

ON THE CONTINUITY OF LINEAR CANONICAL BESSEL WAVELET TRANSFORMATIONS

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Date of Receiving : 15. 03. 2023
 Date of Revision : 28. 07. 2023
 Date of Acceptance : 30. 07. 2023

Abstract. Due to the extra degrees of freedom and simple geometrical manifestation, the linear canonical transform (LCT) has being broadly employed across several disciplines of science and engineering including signal processing, optical and radar systems, electrical and communication systems, quantum physics etc. The main objective of this paper is to study the linear canonical Hankel transformation and the continuous canonical Bessel wavelet transformation and some of their basic properties. The continuous Canonical Bessel wavelet transformation, its inversion formula and Parseval's relation for the continuous Canonical Bessel wavelet transformation are also studied.

1. Introduction

The theory of linear canonical transformation (LCT) was motivated by the work of two different projects by Collins [6] on the field of paraxial optics, on the other hand, Moshinsky and Quesne [11] in the field of nuclear physics in early seventies. The LCT is a four parameter class of linear integral transformation for studying the behaviour of many useful transformations and system responses in physics and engineering in general. Therefore, LCT is found as a powerful mathematical tool in many fields of physics and engineering. In this correspondence, we have defined the continuous Canonical Bessel wavelet transformation and associated properties. A general class of LCT has been studied by [1, 14]. The conventional canonical transformation represents any affine linear transformation in the (x, y) plane and specified by a 2×2 unimodular matrix A (i.e. determinant is one). For the sake of brevity, we may write the matrix as $A = (a, b; c, d)$

2010 *Mathematics Subject Classification.* 33A40; 42C10; 65R10; 44A35.

Key words and phrases. Linear Canonical transformation, Hankel transformation, Linear Canonical Hankel transformation, canonical Bessel wavelet transformation, Bessel function.

Communicated by. Mawardi Bahri

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in the text and linear canonical transform of any $f \in L^2(\mathbb{R})$ with respect to the unimodular matrix $A = (a, b, c, d)$ is defined by

$$\mathcal{L}[f](x) = \begin{cases} \int_{\mathbb{R}} f(t) K^A(t, x) dt & b \neq 0 \\ \sqrt{d} \exp \frac{cdx^2}{2} f(dx) & b = 0. \end{cases}$$

where $K^A(t, x)$ is the kernel of linear canonical transform and is given by

$$K^A(t, x) = \frac{1}{\sqrt{2\pi\iota b}} \exp \left\{ \frac{\iota(at^2 - 2tx + dx^2)}{2b} \right\}, \quad b \neq 0$$

The linear canonical transform includes several known transforms as special cases. For example, for $A = (1, b, 0, 1)$, we obtain the Fresnel transform, for $A = (\cos \theta, \sin \theta, -\sin \theta, \cos \theta)$, the LCT boils down to the fractional Fourier transform whereas for $A = (0, 1, -1, 0)$, we obtain the classical Fourier transform. Moreover, Bi-lateral Laplace, Gauss-Weierstrass, and Bargmann transform are also special cases of LCT. Similarly, we can define a linear canonical Hankel transformation, fractional Hankel transformation. For more details we refer to [2, 3, 4, 5, 7, 10, 12, 13] and references therein.

Pathak and Dixit [11] introduced continuous and discrete Bessel wavelet transformations and studied their properties by exploiting the Hankel convolution of Haimo [8] and Hirschman [9]. Upadhyay et al. [15] studied the continuous Bessel wavelet transformation associated with the Hankel-Hausdorff operator.

Let $L^p(\mathbb{R})$ denote the measurable functions f on $\mathbb{R}$ such that the integral $\int_{\mathbb{R}} |f(x)|^p dx$ is a finite. Also let $L^\infty(\mathbb{R})$ be a collection of almost bounded functions endowed with norm

$$\|f\|_{L^p} = \begin{cases} (\int_{\mathbb{R}} |f(x)|^p dx)^{1/p}, & 1 \leq p < \infty \\ \text{esssup}_{x \in \mathbb{R}} |f(x)|, & p = \infty. \end{cases}$$

The Hankel transformation of H_μ of conventional function $f \in L^1(\mathbb{R}_+)$, $\mathbb{R}_+ = (0, \infty)$ is usually defined by:

$$\hat{f}(y) = (H_\mu f)(y) = \int_0^\infty (xy)^{1/2} J_\mu(xy) f(x) dx \quad x \in \mathbb{R}_+, \mu \geq -1/2,$$

and its inverse formula is given by

$$f(x) = (H_\mu \hat{f})(x) = \int_0^\infty (xy)^{1/2} J_\mu(xy) \hat{f}(y) dy \quad y \in \mathbb{R}_+.$$

where J_μ is the Bessel function of the first kind of order μ .

The Linear Canonical Hankel transform (LCHT) is the generalization of conventional Hankel transformation and it is characterized by a unimodal matrix $A = (a, b, c, d)$, where a, b, c and d are four real or complex parameters. The earliest work on canonical Hankel transform was published by Wolf and Bultheel et al. We define a one-dimensional linear canonical Hankel transformation (LCHT) as:

$$\hat{f}_\mu^A(y) = (H_\mu^A f)(y) = \int_0^\infty K_\mu^A(x, y) f(x) dx$$

where the kernel

$$K_\mu^A(x, y) = \frac{e^{-i\pi/2(1+\mu)}}{b} e^{i/2b(ax^2+dy^2)} (xy)^{1/2} J_\mu\left(\frac{xy}{b}\right)$$

The inversion formula of one dimensional LCHT is given by

$$f(x) = ((H_\mu^A)^{-1} \hat{f})(x) = \int_0^\infty \overline{K_\mu^A(x, y)} (H_\mu^A f)(y) dy$$

where

$$\overline{K_\mu^A(x, y)} = \frac{e^{i\pi/2(1+\mu)}}{b} e^{-i/2b(ax^2+dy^2)} (xy)^{1/2} J_\mu\left(\frac{xy}{b}\right)$$

From [7], wavelets are a family of functions constructed from translation and dilation of a single function ψ are called the mother wavelet defined by

$$\psi_{t,s} = \frac{1}{\sqrt{s}} \psi\left(\frac{x-t}{s}\right), \quad t \in \mathbb{R}, s > 0$$

where s is called the scaling parameter which measures the degree of compression or scale and t is a translation parameter which determines the time location of the wavelet. The linear canonical mother wavelet is defined as

$$\psi_{t,s}^A = \frac{1}{\sqrt{s}} \psi\left(\frac{x-t}{s}\right) e^{\frac{-i\alpha}{2b}(x^2-t^2)}$$

for each s, t and A as above. As per [12, 13] the canonical Hankel convolution of $f, \psi \in L^1(\mathbb{R}_+)$ is defined as:

$$(f *_A \psi)(x) = \frac{e^{\frac{-i\pi}{2}(1+\mu)}}{b} \int_0^\infty f(x) (\tau_x^A \psi)(y) e^{\frac{i\alpha}{2b} y^2} dy$$

where the canonical Hankel translation τ_x^A is given as:

$$\begin{aligned} (\tau_x^A \psi)(y) &= \psi^A(x, y) \\ &= \frac{e^{\frac{-i\pi}{2}(1+\mu)}}{b} \int_0^\infty \psi(z) D_\mu^A(x, y, z) e^{\frac{i\alpha}{2b} z^2} dz \end{aligned}$$

Where

$$\begin{aligned} D_\mu^A(x, y, z) &= \frac{e^{\frac{-i\pi}{2}(1+\mu)}}{b} \int_0^\infty (x\xi)^{\frac{1}{2}} J_\mu\left(\frac{x\xi}{b}\right) (y\xi)^{\frac{1}{2}} J_\mu\left(\frac{y\xi}{b}\right) (z\xi)^{\frac{1}{2}} \\ &\quad \times J_\mu\left(\frac{z\xi}{b}\right) e^{\frac{-i\alpha}{2b}(x^2+y^2+z^2)} \xi^{1+2\mu} d\xi. \end{aligned}$$

and

$$\frac{e^{\frac{-i\pi}{2}(1+\mu)}}{b} \int_0^\infty D_\mu^A(x, y, z) e^{\frac{i\alpha}{2b} z^2} z^{1+2\mu} dz = \frac{e^{\frac{-i\alpha}{2b}(x^2+y^2)}}{(2b)^\mu \Gamma(\mu+1)}. \quad (1.1)$$

Lemma 1.1. If $f \in L^2(\mathbb{R}_+)$, then

$$\left\| x^{-\mu-1/2} (\tau_x^A f)(y) \right\|_{L^2} = \frac{1}{|b^{\mu+1/2} 2^\mu \Gamma(\mu+1)|} \|f\|_{L^2}$$

Proof. Since

$$\begin{aligned} (\tau_x^A f)(y) &= f^A(x, y) \\ &= \frac{e^{-\frac{\iota\pi}{2}(1+\mu)}}{b} \int_0^\infty f(z) D_\mu^A(x, y, z) e^{\frac{\iota a}{2b} z^2} dz \end{aligned}$$

Using (1.1), we have

$$\begin{aligned} &|(\tau_x^A f)(y)| \\ &\leq \int_0^\infty \left| f(z) z^{-1/2(\mu+1/2)} \{D_\mu^A(x, y, z)\}^{1/2} z^{1/2(\mu+1/2)} \{D_\mu^A(x, y, z)\}^{1/2} \right| dz \\ &\leq \left(\int_0^\infty z^{-(\mu+1/2)} |f(z)|^2 |D_\mu^A(x, y, z)| dz \right)^{1/2} \\ &\quad \left(\int_0^\infty z^{(\mu+1/2)} |f(z)|^2 |D_\mu^A(x, y, z)| dz \right)^{1/2} \\ &\leq \left(\frac{(xy)^{\mu+1/2}}{|(b)^{\mu+1/2}| 2^\mu \Gamma(\mu+1)} \right)^{1/2} \\ &\quad \left(\int_0^\infty z^{-(\mu+1/2)} |f(z)|^2 |D_\mu^A(x, y, z)| dz \right)^{1/2} \end{aligned} \tag{1.2}$$

so that

$$\begin{aligned} &\int_0^\infty |(\tau_x^A f)(y)|^2 dy \\ &\leq \frac{x^{\mu+1/2}}{|(b)^{\mu+1/2}| 2^\mu \Gamma(\mu+1)} \int_0^\infty z^{-(\mu+1/2)} |f(z)|^2 dz \int_0^\infty |D_\mu^A(x, y, z)| y^{\mu+1/2} dy \\ &\leq \frac{x^{\mu+1/2}}{|(b)^{\mu+1/2}| 2^\mu \Gamma(\mu+1)} \int_0^\infty z^{-(\mu+1/2)} |f(z)|^2 \frac{zx^{\mu+1/2}}{|(b)^{\mu+1/2}| 2^\mu \Gamma(\mu+1)} dz \\ &= \frac{x^{2\mu+1/2}}{(|(b)^{\mu+1/2}| 2^\mu \Gamma(\mu+1))^2} \int_0^\infty |f(z)|^2 dz \end{aligned}$$

therefore

$$\left\| x^{-\mu-1/2} (\tau_x^A f)(y) \right\|_{L^2} = \frac{1}{|b^{\mu+1/2} 2^\mu \Gamma(\mu+1)|} \|f\|_{L^2}.$$

□

Remark 1.2. If $f \in L^2(\mathbb{R}_+)$, then

$$\int_0^\infty |(\tau_y^A f)(x)|^2 dx \leq \frac{y^{2\mu+1/2}}{(|(b)^{\mu+1/2}| 2^\mu \Gamma(\mu+1))^2} \int_0^\infty |f(z)|^2 dz,$$

and

$$\left\| y^{-\mu-1/2}(\tau_y^A f)(x) \right\|_{L^2} = \frac{1}{|b^{\mu+1/2} 2^\mu \Gamma(\mu+1)|} \|f\|_{L^2}.$$

Proposition 1.3. Let $(H_\mu^A f)(y)$ denote the canonical Hankel transformations of function f then

$$\int_0^\infty f(x) \overline{g(x)} dx = b \int_0^\infty (H_\mu^A f)(y) \overline{(H_\mu^A g)(y)} dy \quad (1.3)$$

and

$$\int_0^\infty |f(x)|^2 dx = b \int_0^\infty |(H_\mu^A f)(y)|^2 dy. \quad (1.4)$$

Proof. We have

$$\begin{aligned} & \langle f, g \rangle \\ &= \int_0^\infty f(x) \overline{g(x)} dx \\ &= \int_0^\infty f(x) \left(\left(\frac{e^{\frac{i\pi}{2}(1+\mu)}}{b} \right) b \int_0^\infty e^{\frac{-i}{2b}(ax^2+dy^2)} \left(\frac{xy}{b} \right)^{1/2} J_\mu \left(\frac{xy}{b} \right) (H_\mu^A g)(y) dy \right) \\ &= b \int_0^\infty \left(\frac{e^{\frac{-i\pi}{2}(1+\mu)}}{b} \right) \overline{(H_\mu^A g)(y)} \left(\int_0^\infty e^{\frac{i}{2b}(ax^2+dy^2)} \left(\frac{xy}{b} \right)^{1/2} J_\mu \left(\frac{xy}{b} \right) f(x) dx \right) dy \\ &= b \int_0^\infty (H_\mu^A f)(y) \overline{(H_\mu^A g)(y)} dy \end{aligned}$$

if $f = g$, then

$$\int_0^\infty |f(x)|^2 dx = b \int_0^\infty |(H_\mu^A f)(y)|^2 dy.$$

□

2. The continuous Linear Canonical Bessel wavelet transformation (CLCBWT)

The continuous linear canonical Bessel wavelet transformation (CLCBWT) is a generalization of the ordinary continuous Bessel wavelet transformation (CBWT). In this section, we define the continuous linear canonical Bessel wavelet transformation and study some of its properties using the theory of linear canonical Hankel convolution.

A Canonical Bessel wavelet is a function $\psi \in L^2(\mathbb{R})$ which satisfies the condition

$$\mathcal{C}_{\mu,\psi}^A = \int_0^\infty \frac{|(H_\mu^A \psi)(x)|^2}{x^{2(\mu-1)}} dx < \infty, \quad \mu \geq -1/2$$

where $\mathcal{C}_{\mu,\psi}^A$ is called the admissibility condition of the canonical Bessel wavelet and $(H_\mu^A \psi)$ is the canonical Bessel transformation of ψ . The canonical Bessel wavelets $\psi_{t,s}^A$,

are generated from one single function $\psi \in L^2(\mathbb{R}_+)$ by dilation and translation with parameters $s > 0$ and $t > 0$ respectively by

$$\begin{aligned}\psi_{t,s}^A(x) &= s^{-1/2} D_s \tau_t^A \psi(x) = s^{-1/2} D_s \psi^A(t, x) = s^{-1/2} \psi^A\left(\frac{t}{s}, \frac{x}{s}\right) \\ &= s^{-1/2} e^{\frac{-\iota a}{2b}(\frac{t^2}{s^2} + \frac{x^2}{s^2})} \int_0^\infty \psi(z) D_\mu^A\left(\frac{t}{s}, \frac{x}{s}, z\right) dz.\end{aligned}$$

Lemma 2.1. *If $\psi \in L^2(\mathbb{R}_+)$, then*

$$\|\psi_{t,s}^A\|_{L^2} = \frac{t^{(\mu+1/2)} s^{-(\mu+1/2)}}{|b^{\mu+1/2} 2^\mu \Gamma(\mu+1)|} \|\psi\|_{L^2}$$

Proof. Since

$$\psi_{t,s}^A = s^{-1/2} e^{\frac{-\iota a}{2b}(\frac{t^2}{s^2} + \frac{x^2}{s^2})} \int_0^\infty \psi(z) D_\mu^A\left(\frac{t}{s}, \frac{x}{s}, z\right) dz$$

Now,

$$\begin{aligned}|\psi_{t,s}^A| &\leq s^{-1/2} \int_0^\infty |\psi(z)| z^{-1/2(\mu+1/2)} \left| \left\{ D_\mu^A\left(\frac{t}{s}, \frac{x}{s}, z\right) \right\}^{1/2} \right| z^{1/2(\mu+1/2)} \\ &\quad \times \left| \left\{ D_\mu^A\left(\frac{t}{s}, \frac{x}{s}, z\right) \right\}^{1/2} \right| dz \\ &\leq s^{-1/2} \left(\int_0^\infty |\psi(z)|^2 z^{-(\mu+1/2)} \left| D_\mu^A\left(\frac{t}{s}, \frac{x}{s}, z\right) \right| dz \right)^{1/2} \\ &\quad \times \left(\int_0^\infty \left| D_\mu^A\left(\frac{t}{s}, \frac{x}{s}, z\right) \right| z^{(\mu+1/2)} dz \right)^{1/2} \\ &\leq s^{-1/2} \left(\frac{(tx)^{\mu+1/2}}{s^{2(\mu+1/2)} |b^{\mu+1/2} 2^\mu \Gamma(\mu+1)|} \right)^{1/2} \\ &\quad \times \left(\int_0^\infty |\psi(z)|^2 z^{-(\mu+1/2)} \left| D_\mu^A\left(\frac{t}{s}, \frac{x}{s}, z\right) \right| dz \right)^{1/2}.\end{aligned}$$

Thus

$$\begin{aligned}&\int_0^\infty |\psi_{t,s}^A(x)|^2 dx \\ &\leq \frac{t^{2(\mu+1/2)}}{s^{2\mu+2} |b^{\mu+1/2} 2^\mu \Gamma(\mu+1)|} \int_0^\infty z^{-(\mu+1/2)} |\psi(z)|^2 dz \int_0^\infty \left| D_\mu^A\left(\frac{t}{s}, \frac{x}{s}, z\right) \right| x^{\mu+1/2} dx \\ &\leq \frac{t^{2(\mu+1/2)}}{(s^{\mu+1/2} |b^{\mu+1/2} 2^\mu \Gamma(\mu+1)|)^2} \int_0^\infty |\psi(z)|^2 dz.\end{aligned}$$

Hence

$$\|\psi_{t,s}^A\|_{L^2} \leq \frac{t^{(\mu+1/2)} s^{-(\mu+1/2)}}{(|b^{\mu+1/2} 2^\mu \Gamma(\mu+1)|)^2} \|\psi\|_{L^2}.$$

□

Theorem 2.2. Let $f, \psi \in L^2(\mathbb{R}_+)$. Then the continuous linear canonical Bessel wavelet transform B_ψ^A is defined on f by

$$(B_\psi^A f)(t, s) = e^{-\iota\pi/2(1+\mu)} s^{-\mu} \int_0^\infty e^{\frac{-\iota a}{2b}(\frac{1}{s^2}-1)(sx)^2} x^{-\mu-1/2} \left(\frac{tx}{b}\right)^{1/2} \\ \times J_\mu\left(\frac{tx}{b}\right) (H_\mu^A e^{\frac{\iota a}{2b}(\cdot)^2} f)(x) \overline{(H_\mu^A \psi)(sx)} dx.$$

Proof. We have

$$\begin{aligned} & (B_\psi^A f)(t, s) \\ &= \langle f, \psi_{t,s}^A \rangle \\ &= \int_0^\infty f(\xi) \overline{\psi_{t,s}^A(\xi)} d\xi \\ &= \int_0^\infty f(\xi) \overline{\left(s^{-1/2} e^{\frac{-\iota a}{2b}(\frac{t^2}{s^2} + \frac{\xi^2}{s^2})} \int_0^\infty \psi(z) D_\mu^A\left(\frac{t}{s}, \frac{\xi}{s}, z\right) dz \right)} d\xi \\ &= \frac{b}{e^{\iota\pi/2(1+\mu)} s^{1/2}} \int_0^\infty \left(\frac{e^{\iota\pi/2(1+\mu)}}{b} \int_0^\infty e^{\frac{\iota a}{2b}(\xi^2 + \frac{\omega^2}{s^2})} \left(\frac{\xi\omega}{sb}\right)^{1/2} J_\mu\left(\frac{\xi\omega}{sb}\right) e^{\frac{-\iota a\xi^2}{2b}} f(\xi) d\xi \right) \\ & \quad \times e^{\frac{-\iota a\omega^2}{b}(\frac{1}{s^2}-1)} \omega^{(-\mu-1/2)} \left(\frac{t\omega}{sb}\right)^{1/2} J_\mu\left(\frac{t\omega}{sb}\right) \overline{(H_\mu^A \psi(z))(\omega)} d\omega \\ &= \frac{b}{e^{\iota\pi/2(1+\mu)} s^{1/2}} \int_0^\infty e^{\frac{-\iota a\omega^2}{b}(\frac{1}{s^2}-1)} \omega^{(-\mu-1/2)} \left(\frac{t\omega}{sb}\right)^{1/2} \\ & \quad \times J_\mu\left(\frac{t\omega}{sb}\right) (H_\mu^A e^{\frac{-\iota a}{2b}(\cdot)^2} f)\left(\frac{\omega}{s}\right) \overline{(H_\mu^A \psi)(\omega)} d\omega. \end{aligned}$$

By taking $\frac{\omega}{s} = x$ the continuous linear canonical Bessel wavelet transformation can be written as

$$(B_\psi^A f)(t, s) = e^{-\iota\pi/2(1+\mu)} s^{-\mu} \int_0^\infty e^{\frac{-\iota a}{2b}(\frac{1}{s^2}-1)(sx)^2} x^{-\mu-1/2} \left(\frac{tx}{b}\right)^{1/2} \\ \times J_\mu\left(\frac{tx}{b}\right) (H_\mu^A e^{\frac{\iota a}{2b}(\cdot)^2} f)(x) \overline{(H_\mu^A \psi)(sx)} dx.$$

This means

$$H_\mu^A \left\{ e^{\frac{-\iota a}{2b}t^2} (B_\psi^A f)(t, s) \right\} = bs^{-\mu} (x^{-\mu-1/2} e^{\frac{\iota a}{2b}ax^2} (H_\mu^A e^{-\iota\frac{a}{2b}(\cdot)^2} f)(x) \overline{(H_\mu^A \psi)(sx)}).$$

□

Theorem 2.3. If ψ_1 and ψ_2 are two wavelets and $(B_{\psi_1}^A f)(t, s)$ and $(B_{\psi_2}^A g)(t, s)$ denote the continuous linear canonical Bessel wavelet transformations of $f, g \in L^2(\mathbb{R}_+)$ respectively, then

$$\int_0^\infty (B_{\psi_1}^A f)(t, s) \overline{(B_{\psi_2}^A g)(t, s)} \frac{dt ds}{s^2} = b^2 \mathcal{C}_{\mu, \psi_1, \psi_2}^A \langle f, g \rangle,$$

where

$$\mathcal{C}_{\mu, \psi_1, \psi_2}^A = \int_0^\infty \frac{\overline{(H_\mu^A \psi_1)}(s)(H_\mu^A \psi_2)(s)}{s^{2(\mu+1)}} ds < \infty.$$

Proof. Since

$$\begin{aligned} (B_\psi^A f)(t, s) &= e^{-\iota\pi/2(1+\mu)} s^{-\mu} \int_0^\infty e^{\frac{-\iota a}{2b}(\frac{1}{s^2}-1)(sx)^2} x^{-\mu-1/2} \left(\frac{tx}{b}\right)^{1/2} \\ &\quad \times J_\mu\left(\frac{tx}{b}\right) (H_\mu^A e^{\frac{\iota a}{2b}(\cdot)^2} f)(x) \overline{(H_\mu^A \psi_1)}(sx) dx. \end{aligned}$$

Now

$$\begin{aligned} &\int_0^\infty (B_{\psi_1}^A f)(t, s) \overline{(B_{\psi_2}^A g)}(t, s) \frac{dt ds}{s^2} \\ &= \int_0^\infty \int_0^\infty \left[e^{-\iota\pi/2(1+\mu)} s^{-\mu} \int_0^\infty e^{\frac{-\iota a}{2b}(\frac{1}{s^2}-1)(sx)^2} x^{-\mu-1/2} \left(\frac{tx}{b}\right)^{1/2} \right. \\ &\quad \times J_\mu\left(\frac{tx}{b}\right) (H_\mu^A e^{\frac{\iota a}{2b}(\cdot)^2} f)(x) \overline{(H_\mu^A \psi_1)}(sx) dx \Big] \\ &\quad \times \overline{\left[e^{-\iota\pi/2(1+\mu)} s^{-\mu} \int_0^\infty e^{\frac{-\iota a}{2b}(\frac{1}{s^2}-1)(sx)^2} x^{-\mu-1/2} \left(\frac{tx}{b}\right)^{1/2} \right.} \\ &\quad \times J_\mu\left(\frac{ty}{b}\right) (H_\mu^A e^{\frac{\iota a}{2b}(\cdot)^2} f)(y) \overline{(H_\mu^A \psi_2)}(sy) dy \Big] dt \frac{ds}{s^2} \\ &= \int_0^\infty \int_0^\infty b^3 s^{-2(\mu+1)} e^{\frac{-\iota a}{2b}(2-s^2)x^2} x^{-\mu-1/2} (H_\mu^A e^{\frac{-\iota a}{2b}(\cdot)^2} f)(x) \overline{(H_\mu^A \psi_1)}(sx) \\ &\quad \times \left\{ \frac{e^{\iota\pi/2(1+\mu)}}{b} \int_0^\infty e^{\frac{\iota a}{2b}(t^2+x^2)} \left(\frac{tx}{b}\right)^{1/2} J_\mu\left(\frac{tx}{b}\right) \right. \\ &\quad \times \left(\frac{e^{-\iota\pi/2(1+\mu)}}{b} \int_0^\infty e^{\frac{-\iota a}{2b}(t^2+y^2)} \left(\frac{tx}{b}\right)^{1/2} J_\mu\left(\frac{ty}{b}\right) \right. \\ &\quad \times e^{\frac{-\iota a}{2b}(2-s^2)y^2} y^{-\mu-1/2} \overline{(H_\mu^A e^{\frac{-\iota a}{2b}(\cdot)^2} g)}(y) (H_\mu^A \psi_2)(sy) dy \Big) dt \Big\} dx ds \\ &= \int_0^\infty \int_0^\infty b^3 s^{-2(\mu+1)} e^{\frac{-\iota a}{2b}(2-s^2)x^2} x^{-\mu-1/2} (H_\mu^A e^{\frac{-\iota a}{2b}(\cdot)^2} f)(x) \overline{(H_\mu^A \psi_1)}(sx) \\ &\quad \times \left\{ \frac{e^{\iota\pi/2(1+\mu)}}{b} \int_0^\infty e^{\frac{\iota a}{2b}(t^2+x^2)} \left(\frac{tx}{b}\right)^{1/2} J_\mu\left(\frac{tx}{b}\right) (H_\mu^A)^{-1} \left(e^{\frac{-\iota a}{2b}(2-s^2)y^2} \right. \right. \\ &\quad \times y^{-\mu-1/2} \overline{(H_\mu^A e^{\frac{-\iota a}{2b}(\cdot)^2} g)}(y) (H_\mu^A \psi_2)(sy) \Big) (t) dt \Big\} dx ds. \end{aligned}$$

$$\begin{aligned}
&= b^3 \int_0^\infty \int_0^\infty s^{-2(\mu+1)} e^{\frac{-\iota a}{2b}(2-s^2)x^2} x^{-\mu-1/2} (H_\mu^A e^{\frac{-\iota a}{2b}(\cdot)^2} f)(x) \overline{(H_\mu^A \psi_1)(sx)} \\
&\quad \times (H_\mu^A)(H_\mu^A)^{-1} \left(e^{\frac{-\iota a}{2b}(2-s^2)y^2} y^{-\mu-1/2} \overline{(H_\mu^A e^{\frac{-\iota a}{2b}(\cdot)^2} g)}(y) (H_\mu^A \psi_2)(sy) \right) (x) dx ds \\
&= b^3 \int_0^\infty \int_0^\infty s^{-2(\mu+1)} x^{-2\mu-1} (H_\mu^A e^{\frac{-\iota a}{2b}(\cdot)^2} f)(x) \\
&\quad \times \overline{(H_\mu^A e^{\frac{-\iota a}{2b}(\cdot)^2} g)}(x) (H_\mu^A \psi_2)(sx) dx ds \\
&= b^3 \int_0^\infty (H_\mu^A e^{\frac{-\iota a}{2b}(\cdot)^2} f)(x) \overline{(H_\mu^A e^{\frac{-\iota a}{2b}(\cdot)^2} g)}(x) \\
&\quad \times \left(\int_0^\infty (sx)^{-2(\mu+1)} \overline{(H_\mu^A \psi_1)}(sx) (H_\mu^A \psi_2)(sx) x ds \right) dx \\
&= b^3 C_{\mu, \psi_1, \psi_2}^A \int_0^\infty (H_\mu^A f)(x) \overline{(H_\mu^A g)}(x) dx \\
&= b^3 C_{\mu, \psi_1, \psi_2}^A \langle H_\mu^A f, H_\mu^A g \rangle \\
&= b^2 C_{\mu, \psi_1, \psi_2}^A \langle f, g \rangle
\end{aligned}$$

Hence the proof is completed. $\square$

Theorem 2.4. If ψ is a wavelet and $(B_\psi^A f)(t, s)$ and $(B_\psi^A g)(t, s)$ are the continuous linear canonical Bessel wavelet transform of $f, g \in L^2(\mathbb{R}_+)$ respectively, Then

$$\int_0^\infty \int_0^\infty (B_\psi^A f)(t, s) \overline{(B_\psi^A g)}(t, s) \frac{dt ds}{s^2} = b^2 C_{\mu, \psi}^A \langle f, g \rangle.$$

Proof. If we take $\psi_1 = \psi_2 = \psi$ in Theorem 2.3 we get the proof of Theorem 2.6 $\square$

Remark 2.5. If $f = g$ and $\psi_1 = \psi_2 = \psi$ then from Theorem 2.6, we have

$$\int_0^\infty \int_0^\infty |(B_\psi^A f)(t, s)|^2 \frac{dt ds}{s^2} = b^2 C_{\mu, \psi}^A \|f\|_2^2$$

Theorem 2.6. Let $f \in L^2(\mathbb{R}_+)$. Then f can be reconstructed by the formula

$$f(\xi) = \frac{1}{b^2 C_{\mu, \psi}^A} \int_0^\infty \int_0^\infty (B_\psi^A f)(t, s) \psi_{t, s}^A(\xi) \frac{dt ds}{s^2}, \quad s > 0$$

Proof. For any $g \in L^2(\mathbb{R}_+)$, we have

$$\begin{aligned} b^2 C_{\mu, \psi}^A \langle f, g \rangle &= \int_0^\infty \int_0^\infty (B_\psi^A f)(t, s) \overline{(B_\psi^A g)(t, s)} \frac{dt ds}{s^2} \\ &= \int_0^\infty \int_0^\infty (B_\psi^A f)(t, s) \overline{\left(\int_0^\infty g(\xi) \overline{\psi_{t,s}^A(\xi)} d\xi \right)} \frac{dt ds}{s^2} \\ &= \int_0^\infty \left[\int_0^\infty \int_0^\infty (B_\psi^A f)(t, s) \psi_{t,s}^A(\xi) \frac{dt ds}{s^2} \right] \overline{g(\xi)} d\xi \\ &= \left\langle \int_0^\infty \int_0^\infty (B_\psi^A f)(t, s) \psi_{t,s}^A(\xi) \frac{dt ds}{s^2}, g(\xi) \right\rangle. \end{aligned}$$

Therefore,

$$f(\xi) = \frac{1}{b^2 C_{\mu, \psi}^A} \int_0^\infty \int_0^\infty (B_\psi^A f)(t, s) \psi_{t,s}^A(\xi) \frac{dt ds}{s^2}.$$

□

Theorem 2.7. If $\psi \in L^2(\mathbb{R}_+)$, then

$$\int_0^\infty \left[(B_\psi^A f)(t, s) \overline{(B_\psi^A g)(t, s)} \right] dt = b^2 s^{-2\mu} \langle \mathcal{P}, Q \rangle,$$

where

$$\mathcal{P}(x) = e^{\frac{-\iota a}{2b}(2-s^2)x^2} x^{-\mu-1/2} (H_\mu^A e^{\frac{-\iota a}{2b}(\cdot)^2} f)(x) \overline{(H_\mu^A \psi)(sx)},$$

$$\mathcal{Q}(x) = e^{\frac{-\iota a}{2b}(2-s^2)y^2} y^{-\mu-1/2} (H_\mu^A e^{\frac{-\iota a}{2b}(\cdot)^2} \bar{g})(y) \overline{(H_\mu^A \psi)(sy)}.$$

Proof. Using Theorem 2.2 and Theorem 2.7, we have

$$\begin{aligned} &\int_0^\infty \left[(B_\psi^A f)(t, s) \overline{(B_\psi^A g)(t, s)} \right] dt \\ &= \int_0^\infty \langle f, \psi_{t,s}^A \rangle \overline{\langle g, \psi_{t,s}^A \rangle} dt \\ &= \int_0^\infty \left(\int_0^\infty f(\xi) \overline{\psi_{t,s}^A(\xi)} d\xi \right) \left(\int_0^\infty \bar{g}(\eta) \psi_{t,s}^A(\eta) d\eta \right) dt \\ &= \int_0^\infty \left[\int_0^\infty f(\xi) \left(s^{-1/2} e^{\frac{-\iota a}{2b}(\frac{t^2}{s^2} + \frac{\xi^2}{s^2})} \int_0^\infty \psi(z) D_\mu^A \left(\frac{t}{s}, \frac{\xi}{s}, z \right) dz \right) d\xi \right] \\ &\quad \times \left[\int_0^\infty \bar{g}(\eta) \left(s^{-1/2} e^{\frac{-\iota a}{2b}(\frac{t^2}{s^2} + \frac{\eta^2}{s^2})} \int_0^\infty \psi(z) D_\mu^A \left(\frac{t}{s}, \frac{\eta}{s}, z \right) dz \right) d\eta \right] dt \end{aligned}$$

$$\begin{aligned}
&= b^3 s^{-2\mu} \int_0^\infty H_\mu^A(e^{-\frac{t\alpha}{2b}(2-s^2)} x^2 x^{-\mu-1/2} (H_\mu^A e^{-\frac{t\alpha}{2b}(\cdot)^2} f)(x) (\overline{H_\mu^A \psi}(sx))(t) \\
&\quad \times \overline{H_\mu^A(e^{-\frac{t\alpha}{2b}(2-s^2)} y^2 y^{-\mu-1/2} (H_\mu^A e^{-\frac{t\alpha}{2b}(\cdot)^2} \bar{g})(y) (\overline{H_\mu^A \psi}(sy))(t) dt} \\
&= b^3 s^{-2\mu} \int_0^\infty (H_\mu^A \mathcal{P})(t) (\overline{H_\mu^A \mathcal{Q}})(t) dt \\
&= b^3 s^{-2\mu} \langle H_\mu^A \mathcal{P}, H_\mu^A \mathcal{Q} \rangle \\
&= b^3 s^{-2\mu} \langle \mathcal{P}, \mathcal{Q} \rangle.
\end{aligned}$$

This completes the proof of the theorem. $\square$

Theorem 2.8. *If ψ be in $L^2(\mathbb{R}_+)$ is a Bessel wavelet and f is a function which is bounded and integrable, then the convolution $(\psi *_A f)(x)$ is linear canonical Bessel wavelet.*

Proof. We know that

$$(\psi *_A f)(x) = \int_0^\infty (\tau_x^A \psi)(y) y^{-(\mu+1/2)} f(y) dy.$$

Therefore,

$$|(\psi *_A f)(x)| \leq \int_0^\infty |(\tau_x^A \psi)(y) y^{-(\mu+1/2)} f^{1/2}(y)| |f^{1/2}(y)| dy,$$

$$\begin{aligned}
&\Rightarrow \int_0^\infty |(\psi *_A f)(x)|^2 dx \\
&\leq \int_0^\infty \left(\int_0^\infty |y^{-(\mu+1/2)} f^{1/2}(y) (\tau_x^A \psi)(y)| |f^{1/2}(y)| dy \right)^2 dx \\
&\leq \int_0^\infty \left[\left(\int_0^\infty |f(y)| |y^{-(\mu+1/2)} (\tau_x^A \psi)(y)|^2 dy \right)^{1/2} \left(\int_0^\infty |f(y)| dy \right)^{1/2} \right]^2 dx \\
&= \left(\int_0^\infty |f(y)| dy \right) \left(\int_0^\infty \left(\int_0^\infty |f(y)| |y^{-(\mu+1/2)} (\tau_x^A \psi)(y)|^2 dx \right) |f(y)| dy \right) \\
&\leq \left(\int_0^\infty |f(y)| dy \right)^2 \left(\int_0^\infty |y^{-(\mu+1/2)} (\tau_y^A \psi)(x)|^2 dx \right)
\end{aligned}$$

This implies that

$$\|(\psi *_A f)(x)\|_{L^2} \leq \frac{1}{|(b)^{\mu+1/2}| 2^\mu \Gamma(\mu+1)} \|\psi\|_{L^2} \|f\|_{L^1} < \infty.$$

We have $(\psi *_A f) \in L^2(\mathbb{R}_+)$. Moreover,

$$\begin{aligned} & \int_0^\infty x^{-2(\mu+1)} |H_\mu^A(\psi *_A f)(x)|^2 dx \\ & \leq \frac{1}{|b|} \int_0^\infty \left| x^{-3\mu-5/2} (H_\mu^A f)\left(\frac{x}{b}\right) (H_\mu^A \psi)\left(\frac{x}{b}\right) \right|^2 dx \\ & \leq \frac{1}{|b|} \sup \left(x^{-\mu-1/2} \left| (H_\mu^A f)\left(\frac{x}{b}\right) \right|^2 \right) \\ & \quad \times \int_0^\infty x^{-2(\mu+1)} \left| (H_\mu^A \psi)\left(\frac{x}{b}\right) \right|^2 dx. \\ & = C_{\mu,\psi}^A \sup \left(x^{-\mu-1/2} \left| (H_\mu^A f)\left(\frac{x}{b}\right) \right|^2 \right) < \infty. \end{aligned}$$

Thus, the convolution function $(\psi *_A f)$ is a linear canonical Bessel wavelet. $\square$

Theorem 2.9. Let f, ψ are in $L^2(\mathbb{R}_+)$ and $(B_\psi^A f)(t, s)$ is a continuous linear canonical Bessel wavelet transform (CLCBWT), then

- (1) $(B_\psi^A f)(t, s)$ is continuous on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$.
- (2) $\|(B_\psi^A f)(t, s)\|_{L^\infty} \leq \frac{t^{\mu+1/2} s^{-(\mu+1/2)}}{|b^{\mu+1/2}| 2^\mu \Gamma(\mu+1)} \|f\|_L^2 \|\psi\|_{L^2}$

Proof. (1) Let (t_0, s_0) be an arbitrary but fixed in $\mathbb{R}_+$. Then using Hölders inequality we have:

$$\begin{aligned} & |(B_\psi^A f)(t, s) - (B_\psi^A f)(t_0, s_0)| \\ & \leq s^{-1/2} \int_0^\infty \int_0^\infty \left| f(x) \psi(z) \left[D_\mu^A \left(\frac{t}{s}, \frac{x}{s}, z \right) \right. \right. \\ & \quad \left. \left. - D_\mu^A \left(\frac{t_0}{s_0}, \frac{x}{s_0}, z \right) \right] \right| dx dz \\ & = s^{-1/2} \left(\int_0^\infty x^{-\mu-1/2} |f(x)|^2 dx \int_0^\infty z^{\mu+1/2} \right. \\ & \quad \times \left[\left| D_\mu^A \left(\frac{t}{s}, \frac{x}{s}, z \right) - D_\mu^A \left(\frac{t_0}{s_0}, \frac{x}{s_0}, z \right) \right| dz \right]^{1/2} \\ & \quad \times \left(\int_0^\infty z^{-\mu-1/2} |\psi(z)|^2 dz \int_0^\infty x^{\mu+1/2} \right. \\ & \quad \times \left[\left| D_\mu^A \left(\frac{t}{s}, \frac{x}{s}, z \right) - D_\mu^A \left(\frac{t_0}{s_0}, \frac{x}{s_0}, z \right) \right| dx \right]^{1/2} \end{aligned}$$

As

$$\begin{aligned} & \int_0^\infty z^{\mu+1/2} \left[\left| D_\mu^A \left(\frac{t}{s}, \frac{x}{s}, z \right) - D_\mu^A \left(\frac{t_0}{s_0}, \frac{x}{s_0}, z \right) \right| dz \right] \\ & \leq \frac{x^{\mu+1/2}}{|b^{\mu+1/2}| 2^\mu \Gamma(\mu+1)} \left[\frac{t^{(\mu+1/2)}}{s^{2(\mu+1/2)}} - \frac{t_0^{(\mu+1/2)}}{s_0^{2(\mu+1/2)}} \right], \end{aligned}$$

And

$$\begin{aligned} \int_0^\infty x^{\mu+1/2} \left[\left| D_\mu^A \left(\frac{t}{s}, \frac{x}{s}, z \right) - D_\mu^A \left(\frac{t_0}{s_0}, \frac{x}{s_0}, z \right) \right| \right] dz \\ \leq \frac{z^{\mu+1/2}}{|b^{\mu+1/2}| 2^\mu \Gamma(\mu+1)} \left[st^{(\mu+1/2)} - s_0 t_0^{(\mu+1/2)} \right], \end{aligned}$$

Hence by dominated convergence theorem and by the continuity of $D_\mu^A \left(\frac{t}{s}, \frac{x}{s}, z \right)$ in the variables t and s , we have

$$\lim_{t \rightarrow t_0} \lim_{s \rightarrow s_0} |(B_\psi^A f)(t, s) - (B_\psi^A f)(t_0, s_0)| = 0$$

Which proves $(B_\psi^A f)(t, s)$ is continuous on $\mathbb{R}_+ \times \mathbb{R}_+$.

□

Proof. (2) We have

$$(B_\psi^A f)(t, s) = \int_0^\infty f(x) \overline{\left(s^{-1/2} e^{\frac{-ia}{2b} \left(\frac{t^2}{s^2} + \frac{x^2}{s^2} \right)} \int_0^\infty \psi(z) D_\mu^A \left(\frac{t}{s}, \frac{x}{s}, z \right) dz \right)} dx$$

Therefore, by the Hölders inequality, we have

$$\begin{aligned} |(B_\psi^A f)(t, s)| \\ \leq s^{-1/2} \left(\int_0^\infty \int_0^\infty x^{-(\mu+1/2)} |f(x)|^2 z^{(\mu+1/2)} \left| D_\mu^A \left(\frac{t}{s}, \frac{x}{s}, z \right) \right| dx dz \right)^{1/2} \\ \times \left(\int_0^\infty \int_0^\infty z^{-(\mu+1/2)} |\psi(z)|^2 x^{(\mu+1/2)} \left| D_\mu^A \left(\frac{t}{s}, \frac{x}{s}, z \right) \right| dx dz \right)^{1/2} \\ = s^{-1/2} \left(\int_0^\infty x^{-(\mu+1/2)} |f(x)|^2 dx \int_0^\infty \left| D_\mu^A \left(\frac{t}{s}, \frac{x}{s}, z \right) \right| z^{(\mu+1/2)} dz \right)^{1/2} \\ \times \left(\int_0^\infty z^{-(\mu+1/2)} |\psi(z)|^2 dz \int_0^\infty \left| D_\mu^A \left(\frac{t}{s}, \frac{x}{s}, z \right) \right| x^{(\mu+1/2)} dx \right)^{1/2} \\ \leq s^{-1/2} \left(\frac{t^{\mu+1/2}}{s^{2(\mu+1/2)} |b^{\mu+1/2}| 2^\mu \Gamma(\mu+1)} \right)^{1/2} \left(\int_0^\infty |f(x)|^2 dx \right)^{1/2} \end{aligned}$$

$$\begin{aligned}
& \times \left(\frac{st^{\mu+1/2}}{|b^{\mu+1/2}|2^\mu\Gamma(\mu+1)} \right)^{1/2} \left(\int_0^\infty |\psi(z)|^2 dz \right)^{1/2} \\
& = \frac{t^{\mu+1/2}}{s^{(\mu+1/2)}|b^{\mu+1/2}|2^\mu\Gamma(\mu+1)} \|f\|_{L^2} \|\psi\|_{L^2} \\
& = \frac{t^{\mu+1/2}s^{-(\mu+1/2)}}{s^{(\mu+1/2)}|b^{\mu+1/2}|2^\mu\Gamma(\mu+1)} \|f\|_{L^2} \|\psi\|_{L^2}.
\end{aligned}$$

This completes the proof. $\square$

Acknowledgements

This work is supported by the Research project (JKST&IC/SRE/J/357-60) provided by JKST&IC, UT of J&K, India.

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