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DIFFERENT CLASSES OF CONTINUITY IN CONE METRIC SPACES

SHALLU SHARMA† AND IQBAL KOUR

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Abstract. The different forms of continuity are discussed thoroughly on cone metric spaces. A new notion namely Lebesgue cone metric spaces is also introduced in this paper. We established certain equivalent conditions on Lebesgue cone metric spaces.

1. Introduction

Huang and Zhang [10] revived the concept of a cone metric, which had been known since the mid-twentieth century(see [9, 14, 17]). Cone metric space is an intriguing generalisation of metric space in which the set of real numbers is replaced with an ordered Banach space. In [10] the authors introduced completeness and discussed convergence in cone metric spaces. Cone metric spaces are generalisations of metric spaces that play an essential role in fixed point theory, computer science, and certain other areas of general topology (see, for instance, [1, 2, 3, 4, 5, 7, 8, 13, 16, 6]). Yaari et al. [18] established new notions of continuity between cone metric spaces, quasi cone metric spaces and vice-versa. In this paper we have generalized the notions of uniform continuity, almost uniform continuity, Cauchy continuity and other related notions from [11] and [12] from metric spaces to cone metric spaces. Uniform continuity in a cone metric space and TVS-cone metric space is discussed in [19] and [5], respectively. In this paper we have studied uniform continuity in two distinct cone metric spaces.

2. Preliminaries

Definition 2.1. [10] Consider a real Banach space E and P be a set contained in E. The set P is called a cone if

(i) $P \neq \{0\}, P \neq \phi$ and $Cl(P) = P$.

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† *Corresponding author*

(ii) $\alpha u + \beta v \in P \ \forall\ \alpha, \beta \in \mathbb{R}, \alpha, \beta \geq 0$ and $u, v \in P$.
(iii) $P \cap (-P) = \{0\}$.

For a cone P contained in E, the partial ordering $\leq$ with respect to P is defined as: $x \leq y$ if and only if $y - x \in P$. The partial ordering $x < y$ denotes that $x \leq y$, $x \neq y$ whereas $x \ll y$ denotes that $y - x \in int(P)$.

Definition 2.2. [10] To define a cone metric space let $X \neq \phi$ be a set, E be a real Banach space and define a map $d : X \times X \to E$ that satisfies:
(i) $d(x, y) \geq 0,\ \forall\ x, y \in X$.
(ii) $d(x, y) = 0$ if and only if $x = y$.
(iii) $d(x, y) = d(y, x),\ \forall\ x, y \in X$.
(iv) $d(x, y) \leq d(x, z) + d(z, y),\ \forall\ x, y, z \in X$.

Then (X, d) is said to be a cone metric space with a cone metric d.

Throughout this paper for a set $B \subseteq X$, where (X, d) is a cone metric space with cone metric d, closure of B and interior of B are denoted as $Cl(B)$ and $int(B)$ respectively. Also, diameter of B is denoted as $diam(B) = sup\{d(x, y) : x, y \in B\}$. E, E', E'' are Banach spaces associated with cone metric spaces $(X, d_X), (Y, d_Y)$ and (Z, d_Z) respectively.

Definition 2.3. [15] Consider the cone metric space (X, d). Let $x \in X, v \in E, v \gg 0$ and $B(x, v) = \{y \in X : d(x, y) \ll v\}$. Then the basis for the topology T_v on X is defined by the family $\mathcal{B} = \{B(x, v)\}$.

Definition 2.4. [10] A cone metric space (X, d) is said to be complete if every Cauchy sequence converges in X.

Definition 2.5. [16] Let (X, d) be a cone metric space. A set B contained in X is said to be compact if every open cover of B admits a finite subcover.

Definition 2.6. [10] Let (X, d) be a cone metric space. X is said to be sequentially compact cone metric space if there exists a subsequence $\{x_{n_j}\}$ of any sequence $\{x_n\}$ in X such that $\{x_{n_j}\}$ converges in X.

Definition 2.7. [16] Let (X, d) be a cone metric space and B be a set contained in X. The set B is said to be totally bounded if there are bounded sets $B_1, B_2, \ldots, B_l$ contained in X such that $B \subset \bigcup_{j=1}^{l} B_j$ and $d(x, y) \ll v$ whenever, $x, y \in B_j, 1 \leq j \leq l$.

Definition 2.8. [16] Consider a cone metric space (X, d). For a subset B of X an element $v \gg 0, v \in E$ is known as a Lebesgue element for the open cover $\mathcal{H} = \{H_j\}$ of B if for each subset A of B with $diam(A) < v$ there exists $H_{j_0} \in \mathcal{H}$ such that $A \subset H_{j_0}$.

Definition 2.9. [16] Consider a cone metric space (X, d) and a map $f : (X, d) \to (X, d)$. Then f is said to be continuous at $x \in X$, if for each open set V containing $f(x)$ there exists an open set U containing x such that $f(U) \subset V$.
The map f is said to be continuous if f is continuous at each $x \in X$.

Definition 2.10. [16] Let (X, d) be a cone metric space and $v \gg 0, v \in E$. A finite set $S = \{s_1, s_2, \ldots, s_n\}$ contained in X is said to be a v-net for B if there is an $s_{j_0} \in S$ such that $d(q, s_{j_0}) \ll v$ for every $q \in B$.

Theorem 2.11. [15] *Every cone metric space is T_2-space.*

3. Main Results

Lemma 3.1. *A cone metric space (X, d) is complete if every Cauchy sequence has a convergent subsequence.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in X. Then there exists a subsequence $\{x_{n_l}\}$ that converges to $x \in X$. Let $v \gg 0, v \in E$. Then there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) \ll \frac{v}{2}$, $\forall\, m \geq n \geq N$. Let $l \in \mathbb{N}$ such that $n_l \geq N$ and $d(x, x_{n_l}) \ll \frac{v}{2}$. Then for $n \geq n_l \geq N$, we have

$$\begin{array}{rcl} d(x, x_n) & \leq & d(x, x_{n_l}) + d(x_{n_l}, x_n) \\ & \ll & \dfrac{v}{2} + \dfrac{v}{2} \\ & = & v. \end{array}$$

This implies that $\{x_n\}$ is convergent to x. Hence X is complete metric space. □

Lemma 3.2. [16] *Let (X, d) be a cone metric space. Then every sequentially compact subset of X is compact.*

Lemma 3.3. [16] *Let (X, d) be a cone metric space and $B \subset X$. For each $v \gg 0, v \in E, B$ has a v-net if and only if B is totally bounded.*

Lemma 3.4. [16] *Let (X, d) be a cone metric space. Then for every $u_1 \gg 0$ and $u_2 \gg 0, u_1, u_2 \in E$ there exists $u \gg 0$ such that $u \ll u_2$ and $u \ll u_1$.*

Theorem 3.5. *A cone metric space (X, d) is compact if and only if it is complete and totally bounded.*

Proof. Suppose that X is compact. Then X is sequentially compact as X is a topological space. This implies that every Cauchy sequence has a convergent subsequence. Then from Lemma 3.1 we infer that X is complete. To prove that X is totally bounded let $v \gg 0, v \in E$ and $x_1 \in X$. If the complement of $B(x_1, v)$ in X is empty, then for $X, \{x_1\}$ will be a v-net. In this case X is totally bounded. Next we assume that the complement of $B(x_1, v)$ in X is a non-empty set containing the element x_2. Therefore, if the complement of $B(x_1, v) \cup B(x_2, v)$ in X is empty, then $\{x_1, x_2\}$ is a v-net for X. By induction, $\{x_1, x_2, x_3\}$ is a v-net for X. We shall show that this process ends after a limited number of steps. If this process continues, the set $\{x_1, x_2, \ldots\}$ will become infinite and $d(x_j, x_k) \geq v$. Hence there is no Cauchy subsequence, a contradiction. Conversely, suppose that X is totally bounded and complete. To prove that X is compact let $\{x_n\}$ be a sequence in X. Owing to the total boundedness of X, there exists a v-net in X for each $v \gg 0, v \in E$. Therefore, finite union of balls of radius v contain X and infinitely many elements of $\{x_n\}$ will be contained in at least one of these balls. Suppose $M_1 = \{x_1^1, x_2^1, \ldots, x_n^1, \ldots\}$ is a subset of $\{x_n\}$ so that for some $b_1 \in X, B(b_1, v_1)$ contains M_1. Further, using Lemma 3.4 let $v_2 \ll v_1$. Then union of finite number of balls of radius v_2 contains M_1. Also, infinite number of elements of M_1 are contained in one of the balls of radius v_2. Take a subset $M_2 = \{x_1^2, x_2^2, \ldots, x_n^2, \ldots\}$ of M_1 so that for some $b_2 \in X, B(b_2, v_2)$ contains M_2. Proceeding similarly we get a subsequence $M_l = \{x_1^l, x_2^l, \ldots,\}$ of M_{l-1} so that $B(b_l, v_l), v_l \gg 0, v_l \in E$ contains M_l. Let $M = \{x_1^1, x_2^2, \ldots\} = \{x_l^l\}$ be a subsequence of $\{x_n\}$. We assert that $M = \{x_l^l\}$is convergent. For this, let $p, q \in \mathbb{N}$ such that $p \geq q \geq l$. Then $B(b_l, v_l)$ contains the elements x_p^k, x_q^k. Therefore, $d(x_p^k, x_q^k) \ll v_l, \ \forall \ p \geq q \geq l$. Therefore, $\{x_l^l\}$ is a Cauchy sequence in X. Hence $\{x_l^l\}$ is convergent in X since X is complete. As a result each sequence in X has a convergent subsequence. Hence the proof. $\square$

Theorem 3.6. *Let (X, d_X) and (Y, d_Y) two cone metric spaces. Then f is a continuous map from X to Y if and only if for each $x \in X$, and each $v \gg 0, v \in E'$ there exists $w \gg 0, w \in E$ such that*

$$d_Y(f(x), f(y)) \ll v, \ \text{whenever } d_X(x, y) \ll w.$$

Proof. The proof is trivial hence omitted.

$\square$

Definition 3.7. Let (X, d) be a cone metric space and $A \subset X$ be any set. A point $x \in X$ is said to be sequential accumulation point of A if there exists a sequence $\{x_n\}$ in $A \setminus \{x\}$ such that $\{x_n\}$ is convergent to x.

Definition 3.8. Let (X, d) be a cone metric space and $A \subset X$ be any set. A is said to be sequentially countably compact if there is at least one sequential accumulation point for any infinite subset of A.

Theorem 3.9. *Let (X, d) be a cone metric space and $A \subset X$. Then A is sequentially compact if and only if A is sequentially countably compact.*

Proof. Suppose that the set $A \subset X$ is sequentially countably compact and $H \subset A$ be an infinite set. Consider a sequence $\{x_n\}$ in H such that the points of sequence $\{x_n\}$ are distinct. Then $\{x_n\}$ has a subsequence $\{x_{n_p}\}$ that is convergent to a point x as A is sequentially countably compact. The limit point of H is then x. This proves the direct part. To prove the converse part let A be sequentially countably compact. Consider a sequence $\{x_n\}$ in A. Let $M = \{x_n : n \in \mathbb{N}\}$. There is nothing to prove in case M is a finite set. Hence suppose that M is infinite. The set M has a sequential accumulation point x in A since A is sequentially countably compact. We can easily find a subsequence $\{x_{n_p}\}$ of $\{x_n\}$ such that $\{x_{n_p}\}$ converges to x. Hence the proof. $\square$

Theorem 3.10. *Continuous image of a sequentially countably compact subset of a cone metric space (X, d) is sequentially countably compact.*

Proof. Suppose that (X, d_X) and (Y, d_Y) are cone metric spaces and f be a continuous map from X to Y. Let B be a sequentially countably compact subset of X. Consider an infinite set $A \subset f(B)$ such that the set $A_n = \{a_n : n \in \mathbb{N}\}$ is contained in A. Then there is a set $B_n = \{b_n : n \in \mathbb{N}\}$ contained in B so that $f(b_n) = a_n$ for all $n \in \mathbb{N}$. Therefore, B_n has an accumulation point, say b_0 in B. That is, b_n is convergent to b_0. The continuity of f implies that $f(b_n)$ is convergent to $f(b_0)$ in $f(B)$. Hence $\{a_n\}$ converges to $f(b_0)$ in $f(B)$. As a result, $f(b_0)$ is sequential accumulation point of $\{b_n\}$ in $f(B)$. This proves that $f(B)$ is sequentially countably compact. $\square$

Definition 3.11. Let (X, d_X) and (Y, d_Y) be cone metric spaces and f be a function from X to Y. Then f is said to be uniformly continuous for every $v' \gg 0, v' \in E'$, there exists $u \gg 0, u \in E$ such that $d_Y(f(x_1), f(x_2)) \ll v'$, whenever $d_X(x_1, x_2) \ll u$.

Theorem 3.12. *(Uniform Continuity Theorem) Let (X, d_X) and (Y, d_Y) be cone metric spaces where (X, d_X) is compact. Then a continuous map $f : X \to Y$ is uniformly continuous.*

Proof. Let $u' \in E', \frac{u'}{2} \gg 0$. Then the family $\mathcal{B} = \{B(y, \frac{u'}{2}) : y \in Y\}$ is an open cover of Y. Consider the open covering $\mathcal{S}$ of X by open balls B' such that $f(B') \subset B(y, \frac{u'}{2})$. Let δ be a Lebesgue number of the open cover $\mathcal{S}$. Choose $x_1, x_2 \in X$ such that $d_X(x_1, x_2) \ll \delta, diam\{x_1, x_2\} < \delta$. The set $\{f(x_1), f(x_2)\}$ has its image in $B(y, \frac{u'}{2})$ as $\mathcal{B}$ is an open cover of Y. Hence $d_Y(f(x_1), f(x_2)) \ll u'$. This proves that f is uniformly continuous. □

Theorem 3.13. *Let (X, d_X) and (Y, d_Y) be cone metric spaces and f be a uniformly continuous function from X to Y. Then $\{f(x_n)\}$ is a Cauchy sequence in Y for any Cauchy sequence $\{x_n\}$ in X.*

Proof. Let $v' \gg 0, v' \in E'$. By uniform continuity of f there exists $u \gg 0, u \in E$ such that $d_X(x, y) \ll u$ implies that $d_Y(f(x), f(y)) \ll v'$. Since $\{x_n\}$ is a Cauchy sequence in X. Then there exists $M \in \mathbb{N}$ such that $d_X(x, y) \ll u, \ \forall \ m, n \geq M$. Therefore, we have $d_Y(f(x), f(y)) \ll v', \ \forall \ m, n \geq M$. This proves the theorem. □

Theorem 3.14. *Composition of two uniformly continuous functions is uniformly continuous.*

Proof. Let $(X, d_X), (Y, d_Y)$ and (Z, d_Z) be cone metric spaces. Let f and g be two uniformly continuous functions from (X, d_X) to (Y, d_Y) and (Y, d_Y) to (Z, d_Z) respectively. Then we shall show that $g \circ f$ is a uniformly continuous function from (X, d_X) to (Z, d_Z). By the uniform continuity of g for every $v'' \in E'', v'' \gg 0$ there exists $u' \in E', u' \gg 0$ such that $d_Y(f(x), f(y)) \ll u'$ implies that $d_Z((g(f(x)), (g(f(y)))) \ll v'', \ \forall \ f(x), f(y)$ in Y. Also, f is uniformly continuous. Then for above $u' \in E', u' \gg 0$ there exists $w \in E, w \gg 0$ so that $d_X(x, y) \ll w$ implies that $d_Y(f(x, y)) \ll u', \ \forall \ x, y \in X$. Therefore, for each $v'' \in E'', v'' \gg 0$, there exists $w \in E, w \gg 0$ such that $d_X(x, y) \ll w$ implies that $d_Z((g \circ f)(x), (g \circ f)(y))) \ll v''$. Hence the proof. □

Definition 3.15. [16] Let (X, d) be a cone metric space. Suppose that $\mathcal{H} = \{H_j\}, j \in \mathbb{N}$ is an open cover of a set $B \subset X$. An element $\epsilon \gg 0, \epsilon \in E$ is said to be a Lebesgue element of $\mathcal{H}$ for B if for every set $A \subset B, diam \ A < \epsilon, \exists H_{j_0} \in \mathcal{H}, 1 \leq j \leq m$ so that $A \subseteq H_{j_0}$.

Definition 3.16. Let (X, d) be a cone metric space. X is called Lebesgue if each open cover of X has a Lebesgue element.

Definition 3.17. Let (X, d) be a cone metric space. X is called countably Lebesgue if each countable open cover of X has a Lebesgue element $\epsilon \gg 0, \epsilon \in E$.

Definition 3.18. Let (X, d) be a cone metric space. X is called weakly Lebesgue if each open cover of X has a weak Lebesgue element $\epsilon \gg 0, \epsilon \in E$.

Definition 3.19. Let (X, d_X) and (Y, d_Y) be two cone metric spaces and consider the mapping $f : X \to Y$. The map f is said to be almost bounded if and only if for each $u' \in E', u' \gg 0$ there exists $u \in E, u \gg 0$ such that for each subset B of X such that $diam(B) < u$, there exists a finite set $A \subset Y$ that satisfies: For each $b \in B$ there exists $a \in A$ so that $B(a, u')$ contains $f(b)$.

Definition 3.20. Let (X, d_X) and (Y, d_Y) be two cone metric spaces and consider the mapping $f : X \to Y$. Then the map f is said to be almost uniformly continuous if f is continuous and almost bounded.

Definition 3.21. Let (X, d_X) and (Y, d_Y) be two cone metric spaces and consider the mapping $f : X \to Y$. Then the map f is said to be Cauchy continuous if $\{f(x_n)\}$ is a Cauchy sequence in Y for any Cauchy sequence $\{x_n\}$ in X.

Theorem 3.22. *Let (X, d_X) and (Y, d_Y) be two cone metric spaces and $f : X \to Y$ be a uniformly continuous function. Then f is Cauchy continuous.*

Proof. The proof trivially follows by the definition of Cauchy continuity and Theorem 3.13. □

Proposition 3.23. [16] Let (X, d) be a cone metric space and the set $B \subset X$ is sequentially compact. Then B is compact.

Proposition 3.24. [16] Let (X, d) be a cone metric space and the set B be a sequentially compact subset of X. Then B is totally bounded.

Proposition 3.25. [16] Let (X, d) be a cone metric space and $B \subset X$ be sequentially compact. Then each open cover $\mathcal{H} = \{H_j\}$ for B has a Lebesgue element.

Proposition 3.26. Let (X, d) be a cone metric space. X is Lebesgue and totally bounded if X is compact.

Proof. Owing to the fact that a cone metric space is a topological space and Proposition 3.23 we see that in a cone metric space both compactness and sequential compactness are equivalent. Hence the proof trivially follows from Proposition 3.24 and Proposition 3.25. □

Lemma 3.27. *A cone metric space (X, d) is countably Lebesgue if (X, d) is Lebesgue.*

Proof. Straightforward hence omitted. □

Definition 3.28. Let (X, d) be a cone metric space. Then X is said to be countably compact if every countable open cover of X admits a finite subcover.

Lemma 3.29. *Let (X, d) be a cone metric space. X is countably compact iff X is sequentially countably compact.*

Proof. The proof trivially follows since X is a T_1-space and in a T_1-space countable compactness and sequential countable compactness are equivalent. □

Theorem 3.30. *Let (X, d) be a cone metric space. Then the following conditions are equivalent:*

(i) *X is compact.*
(ii) *X is sequentially countably compact.*
(iii) *X is sequentially compact.*

Proof. $(i) \implies (ii)$
Suppose that X is compact. Let B be a subset of X. To prove that B is sequentially countably compact we have to show that B has an accumulation point if B is infinite. For this it is sufficient to prove the contrapositive. We shall show that if B has no

accumulation point, then B is finite. Consider the case when B has no accumulation point. Then B is closed. Moreover, for every $b \in B$ there exists a neighborhood of b, say V_b such that $V_b \cap B = \{b\}$. Therefore, union of the sets $X \setminus B$ and V_b forms an open covering of X. A finite subcover of this cover is also a covering of X since X is compact. Further, $(X \setminus B) \cap B = \phi$ and only one point of B is in V_b. Hence B is finite.
$(ii) \implies (iii)$
The proof follows from Theorem 3.9.
$(iii) \implies (i)$
The proof follows from Proposition 3.23. □

Theorem 3.31. *Let (X, d) be a cone metric space. Then the following conditions are equivalent:*

(i) *X is compact.*
(ii) *X is weakly Lebesgue and totally bounded.*
(iii) *X is weakly countably Lebesgue and totally bounded.*

Proof. Clearly, if X is Lebesgue, then X is weakly Lebesgue.
$(i) \implies (ii)$
Proof follows from Proposition 3.26.
$(ii) \implies (iii)$
Straightforward.
$(iii) \implies (i)$
Using Lemma 3.29 and Theorem 3.30 it is sufficient to show that X is countably compact. Choose an open cover $\mathcal{V}$ of X. Since X is weakly Lebesgue. Then there exists $u \in E, u \gg 0$ such that countable number of members of $\mathcal{V}$ cover $B(x, u)$ for every $x \in X$. Moreover, we have $X = \bigcup_{j=1}^{m} B(x_j, u), x_j \in X, 1 \leq j \leq m$ as X is totally bounded. Then countable number of members of $\mathcal{V}$ cover each $B(x_j, u)$. Therefore, the open cover $\mathcal{V}$ admits a finite subcover. This proves that X is compact. □

Proposition 3.32. Let (X, d_X) and (Y, d_Y) be two cone metric spaces. Consider a continuous function $f : X \to Y$ and a dense set $S \subset Y$. Then f is almost uniformly continuous iff for each $u' \gg 0, u' \in E'$ there exists $u \gg 0, u \in E$ such that for every set $B \subset X$ such that $diam(B) < u$, there exists a finite set $P \subset S$ that satisfies: For each $b \in B$ there exists $p \in P$ such that $B(p, u')$ contains $f(b)$.

Proof. Obviously the converse is true. To prove the direct part choose $u' \in E', u' \gg 0$. Then by hypothesis take $u \gg 0$ such that for each $B \subset X$ with $diam(B) < u$ there exists a finite set $A \subset Y$ and for each $b \in B$ there exists $a \in A$ such that $d_Y(f(b), a) \ll \frac{u'}{2}$. Since S is dense in X. Then for each $a \in A$ choose $p_b \in S$ such that $d_Y(a, p_a) \ll \frac{u'}{2}$ and let $S = \{p_a : a \in A\}$. It is clear that P is a finite set contained in S. Therefore,

$$\begin{aligned} d_Y(f(b), p_a) &\leq d_Y(f(b), a) + d_Y(a, p_a) \\ &\ll \frac{u'}{2} + \frac{u'}{2} \\ &= u'. \end{aligned}$$

Hence the proof. □

Proposition 3.33. Let (X, d_X) and (Y, d_Y) be two cone metric spaces and consider the mapping $f : X \to Y$. Then the following hold:

(i) f is almost-uniformly continuous if f is continuous.

(ii) If $f(X)$ is totally bounded then f is almost-bounded.

Proof. (i) Let us fix a map $f : X \to Y$ that is uniformly continuous. To show that f is almost uniformly continuous let $u' \gg 0, u' \in E'$ and $u \gg 0, u \in E$ such that $d_Y(f(x), f(y)) \ll u'$ whenever $d_X(x, y) \ll u$ for all $x, y \in X$. Let us fix a set B contained in X such that $diam(B) < u$. Choose $b \in B$ and $P = \{f(b)\}$. Then for each $x \in B$ we see that $d_X(x, b) \ll u$. Therefore, $d_Y(f(x), f(b)) \ll u'$ implies that $B(f(b), u')$ contains $f(x)$. This proves that f is uniformly continuous.

(ii) Straightforward hence omitted.

□

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(Shallu Sharma) Department of Mathematics, University of Jammu, J&K-180006, India

E-mail address: `shallujamwal09@gmail.com`

(Iqbal Kour) Department of Mathematics, University of Jammu, J&K-180006, India

E-mail address: `iqbalkour208@gmail.com`