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ON PROPERTY (BW1)

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Abstract. In this paper we study property (Bw1) defined in [18]. We establish for a bounded linear operator defined on a Banach space the necessary and sufficient conditions for which property (Bw1) holds. We discuss the property for operators satisfying the single valued extension property (SVEP). We also study the preservation of property under perturbations by finite rank and nilpotent operators. Certain conditions are explored on Hilbert space operators T and S so that $T \oplus S$ obeys property (Bw1).

1. Introduction and Preliminaries

Let $B(X)$ denote the algebra of all bounded linear operators on an infinite-dimensional complex Banach space X. For $T \in B(X)$, we denote by T^*, $\sigma(T)$, $\sigma_{iso}(T)$, $N(T)$ and $R(T)$ the adjoint, the spectrum, the isolated points of $\sigma(T)$, the null space and the range space of T, respectively. Let us denote by $\propto (T)$ the dimension of the kernel $N(T)$ and by $\beta(T)$ the codimension of the range $R(T)$. Recall that the operator $T \in B(X)$ is said to be an upper semi-Fredholm, if the range $R(T)$ of T is closed and $\propto (T) < \infty$, while $T \in B(X)$ is said to be lower semi-Fredholm if $\beta(T) < \infty$.

An operator $T \in B(X)$ is said to be semi-Fredholm if T is either an upper or a lower semi-Fredholm and Fredholm operator if it is both upper and lower semi-Fredholm. If T is semi-Fredholm, then the index of T is defined by

$$ind(T) = \propto (T) - \beta(T).$$

Let $p(T) := asc(T)$ be the ascent of an operator T i.e., the smallest nonnegative integer n such that $N(T^n) = N(T^{n+1})$. If such an integer does not exist we put $asc(T) = \infty$. Analogously, let $q(T) := dsc(T)$ be the descent of an operator T i.e. the smallest non negative integer such that $R(T^n) = R(T^{n+1})$ and if such an integer does not exist we put $dsc(T) = \infty$. It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$.

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An operator $T \in B(X)$ is called a Weyl if it is a Fredholm operator of index 0 and a Browder if T is Fredholm operator of finite ascent and descent. The Weyl spectrum $\sigma_W(T)$ and Browder spectrum $\sigma_b(T)$ of T are defined by

$$\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\},$$
$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\}.$$

Let $E_0(T) = \{\lambda \in iso\, \sigma(T) : 0 < \alpha(T - \lambda I) < \infty\}$. Following Coburn [8], we say Weyl's theorem holds for $T \in B(X)$ if $\sigma(T) \setminus \sigma_W(T) = E_0(T)$, and that Browder's theorem holds for T if $\sigma(T) \setminus \sigma_W(T) = \pi_0(T)$ where $\pi_0(T)$ is set of poles of finite rank. Here and elsewhere in this paper, for $A \subset \mathbb{C}$, $iso\, A$ denotes the set of all isolated points of A.

For a bounded linear operator $T \in B(X)$ and a non-negative integer n, we define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into itself (in particular $(T_0 = T)$). If for some integer n, the range space $R(T^n)$ is closed and T_n is an upper (resp., a lower) semi-Fredholm operator, then T is called upper (resp., a lower) semi B-Fredholm operator. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator. From [7, Proposition 2.1] if T_n is a semi-Fredholm operator then T_m is also a semi-Fredholm operator for each $m \geq n$ and $ind(T_m) = ind(T_n)$. Thus, the index of a semi-B-fredholm operator T is defined as the index of the semi-Fredholm operator T_n (see [6, 7]).

An operator $T \in B(X)$ is called a B-Weyl operator if it is a B-Fredholm operator of index 0.

The B-Weyl spectrum $\sigma_{BW}(T)$ of T is defined as

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ if not B-Weyl operator}\}$$

Given $T \in B(X)$, we say that generalized Weyl's theorem holds for T if

$$\sigma(T) \setminus \sigma_{BW}(T) = E(T),$$

where $E(T)$ is the set of isolated eigen values of T, and that generalized Browder's theorem holds for T if $\sigma(T) \setminus \sigma_{BW}(T) = \pi(T)$, where $\pi(T)$ is the set of poles of T [6, Definition 2.13]. Generalized Weyl's theorem and its variations have been studied in [2, 3, 4, 5, 6, 7, 16, 17] and the references therein. Berkani and Koliha [6] proved that generalized Weyl's theorem $\Rightarrow$ Weyl's theorem.

The single valued extension property was introduced by Dunford ([10], [11]) and it plays an important role in local spectral theory and Fredholm theory ([1], [14]).

The operator $T \in B(X)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at $\lambda_0 \in \mathbb{C}$), if for every open disc U of λ_0 the only analytic function $f : U \to X$ which satisfies the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in U$, is the function $f \equiv 0$.

An operator $T \in B(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$. An operator $T \in B(X)$ has SVEP at every point of the resolvent $\rho(T) = \mathbb{C} \setminus \sigma(T)$. Every operator T has SVEP at an isolated point of the spectrum. Duggal [9] gave the following important result:

Theorem 1.1 ([9, Proposition 3.9]). *The following statements are equivalent.*

(i) *T satisfies generalized Browder's theorem*
(ii) *T has SVEP at points $\lambda \notin \sigma_{BW}(T)$.*

Remark 1.2. Duggal [9] proved that T^* satisfies generalized Browder's theorem if and only if T satisfies generalized Browder's theorem as $\sigma(T) = \sigma(T^*)$, $\sigma_{BW}(T) = \sigma_{BW}(T^*)$ and $\pi(T) = \pi(T^*)$.

2. Property (Bw1)

Property (Bw) and property (Bw1) have been defined in [13] and [18] respectively as follows:

Definition 2.1 ([13, Definition 2.1])**.** A bounded linear operator $T \in B(X)$ is said to satisfy property (Bw) if $\sigma(T) \setminus \sigma_{BW}(T) = E_0(T)$.

Definition 2.2 ([18, Definition 2.7])**.** A bounded linear operator $T \in B(X)$ is said to satisfy property (Bw1) if

$$\sigma(T) \setminus \sigma_{BW}(T) \subseteq E_0(T).$$

We now study property (Bw1) for bounded linear operators as a variant of Weyl's theorem. We establish necessary and sufficient conditions for which this property holds. We prove that T satisfies property (Bw1) if and only if generalized Browder's theorem holds for T and $\pi(T) \subseteq E_0(T)$.

The following example shows that property (Bw1) does not imply property (Bw) in general.

Example 2.3. If $T \in l^2(N)$ is defined by

$$T(x_0, x_1, \ldots) = \left(\frac{1}{2}x_1, \frac{1}{3}x_2, \ldots\right) \text{ for all } (x_n) \in l^2(N)$$

then T is quasinilpotent. Since $\sigma(T) = \sigma_{BW}(T) = \{0\}$, and $E_0(T) = \{0\}$, then T does not satisfy property (Bw), but T satisfies property (Bw1).

Theorem 2.4. *Property (Bw) holds for T if and only if T satisfies property (Bw1) and*

$$\sigma_{BW}(T) \cap E_0(T) = \phi.$$

Proof. Suppose that T satisfies property (Bw), then property (Bw1) holds for T and $\sigma_{BW}(T) \cap E_0(T) = \phi$. For the converse, if $\lambda \in E_0(T)$, $\lambda \notin \sigma_{BW}(T)$ since $\sigma_{BW}(T) \cap E_0(T) = \phi$. Thus $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Hence $E_0(T) \subseteq \sigma(T) \setminus \sigma_{BW}(T)$. □

Theorem 2.5. *If $T \in B(X)$ satisfies property (Bw1). Then generalized Browder's theorem holds for T and $\sigma(T) = \sigma_{BW}(T) \cup E_0(T)$.*

Proof. By Theorem 1.1, it is sufficient to prove that T has SVEP at every $\lambda \notin \sigma_{BW}(T)$. Let us assume that $\lambda \notin \sigma_{BW}(T)$.

Case (i): If $\lambda \notin \sigma(T)$ then T has SVEP at λ.

Case (ii): If $\lambda \in \sigma(T)$ and suppose that T satisfies property (Bw1) then $\lambda \in \sigma(T) \setminus \sigma_{BW}(T) \subseteq E_0(T)$ and hence $\lambda \in iso\,\sigma(T)$, so also in this case T has SVEP at λ. To prove that $\sigma(T) = \sigma_{BW}(T) \cup E_0(T)$, we observe that $\sigma_{BW}(T) \cup E_0(T) \subseteq \sigma(T)$ for every $T \in B(X)$. For the reverse inclusion, consider $\lambda \in \sigma(T)$. If $\lambda \notin \sigma_{BW}(T)$ then $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. As T satisfies property (Bw1), therefore $\lambda \in E_0(T)$. Hence $\lambda \in E_0(T)$, so $\sigma(T) \subseteq \sigma_{BW}(T) \cup E_0(T)$. Thus $\sigma(T) = \sigma_{BW}(T) \cup E_0(T)$. □

Now we give a characterization of property (Bw1):

Theorem 2.6. *If $T \in B(X)$, then the following statements are equivalent:*
(i) *T satisfies property (Bw1),*
(ii) *generalized Browder's theorem holds for T and $\pi(T) \subseteq E_0(T)$.*

Proof. (i)⇒(ii): Assume that T satisfies property (Bw1). By Theorem 2.5 it is sufficient to prove that $\pi(T) \subseteq E_0(T)$. Let $\lambda \in \pi(T) = \sigma(T) \setminus \sigma_{BW}(T) \subseteq E_0(T)$. (ii)⇒(i): If $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Then generalized Browder's theorem implies that $\lambda \in \pi(T) \subseteq E_0(T)$. Thus $\sigma(T) \setminus \sigma_{BW}(T) \subseteq E_0(T)$. □

Theorem 2.7. *Let $T \in B(X)$. If T or T^* has SVEP at $\sigma(T) \setminus \sigma_{BW}(T)$, then T satisfies property (Bw1) if and only if $\pi(T) \subseteq E_0(T)$.*

Proof. The hypothesis T or T^* has SVEP at $\sigma(T) \setminus \sigma_{BW}(T) = \sigma(T^*) \setminus \sigma_{BW}(T^*)$ implies that T satisfies generalized Browder's theorem (see Theorem 1.1 and Remark 1.2).

Hence if $\pi(T) \subseteq E_0(T)$ then $\sigma(T) \setminus \sigma_{BW}(T) = \pi(T) \subseteq E_0(T)$. □

Definition 2.8. Operators $S, T \in B(X)$ are said to be injectively interwined, denoted, $S \prec_i T$, if there exists an injection $U \in B(X)$ such that $TU = US$.
If $S \prec_i T$, then T has SVEP at a point λ implies S has SVEP at λ.
To see this, let T has SVEP at λ, let U be an open neighbourhood of λ and let $f : U \to X$ be an analytic function such that $(S - \mu)f(\mu) = 0$ for every $\mu \in U$.
Then $U(S - \mu)f(\mu) = (T - \mu)Uf(\mu) = 0 \Rightarrow Uf(\mu) = 0$. Since U is injective, $f(\mu) = 0$, i.e. S has SVEP at λ.

Theorem 2.9. *Let $S, T \in B(X)$. If T has SVEP and $S \prec_i T$, then property (Bw1) holds for S if and only if $\pi(S) \subseteq E_0(S)$.*

Proof. Suppose that T has SVEP. Since $S \prec_i T$, therefore S has SVEP. Hence the result follows from Theorem 2.7. □

Definition 2.10. An operator $T \in B(X)$ is said to be polaroid if all the isolated points of its spectrum are poles $\sigma_{iso}(T) \subseteq \pi(T)$.

Theorem 2.11. *Let $T \in B(X)$ be a polaroid operator and satisfy property (Bw1), then generalized Weyl's theorem holds for T.*

Proof. T is a polaroid and satisfies property (Bw1) if and only if $\sigma(T) \setminus \sigma_{BW}(T) \subseteq E_0(T) \subseteq E(T) = \pi(T) = \sigma(T) \setminus \sigma_{BW}(T)$ (since T satisfies generalized Browder's theorem by Theorem 2.6. □

3. Property (Bw1) and Perturbations

In this section we study the preservation of property (Bw1) under perturbations by finite rank and nilpotent operators. We first discuss property (Bw) under perturbations by finite rank and nilpotent operators using the following:

Definition 3.1 ([12, Definition 2.2])**.** If $T \in B(X)$, and let $s \in \mathbb{N}$. Then T has uniform descent if $R(T) + N(T^n) = R(T) + N(T^s)$ for all $n \geq s$. If in addition $R(T) + N(T^s)$ is closed then T is said to have topological uniform descent for $n \geq s$.

Remark 3.2. Recall from [4] that an operator T is Drazin invertible if it has finite ascent and descent. The Drazin spectrum $\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not Drazin invertible$\}$. We observe $\sigma_D(T) = \sigma(T) \setminus \pi(T)$.

Theorem 3.3 ([13, Theorem 2.4])**.** *If $T \in B(X)$, then the following statements are equivalent:*

(i) *T satisfies property (Bw),*
(ii) *generalized Browder's theorem holds for T and $\pi(T) = E_0(T)$.*

Theorem 3.4. *Let $T \in B(X)$. If T has property (Bw) and F is a finite rank operator in $B(X)$ that commutes with T, then $T + F$ has property (Bw) if and only if $\pi(T + F) = E_0(T + F)$.*

Proof. If $T + F$ has property (Bw) then $\pi(T + F) = E_0(T + F)$. Conversely, if $\pi(T + F) = E_0(T + F)$. F is a finite rank operator $\sigma_{BW}(T) = \sigma_{BW}(T + F)$ and $\sigma_D(T) = \sigma_D(T+F)$. As T satisfies generalized Browder's theorem, therefore $\sigma_{BW}(T) = \sigma_D(T)$. Now $\sigma(T+F)\setminus\sigma_{BW}(T+F) = \sigma(T+F)\setminus\sigma_D(T+F) = \pi(T+F) = E_0(T+F)$. Therefore, $T + F$ satisfies property (Bw). □

Theorem 3.5. *Let $T \in B(X)$ and let N be a finite rank nilpotent operator commuting with T. If T satisfies property (Bw), then $T + N$ also satisfies property (Bw).*

Proof. If $\lambda \notin \sigma_{BW}(T)$, then $T - \lambda I$ is a B-Fredholm operator of index 0. From [4, Proposition 3.3], since N is a finite rank operator, it follows $T + N - \lambda I$ is also a B-Fredholm operator of index 0. So $\lambda \notin \sigma_{BW}(T + N)$. By symmetry $\sigma_{BW}(T + N) = \sigma_{BW}(T)$.

Now $E_0(T + N) = E_0(T)$ [15]. Let $0 \in E_0(T)$ so that $N(T)$ is finite dimensional. Let $(T + N)x = 0$ for some $x \neq 0$. Then $Tx = -Nx$. Since N commutes with T, it follows that for every positive integer

$$(1) \quad T^m x = (-1)^m N^m x.$$

Let n be the smallest positive integer such that $N^n = 0$. The relation (1) shows that for some r with $1 \leq r \leq n$, $T^r x = 0$ and then $T^{r-1}x \in N(T)$. Thus

$$N(T + N) \subset N(T^{n-1}).$$

Therefore, $N(T + N)$ is finite dimensional. Also, if for some $(x \neq 0)T^x = 0$ then $(T + N)^n x = 0$ so that 0 is an eigen value of $T + N$. Again, since $\sigma(T + N) = \sigma(T)$ it follows that $0 \in E_0(T + N)$. By symmetry $0 \in E_0(T + N)$ implies $0 \in E_0(T)$.

Thus, we have

$$\begin{aligned} \sigma(T + N) \setminus \sigma_{BW}(T + N) &= \sigma(T) \setminus \sigma_{BW}(T) \\ &= E_0(T) \qquad \text{(since property (Bw) holds for } T) \\ &= E_0(T + N). \end{aligned}$$

Therefore property (Bw) holds for $T + N$. □

Similar results can be deduced for property (Bw1).

Corollary 3.6. Let $T \in B(X)$. If T has property (Bw1) and F is a finite rank operator in $B(X)$ that commute with T, then $T + F$ has property (Bw1) if and only if $\pi(T + F) \subseteq E_0(T + F)$.

Corollary 3.7. Let $T \in B(X)$ and let N be a finite rank nilpotent operator commuting with T. If T satisfies property (Bw1), then $T + N$ also satisfies property (Bw1).

4. Property (Bw1) for Direct Sums

Let H and K be infinite-dimensional Hilbert spaces. In this section we show that if T and S are two operators on H and K respectively and at least one of them satisfies property (Bw1), then their direct sum $T \oplus S$ obeys property (Bw1). We have also explored various conditions on T and S so that $T \oplus S$ satisfies property (Bw1).

Theorem 4.1. *Suppose that property (Bw1) holds for $T \in B(H)$ and $S \in B(K)$. If T and S are isoloid and $\sigma_{BW}(T \oplus S) = \sigma_{BW}(T) \cup \sigma_{BW}(S)$, then property (Bw1) holds for $T \oplus S$.*

Proof. We know $\sigma(T \oplus S) = \sigma(T) \cup \sigma(S)$ for any pair of operators.
If T and S are isoloid, then

$$E_0(T \oplus S) = [E_0(T) \cap \rho(S)] \cup [\rho(T) \cap E_0(S)] \cup [E_0(T) \cap E_0(S)]$$

where $\rho(\cdot) = \mathbb{C} \setminus \sigma(\cdot)$.
If property (Bw1) holds for T and S, then

$$\begin{aligned}&[\sigma(T) \cup \sigma(S)] \setminus [\sigma_{BW}(T) \cup \sigma_{BW}(S)] \\ &\quad \subseteq [E_0(T) \cap \rho(S)] \cup [\rho(T) \cap E_0(S)] \cup [E_0(T) \cap E_0(S)].\end{aligned}$$

Thus $\sigma(T \oplus S) \setminus \sigma_{BW}(T \oplus S) \subseteq [E_0(T \oplus S)$. Hence property (Bw1) holds for $T \oplus S$. $\square$

Theorem 4.2. *Suppose $T \in B(H)$ has no isolated point in its spectrum and $S \in B(K)$ satisfies property (Bw1). If $\sigma_{BW}(T \oplus S) = \sigma(T) \cup \sigma_{BW}(S)$, then property (Bw1) holds for $T \oplus S$.*

Proof. As $\sigma(T \oplus S) = \sigma(T) \cup \sigma(S)$ for any pair of operators, we have

$$\begin{aligned}\sigma(T \oplus S) \setminus \sigma_{BW}(T \oplus S) &= [\sigma(T) \cup \sigma(S)] \setminus [\sigma(T) \cup \sigma_{BW}(S)] \\ &= \sigma(S) \setminus [\sigma(T) \cup \sigma_{BW}(S)] \\ &= [\sigma(S) \setminus \sigma_{BW}(S)] \setminus \sigma(T) \\ &\subseteq E_0(S) \cap \rho(T)\end{aligned}$$

where $\rho(T) = \mathbb{C} \setminus \sigma(T)$.

Now $\sigma_{iso}(T)$ be the set of isolated points of $\sigma(T)$ and $\sigma_{iso}(T \oplus S)$ is the set of isolated points of $\sigma(T \oplus S) = \sigma(T) \cup \sigma(S)$. If $\sigma_{iso}(T) = \phi$ it implies that $\sigma(T) = \sigma_{acc}(T)$, where $\sigma_{acc}(T) = \sigma(T) \setminus \sigma_{iso}(T)$ is the set of all accumulation points of $\sigma(T)$. Thus, we have

$$\begin{aligned}\sigma_{iso}(T \oplus S) &= [\sigma_{iso}(T) \cup \sigma_{iso}(S)] \setminus [(\sigma_{iso}(T) \cap \sigma_{acc}(S)) \cup (\sigma_{acc}(T) \cap \sigma_{iso}(S))] \\ &= (\sigma_{iso}(T) \setminus \sigma_{acc}(S)) \cup (\sigma_{iso}(S) \setminus \sigma_{acc}(T)) \\ &= \sigma_{iso}(S) \setminus \sigma(T) \\ &= \sigma_{iso}(S) \cap \rho(T).\end{aligned}$$

Let $\sigma_P(T)$ denote the point spectrum of T and $\sigma_{PF}(T)$ denote the set of all eigen values of T of finite multiplicity.

We have that $\sigma_P(T\oplus S) = \sigma_P(T)\cup\sigma_P(S)$ and $\dim N(T\oplus S) = \dim N(T)+\dim N(S)$ for every pair of operators, so that

$$\sigma_{PF}(T\oplus S) = \{\lambda \in \sigma_{PF}(T)\cup\sigma_{PF}(S) : \dim N(\lambda I - T) + \dim N(\lambda I - S) < \infty\}$$

Therefore

$$\begin{aligned} E_0(T\oplus S) &= \sigma_{iso}(T\oplus S)\cap\sigma_{PF}(T\oplus S) \\ &= \sigma_{iso}(S)\cap\rho(T)\cap\sigma_{PF}(S) \\ &= E_0(S)\cap\rho(T). \end{aligned}$$

Thus $\sigma(T\oplus S)\setminus\sigma_{BW}(T\oplus S)\subseteq E_0(T\oplus S)$. Hence $T\oplus S$ satisfies property (Bw1). $\square$

Let $\sigma_1(T)$ denote the compliment of $\sigma_{BW}(T)$ in $\sigma(T)$ i.e. $\sigma_1(T) = \sigma(T)\setminus\sigma_{BW}(T)$. A straight forward application of Theorem 4.2 leads to the following corollaries.

Corollary 4.3. Suppose $T \in B(H)$ is such that $\sigma_{iso}(T) = \phi$ and $S \in B(K)$ satisfies property (Bw1) with $\sigma_{iso}(S)\cap\sigma_{PF}(S) = \phi$ and $\sigma_1(T\oplus S) = \phi$, then $T\oplus S$ satisfies property(Bw1).

Proof. Since S satisfies property (Bw1), therefore given condition $\sigma_{iso}(S)\cap\sigma_{PF}(S) = \phi$ implies that $\sigma(S) = \sigma_{BW}(S)$. Now $\sigma_1(T\oplus S) = \phi$ gives that $\sigma(T\oplus S) = \sigma_{BW}(T\oplus S) = \sigma(T)\cup\sigma_{BW}(S)$. Thus, from Theorem 4.2 we have that $T\oplus S$ satisfies property(Bw1). $\square$

Corollary 4.4. Suppose $T \in B(H)$ is such that $\sigma_1(T)\cup\sigma_{iso}(T) = \phi$ and $S \in B(K)$ satisfies property (Bw1). If $\sigma_{BW}(T\oplus S) = \sigma_{BW}(T)\cup\sigma_{BW}(S)$, then property (Bw1) holds for $T\oplus S$.

Theorem 4.5. *Suppose $T \in B(H)$ is an isoloid operator that satisfy property (Bw1), then $T\oplus S$ satisfies property (Bw1) whenever $S \in B(K)$ is a normal operator and satisfies property (Bw1).*

Proof. If $S \in B(K)$ is normal, then S (also, S^*) has SVEP, and $ind(S-\lambda) = 0$ for every λ such that $S-\lambda$ is B-Fredholm. Observe that $\lambda \notin \sigma_{BW}(T\oplus S) \Leftrightarrow T-\lambda$ and $S-\lambda$ are B-Fredholm and $ind(T-\lambda) + ind\, S-\lambda) = ind(T-\lambda) = 0$.
$\Leftrightarrow \lambda \notin \{\sigma(T)\setminus\sigma_{BW}(T)\}\cap\{\sigma(S)\setminus\sigma_{BW}(S)\}$. Hence $\sigma_{BW}(T\oplus S) = \sigma_{BW}(T)\cup\sigma_{BW}(S)$.

It is well known that the isolated points of the spectrum of a normal operator are simple poles of the resolvent of the operator (implies S is isoloid). Hence the result follows from Theorem 4.1. $\square$

References

[1] P. Aiena, *Fredholm and local spectral theory, with applications to multipliers*, Kluwer Academic Publishers, Dordrecht, 2004.

[2] M. Amouch, Weyl type theorems for operators satisfying the single valued extension property, J. Math. Anal. Appl., 326 (2007), 1476–1484.

[3] M. Berkani, B-Weyl spectrum and poles of the resolvent, J. Math. Anal. Appl., 272 (2002), 596–603.

[4] M. Berkani, Index of B-Fredholm operators and generalization of a Weyl theorem, Proc. Amer. Math. Soc., 130 (2002), 1717–1723.

[5] M. Berkani and A. Arroud, Generalized Weyl's theorem and hyponormal operators, J. Austral. Math. Soc., 76 (2004), 291–302.

[6] M. Berkani and J. J. Koliha, Weyl type theorems for bounded linear operators, Acta Sci. Math. (Szeged), 69 (2003), 359–376.
[7] M. Berkani and M. Sarih, On semi-B-Fredholm operators, Glasgow Math. J., 43 (2001), no. 3, 457–465.
[8] L. A. Coburn, Weyl's theorem for non-normal operators, Michigan Math. J., 13 (1966), 285–288.
[9] B.P. Duggal, Polaroid operators and generalized Browder-Weyl theorems, Math. Proc. R. Ir. Acad., 108 (2008), no. 2, 149–164.
[10] N. Dunford, Spectral theory I, Resolution of the identity, Pacific J. Math., 2 (1952), 559–614.
[11] N. Dunford, Spectral operators, Pacific J. Math., 4 (1954), 321–354.
[12] S. Grabiner, Uniform ascent and descent of bounded operators, J. Math. Soc. Japan, 34 (1982), 317–337.
[13] A. Gupta and N. Kashyap, Property (Bw) and Weyl type theorems, Bull. Math. Anal. Appl., 3 (2011), no. 1, 1–7.
[14] K. B. Laursen and M.M. Neumann, *An Introduction to Local Spectral Theory*, The Clarendon Press, Oxford University Press, New York, 2000.
[15] K. K. Oberai, On the Weyl Spectrum, II, Illinois J. Math., 21 (1977), 84–90.
[16] K. Ben Ouidren, A. Ouahab and H. Zarioh, On a class of (bz)-operators, Rend. Circ. Mat. Palermo (2), 72 (2023), no. 8, 4169–4177.
[17] K. Ben Ouidren and H. Zarioh, Extended Rakočevic property, Func. Annal. Approx. Comp., 13 (2021), no. 1, 27–34.
[18] H. Zariouh and F. Lombarkia, On Browder-type theorems, Acta Math. Univ. Comenian. (N.S.), 83 (2014), no. 2, 281–289.

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