak{L} \to [0, 1] \ni \hbar(J) = \{0\}$ and $\hbar(G) = \{1\}$. Now the disjoint αg_γ-open sets $H_1 = \hbar^{-1}([0, 1/2))$ and $H_2 = \hbar^{-1}((1/2, 1])$ contains J and G respectively. Hence $\mathfrak{L}$ is an αg-γ-normal space. $\square$

7. Conclusion

In this article, various continuities and homeomorphism via operation approach on αg-open sets were studied. Also, we investigated Urysohn's lemma for αg-γ-normal space. This study can be extended to irresolute maps in near future.

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