Poincare Journal of Analysis & Applications
Vol. 10, No. 2 (2023), 297-308

DOI: 10.46753/pjaa.2023.v010i02.007

NORMAL FAMILIES AND ENTIRE FUNCTIONS SHARING ONE VALUE WITH THEIR k-TH DERIVATIVES

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Date of Receiving : 07. 12. 2022
Date of Revision : 24. 07. 2023
Date of Acceptance : 03. 08. 2023

Abstract. In this paper, we use the idea of normal family to investigate the uniqueness problem of entire functions that share one value with their derivatives and obtain a result which improves and generalizes the recent result due to Zhang and Yang [16]. Also we exhibit an example to show that the condition of our result is the best possible.

1. Introduction, definitions and results

In the paper, we assume that the reader is familiar with standard notation and main results of Nevanlinna Theory (see, e.g., [7, 13]). Again we recall that a meromorphic function a is said to be a small function of f if $T(r,a) = S(r,f)$.

Let f and g be two non-constant meromorphic functions and $a \in \mathbb{C}$. If $f(z) - a$ and $g(z) - a$ have the same zeros with the same multiplicities then we say that $f(z)$ and $g(z)$ share a with CM (counting multiplicities) and if we do not consider the multiplicities then we say that $f(z)$ and $g(z)$ share a with IM (ignoring multiplicities). Also if $g(z) - a = 0$ whenever $f(z) - a = 0$, we write $f(z) = a \Rightarrow g(z) = a$. If $f(z) = a \Rightarrow g(z) = a$ and $g(z) = a \Rightarrow f(z) = a$, we then write $f = a \Leftrightarrow g = a$ and we say that f and g share a IM.

We recall that the order $\rho(f)$ of entire function f defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r},$$

where $M(r,f) = \max\limits_{|z|=r} |f(z)|$.

2010 *Mathematics Subject Classification.* 30D35, 30D30.
Key words and phrases. Entire functions, uniqueness, derivative, Nevanlinna theory, normal family.
Communicated by. Kuldeep Singh Charak
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Let f be a non-constant entire function. We know that f can be expressed by the power series $f(z)=\sum\limits_{n=0}^{\infty} a_n z^n$. We denote by

$$\mu(r,f)=\max_{\substack{n\in\mathbb{N}\\|z|=r}}\{|a_n z^n|\} \text{ and } \nu(r,f)=\sup\{n:|a_n|r^n=\mu(r,f)\}.$$

Clearly for a polynomial $P(z)=a_n z^n+a_{n-1}z^{n-1}+\ldots+a_0$, $a_n\neq 0$, we have

$$\mu(r,P)=|a_n|r^n \text{ and } \nu(r,P)=n$$

for all r sufficiently large.

In the general case, $|a_n|r^n \leq \mu(r,f)$ for all $n\geq 0$ and $|a_n|r^n<\mu(r,f)$ for all $n>\nu(r,f)$.

Here it is enough to recall that (see [8])

(1) $\mu(r,f)$ is strictly increasing for all r sufficiently large, is continuous and tends to $+\infty$ as $r\to\infty$;

(2) $\nu(r,f)$ is increasing, piecewise constant, right-continuous and also tends to $+\infty$ as $r\to\infty$;

(3) $\log\nu(r,F)=O(\log r)$, if $\rho(f)<+\infty$.

Next we introduce some basic ideas about normal families. First we recall the chordal metric $\chi(.,,)$ on $\mathbb{C}\cup\{\infty\}$ (see [11]) defined by

$$\chi(z_1,z_2)=\begin{cases}\frac{|z_1-z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}, & \text{if } z_1,z_2\in\mathbb{C},\\ \frac{1}{\sqrt{1+|z_1|^2}}, & \text{if } z_2=\infty.\end{cases}$$

The important property of the chordal metric is that it allows $z=\infty$ to be treated like any other point. It is easy to verify that $\chi(z_1,z_2)\leq|z_1-z_2|$ in $\mathbb{C}$. We now introduce the prime mode of convergence in normal family is as follows.

A sequence of functions $\{f_n\}$ is said to be converge spherically uniformly on compact subsets of D to a function f if for any compact subset $K\subseteq D$ and $\varepsilon>0$, there exists $n_0=n_0(K,\varepsilon)\in\mathbb{N}$ such that $n\geq n_0$ implies $\chi(f_n(z),f(z))<\varepsilon\ \forall\ z\in K$ and $n\geq n_0$.

Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D\subseteq\mathbb{C}$. We say that $\mathcal{F}$ is normal in D in the sense of Montel, if for any sequence $\{f_n\}_n\subseteq\mathcal{F}$ there exists a subsequence $\{f_{n_j}\}$ such that $\{f_{n_j}\}$ converges spherically and uniformly on the compact subsets of D to a meromorphic function or ∞.

The family $\mathcal{F}$ is said to be normal at a point $z_0\in D$ if it is normal in some neighbourhood of z_0. Clearly $\mathcal{F}$ is normal in D if and only if it is normal at every point of D. It follows that if $\mathcal{F}$ is not normal in D, then D contains at least one point at which $\mathcal{F}$ is not normal.

Normal families, in particular, of holomorphic functions often appear in operator theory on spaces of analytic functions.

Following example shows that the limit function may be identically ∞.

Let $f_n(z)=\frac{n}{z}$, $n\in\mathbb{N}$ and $z\in\Delta=\{z\in\mathbb{C}:|z|<1\}$. Then each f_n is meromorphic and $\{f_n\}$ converges spherically uniformly to ∞ in Δ.

The following example is an example of a family of holomorphic functions which is not normal.

Let $\mathcal{F}=\{f_n(z)=nz\colon n\in\mathbb{N}\}$. Then $f_n(0)\to 0$, but $f_n(z)\to\infty$ as $n\to\infty$ for $z\neq 0$. Clearly $\mathcal{F}$ is not normal in Δ.

Next we introduce the notation of the spherical derivative. Let h be a meromorphic function. The spherical derivative of h at $z \in \mathbb{C}$ is given as

$$h^{\#}(z) = \frac{|h'(z)|}{1 + |h(z)|^2}.$$

We remember that h is called a normal function if there exists a positive real number M such that $h^{\#}(z) \leq M \ \forall \ z \in \mathbb{C}$. The following result is the famous Marty's Theorem.
Marty's Theorem: A family $\mathcal{F}$ of meromorphic functions on a domain D is normal if and only if for each compact subset $K \subseteq D$, there exists $M(K) \in \mathbb{R}^+$ such that $f^{\#}(z) \leq M(K) \ \forall \ f \in \mathcal{F}$ and $\forall \ z \in K$.

The Marty criterion is one of the most widely used for determining the normality of a family of meromorphic functions. In the paper, we also use it to deal with the normality of a family of entire functions.

Rubel and Yang [10] were the first to study the entire functions that share values with their derivatives. In 1977 they proved the following important result.

Theorem 1.1. [10] *Let a and b be complex numbers such that $b \neq a$ and let f be a non-constant entire function. If f and f' share the values a and b CM, then $f \equiv f'$.*

Mues and Steinmetz [9] generalized Theorem 1.1 from sharing values CM to IM and obtained the following result.

Theorem 1.2. [9] *Let a and b be complex numbers such that $b \neq a$ and let f be a non-constant entire function. If f and f' share the values a and b IM, then $f \equiv f'$.*

Also Theorem 1.1 has been generalized to sharing values IM independently by G. Gundersen in the case when both shared values are nonzero (see [5]). Since then, in this topics many results have been obtained (see [13]).

A natural question is:
Question A. What is the possible relation between f and f' when an entire function f and it's derivative f' share only one finite value CM ?

Motivated by Question A, Brück [1] obtained the following results in 1996.

Theorem 1.3. [1] *Let f be a non-constant entire function such that $\rho_1(f) < \infty$ and $\rho_1(f)$ is not a positive integer, where*

$$\rho_1(f) := \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}$$

If f and f'share the value 0 CM, then $f' \equiv cf$ for some constant $c \neq 0$.

Theorem 1.4. [1] *Let f be a non-constant entire function. If f and f' share 1 CM, and if $N(r, \frac{1}{f'}) = S(r,f)$, then $f' - 1 \equiv c(f-1)$ for some constant $c \neq 0$.*

In the same paper, Brück [1] made the following conjecture.

Conjecture A . Let f be a non-constant entire function. Suppose $\rho_1(f)$ is not a positive integer or infinite. If f and f' share one finite value a CM, then $f' - a \equiv c(f-a)$, where $c \in \mathbb{C} \setminus \{0\}$.

Also in the same paper, Brück [1] ensured the necessity of the condition $\rho_1(f) \notin \mathbb{N} \cup \{\infty\}$ in Conjecture A, by considering solutions of the following differential equations:

$$\frac{f'(z)-1}{f(z)-1} = e^{z^n} \text{ and } \frac{f'(z)-1}{f(z)-1} = e^{e^z}.$$

In 1998, Gundersen and Yang [6] proved that Conjecture A holds for entire functions of finite order. Also in 2004, Chen and Shon [4] further investigated Conjecture A for the case when f is of infinite order and proved that Conjecture A holds when $\rho_1(f) < \frac{1}{2}$. Recently Cao [2] affirmed Conjecture A when $\rho_1(f) = \frac{1}{2}$.

However, Conjecture A does not hold for meromorphic functions in general (see [6]). For example if $f(z) = \frac{2e^z+z+1}{e^z+1}$, then f and f' share the value 1 CM, but $f'-1 \not\equiv c(f-1)$, where $c \in \mathbb{C} \setminus \{0\}$.

Though Conjecture A remains open when $\rho_1(f) > \frac{1}{2}$, it gives rise to a long course of research on the uniqueness of entire and meromorphic functions sharing a single value with its derivatives. Specially, it was observed by Yang and Zhang [14] that Conjecture A holds if instead of an entire function one considers its suitable power. They proved the following result.

Theorem 1.5. [14] *Let f be a non-constant entire function, $n \in \mathbb{N}$ such that $n \geq 7$ and let $F = f^n$. If F and F' share 1 CM, then $F \equiv F'$ and f assumes the form $f(z) = ce^{\frac{1}{n}z}$, where $c \in \mathbb{C} \setminus \{0\}$.*

In 2010, Zhang and Yang [16] improved the above result in the following manner.

Theorem 1.6. [16] *Let f be a non-constant entire function and $n, k \in \mathbb{N}$. Suppose f^n and $(f^n)^{(k)}$ share 1 CM and $n \geq k+1$. Then $f^n \equiv (f^n)^{(k)}$ and f assumes the form $f(z) = ce^{\frac{\lambda}{n}z}$, where $c \in \mathbb{C} \setminus \{0\}$ and $\lambda^k = 1$.*

2. Main Results

In this paper, we always use $P(z)$ denoting an arbitrary non-zero polynomial as follows:

$$P(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots + a_0, \tag{2.1}$$

where $a_i \in \mathbb{C} (i = 0, 1, \ldots, m)$.

Now observing Theorem 1.6 the following question is inevitable.

Question 1. What will happen if "f^n" is replaced by "$f^n P(f)$" in Theorem 1.6 ?

In this paper, taking the possible answer of the above question into back ground we obtain our main result as follows.

Theorem 2.1. *Let f be a non-constant entire function and let $n, k \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$. Suppose $f^n P(f)$ and $(f^n P(f))^{(k)}$ share 1 CM, where P is defined as in (2.1) and*

$$n > \begin{cases} \max\{k, m\}, & \text{if } k = 1, \\ k, & \text{if } k \geq 2. \end{cases}$$

Then $f^n P(f) \equiv (f^n P(f))^{(k)}$. In this case $P(z)$ reduces to a non-zero monomial, namely $P(z) = a_i z^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$ and so $f^{n+i} \equiv (f^{n+i})^{(k)}$, where f assumes the form $f(z) = ce^{\frac{\lambda}{n+i}z}$, $c \in \mathbb{C} \setminus \{0\}$ and $\lambda^k = 1$.

Remark 2.2. Following example shows that the condition

$$n > \begin{cases} \max\{k, m\}, & \text{if } k = 1, \\ k, & \text{if } k \geq 2. \end{cases}$$

in Theorem 2.1 is sharp.

Example 2.3. Let $f(z) = \frac{1}{c_3}(e^{c_1 z} + c_2)$, where $c_1, c_2, c_3 \in \mathbb{C} \setminus \{0\}$ such that $c_1 \neq 1$ and $c_1 c_2 = c_3(c_1 - 1)$. Then $f - 1$ and $f' - 1$ share 0 CM, but $f \not\equiv f'$.

3. Auxiliary lemmas

Following lemmas play important roles in the proof of our main result.

Lemma 3.1. [3] *Let f be a meromorphic function on $\mathbb{C}$ with finitely many poles. If f has bounded spherical derivative on $\mathbb{C}$, then f is of order at most* 1.

Lemma 3.2. *(Zalcman's Lemma)* [15] *Let $\mathcal{F}$ be a family of meromorphic functions defined in the open unit disc Δ. Then if $\mathcal{F}$ is not normal at a point $z_0 \in \Delta$, then there exist for each real number α satisfying $-1 < \alpha < 1$,*

(i) points $z_n \in \Delta$, $z_n \to z_0$,
(ii) positive numbers ρ_n, $\rho_n \to 0^+$ and
(iii) functions f_n, $f_n \in \mathcal{F}$,

such that $g_n(\zeta) = \rho_n^{\alpha} f_n(z_n + \rho_n \zeta) \to g(\zeta)$ spherically uniformly on compact subsets of $\mathbb{C}$, where g is a non-constant meromorphic function and $g^{\#}(\zeta) \leq g^{\#}(0) = 1$.

Lemma 3.3. *(*[8]*, Lemma 1.3.1.) Let $P(z) = \sum_{i=1}^{n} a_i z^i$, where $a_n \neq 0$. Then $\forall\ \varepsilon > 0$, there exists $r_0 > 0$ such that $\forall\ r = |z| > r_0$ the inequalities $(1-\varepsilon)|a_n| r^n \leq |P(z)| \leq (1+\varepsilon)|a_n| r^n$ hold.*

Lemma 3.4. *(*[8]*, Theorem 3.2.) Let f be a transcendental entire function. Then there exists a set $E \subset (1, +\infty)$ with finite logarithmic measure, we choose z satisfying $|z| = r \notin [0,1] \cup E$ and $|f(z)| = M(r, f)$, such that*

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu(r,f)}{z}\right)^j (1 + o(1)) \ \ for\ j \in \mathbb{N}.$$

Lemma 3.5. [13] *Suppose f_j $(j = 1, 2, \ldots, m+1)$ and g_j $(j = 1, 2, \ldots, m)$ are entire functions satisfying the following conditions*

(i) $\sum\limits_{j=1}^{m} f_j e^{g_j} \equiv f_{m+1}$;
(ii) the order of f_j is less than the order of e^{g_k} for $1 \leq j \leq m+1$, $1 \leq k \leq m$; and furthermore, the order of f_j is less than the order of $e^{g_j - g_k}$ for $m \leq 2$ and $1 \leq j \leq m+1$, $1 \leq l, k \leq m$, $l \neq k$.

Then $f_j \equiv 0$ $(j = 1, 2, \ldots, m+1)$.

Lemma 3.6. *(*[12]*, Proof of Lemma 2) Let f be a non-constant meromorphic function and $P_*(f) = a_n f^n + a_{n-1} f^{n-1} + \ldots + a_1 f + a_0$, where $a_0, a_1, \ldots, a_{n-1}, a_n$ are constants and $a_n \neq 0$. Then $m(r, P_*(f)) = n\, m(r, f) + O(1)$.*

4. Proof of the theorem

Proof of Theorem 2.1. First we see that if f is a polynomial, then $f^n P(f) - 1$ and $(f^n P(f))^{(k)} - 1$ cannot have the same zeros with same multiplicities. Therefore f is a transcendental entire function.

Let $\mathcal{F} = \{f_\omega\}$, where $f_\omega(z) = f(z + \omega)$, $z \in \mathbb{C}$. Clearly $\mathcal{F}$ is a family of entire functions defined on $\mathbb{C}$. We now consider following two cases.

Case 1. Suppose the family $\mathcal{F}$ is normal on $\mathbb{C}$. Then by Marty's theorem $f^{\#}(\omega) = f_{\omega}^{\#}(0) \leq M$ for some $M > 0$ and for all $\omega \in \mathbb{C}$. Hence by Lemma 3.1, we have f is of order at most 1.

Let $H = f^n P(f)$. Now by Lemma 3.6, we have $T(r,H) = (n+m)\, T(r,f) + O(1)$. Therefore H is also a transcendental entire function. On the other hand $\rho(H) = \rho(f)$ and so $\rho(H^{(k)}) = \rho(H) = \rho(f) \leq 1$. Since $H-1$ and $H^{(k)}-1$ share 0 CM, then there exists a polynomial γ such that

$$\frac{H^{(k)}-1}{H-1} = e^{\gamma}. \tag{4.1}$$

We now want to prove that γ is a constant. If not, suppose $\deg(\gamma) = p \geq 1$. Let $\gamma(z) = \sum_{i=0}^{p} b_i z^i$, where $b_i \in \mathbb{C}$ for $i = 0, 1, \ldots, p$ and $b_p \neq 0$. Then from (4.1) we get

$$\gamma = \log \frac{H^{(k)}-1}{H-1} = \log \frac{\frac{H^{(k)}}{H} - \frac{1}{H}}{1 - \frac{1}{H}},$$

where $\log g$ is the principle branch of the logarithm and so by Lemma 3.3 we have

$$|b_p| r^p (1 + o(1)) = |\gamma(z)| = \left| \log \frac{\frac{H^{(k)}(z)}{H(z)} - \frac{1}{H(z)}}{1 - \frac{1}{H(z)}} \right|. \tag{4.2}$$

Since H is a transcendental entire function, we have $M(r,H) \to \infty$ as $r \to \infty$. Again we let

$$M(r,H) = |H(z_r)|, \text{ where } z_r = re^{i\theta} \text{ and } \theta \in [0, 2\pi). \tag{4.3}$$

Now from (4.3) and Lemma 3.4, there exists a subset $E \subset (1, +\infty)$ with finite logarithmic measure such that for some point $z_r = re^{i\theta} (\theta \in [0, 2\pi))$ satisfying $|z_r| = r \notin E$ and $M(r,H) = |H(z_r)|$ we have

$$\frac{H^{(k)}(z)}{H(z)} = \left(\frac{\nu(r,H)}{z} \right)^k (1 + o(1)) \text{ as } r \to \infty. \tag{4.4}$$

Since H is a transcendental entire function, it follows from (4.3) that

$$\lim_{r \to +\infty} \left| \frac{1}{H(z_r)} \right| \leq \lim_{r \to +\infty} \frac{1}{M(r, H(z_r))} = 0. \tag{4.5}$$

Also we know that, if $\rho(f) < +\infty$, then

$$\log \nu(r,f) = O(\log r). \tag{4.6}$$

Therefore from (4.2)-(4.6) we conclude that $|\gamma(z_r)| = O(\log r)$ for $|z_r| = r \notin E$, which is impossible. Hence γ is a constant. Without loss of generality we assume that

$$H^{(k)} - 1 \equiv d(H-1), \tag{4.7}$$

where $d \in \mathbb{C} \setminus \{0\}$.

First we suppose that $d \neq 1$. Then from (4.7), we have

$$H^{(k)} \equiv dH + 1 - d. \tag{4.8}$$

We now consider following two sub-cases.

Sub-case 1.1. Suppose 0 is not a Picard exceptional value of f. Let z_0 be a zero of f. Then $H(z_0) = 0$. Since $n \geq k+1$, it follows that $H^{(k)}(z_0) = 0$. Therefore from (4.8), we arrive at a contradiction.

Sub-case 1.2. Suppose 0 is a Picard exceptional value of f. By Hadamard's Factorization theorem, we may assume that $f(z) = c_1 e^{az}$, where $a, c_1 \in \mathbb{C} \setminus \{0\}$. Then by induction we get

$$a_i\left((f^{n+i})^{(k)} - df^{n+i}\right) = d_i e^{(n+i)az}, \tag{4.9}$$

where d_i $(i = 0, 1, 2, \ldots, m)$ are constants. Since $H^{(k)} - dH \not\equiv 0$, it follows that $d_i \neq 0$ for at least one $i \in \{0, 1, \ldots, m\}$. Now from (4.8) and (4.9) we obtain

$$d_m e^{(n+m)az} + \ldots + d_1 e^{(n+1)az} + d_0 e^{naz} \equiv 1 - d. \tag{4.10}$$

Then from (4.10) and Lemma 3.5, we arrive at a contradiction.

Next we suppose that $d = 1$. Then from (4.7), we obtain $H^{(k)} \equiv H$, i.e.,

$$f^n P(f) \equiv (f^n P(f))^{(k)}. \tag{4.11}$$

We now prove that $P(z) = a_i z^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$. If not, we may assume that $P(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots + a_1 z + a_0$, where at least two of $a_0, a_1, \ldots, a_{m-1}, a_m$ are non-zero. Without loss of generality, we assume that $a_s, a_t \neq 0$, where $s \neq t$, $s, t = 0, 1, 2, \ldots, m$.

Since $n \geq k + 1$, it follows from (4.11) that $f \neq 0$. So we can take $f(z) = e^{az}$, where $a \in \mathbb{C} \setminus \{0\}$. Then by induction, we get

$$a_i\Big(f^{n+i} - (f^{n+i})^{(k)}\Big) = a_i\Big(1 - (n+i)^k a^k\Big)e^{(n+i)az}. \tag{4.12}$$

Now from (4.11) and (4.12), we obtain

$$a_m\Big(1 - (n+m)^k a^k\Big)e^{maz} + \ldots + a_1\Big(1 - (n+1)^k a^k\Big)e^{az} \equiv -a_0\Big(1 - n^k a^k\Big). \tag{4.13}$$

Then from Lemma 3.5 and (4.13), we conclude that $a_i\Big(1 - (n+i)^k a^k\Big) = 0$ for $i = 0, 1, \ldots, m$. As $a_s, a_t \neq 0$, from (4.12), we have $(n+s)^k a^k = 1$ and $(n+t)^k a^k = 1$, which is a contradiction.

Hence $P(z) = a_i z^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$. Therefore from (4.11) we get $f^{n+i} \equiv \left(f^{n+i}\right)^{(k)}$, where $i \in \{0, 1, \ldots, m\}$. Clearly f assumes the form $f(z) = ce^{\frac{\lambda}{n+i}z}$, where $c \in \mathbb{C} \setminus \{0\}$ and $\lambda^k = 1$.

Case 2. Suppose the family $\mathcal{F}$ is not normal on $\mathbb{C}$. Then there exists at least one point $z_0 \in \Delta$ such that $\mathcal{F}$ is not normal at z_0. For the sake of simplicity we assume that $z_0 = 0$. Now by Marty's theorem there exists a sequence of entire functions $\{f(z + \omega_j)\} \subset \mathcal{F}$, where $z \in \Delta$ and $\{\omega_j\} \subset \mathbb{C}$ is some sequence of complex numbers such that $f^{\#}(\omega_j) \to \infty$ as $|\omega_j| \to \infty$. Then by Lemma 3.2, we see that there exist

(i) points $z_j \in \Delta$ such that $z_j \to 0$,
(ii) $\rho_j \in \mathbb{R}^+$ such that $\rho_j \to 0^+$,
(iii) a subsequence $\{f_j(z_j + \rho_j\zeta) = f(\omega_j + z_j + \rho_j\zeta)\}$ of $\{f(\omega_j + z)\}$;

such that

$$h_j(\zeta) = f_j(z_j + \rho_j\zeta) \to h(\zeta) \tag{4.14}$$

spherically uniformly on compact subsets of $\mathbb{C}$, where h is some non-constant entire function such that $h^{\#}(\zeta) \leq h^{\#}(0) = 1$. Now from Lemma 3.1, we conclude that $\rho(h) \leq 1$. Clearly from (4.14), we have

$$h_j^{n+i}(\zeta) = f_j^{n+i}(z_j + \rho_j\zeta) \to h^{n+i}(\zeta), \tag{4.15}$$

$i = 0, 1, \ldots, m$ and so

$$(h_j^{n+i}(\zeta))^{(k)} = \rho_j^k (f_j^{n+i}(z_j + \rho_j \zeta))^{(k)} \to (h^{n+i}(\zeta))^{(k)}, \tag{4.16}$$

$i = 0, 1, \ldots, m$. Now from (4.15), we deduce that

$$\begin{aligned} G_j(\zeta) = H_j(z_j + \rho_j \zeta) &= H(\omega_j + z_j + \rho_j \zeta) \\ &= a_m h_j^{n+m}(\zeta) + a_{m-1} h_j^{n+m-1}(\zeta) + \ldots + a_0 h_j^n(\zeta) \\ &\to a_m h^{n+m}(\zeta) + a_{m-1} h^{n+m-1}(\zeta) + \ldots + a_0 h^n(\zeta) \\ &= h^n(\zeta) P(h(\zeta)), \end{aligned} \tag{4.17}$$

spherically uniformly on compact subsets of $\mathbb{C}$.

Again from (4.16) and (4.17) we have

$$\begin{aligned} (G_j(\zeta))^{(k)} &= \rho_j^k H_j^{(k)}(z_j + \rho_j \zeta) \\ &= a_m \big(h_j^{n+m}(\zeta)\big)^{(k)} + a_{m-1} \big(h_j^{n+m-1}(\zeta)\big)^{(k)} + \ldots + a_0 \big(h_j^n(\zeta)\big)^{(k)} \\ &\to a_m \big(h^{n+m}(\zeta)\big)^{(k)} + a_{m-1} \big(h^{n+m-1}(\zeta)\big)^{(k)} + \ldots + a_0 \big(h^n(\zeta)\big)^{(k)} \\ &= (h^n(\zeta) P(h(\zeta)))^{(k)}, \end{aligned} \tag{4.18}$$

spherically uniformly on compact subsets of $\mathbb{C}$.

Let $g = h^n P(h)$. First we want to prove that $g(\zeta)$ is non-constant. If possible suppose that $g = h^n P(h) \equiv c_0$, where $c_0 \in \mathbb{C}$. Then clearly h has no zeros. Since h is a non-constant entire function of order at most 1, then there exist $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ such that $h(\zeta) = c_1 e^{c_2 \zeta}$. Now using Lemma 3.6, we see that $(n+m)T(r,h) = (n+m)T(r, e^{c_2\zeta}) + O(1) = O(1)$, which is impossible. Hence g is non-constant.

Next we want to prove that $g^{(k)}$ is non-constant. If possible suppose that $g^{(k)} = (h^n P(h))^{(k)} \equiv c_3$, where $c_3 \in \mathbb{C}$. Since $n \geq k+1$, it follows that h has no zeros. Therefore $h(\zeta) = c_4 e^{c_5 \zeta}$, where $c_4, c_5 \in \mathbb{C} \setminus \{0\}$. In this case also we arrive at a contradiction. Hence $g^{(k)}$ is non-constant.

Now we prove that $g = 1 \Rightarrow g^{(k)} = 0$. Let $g(\zeta_0) = 1$. Then by Hurwitz's theorem there exists a sequence $\{\zeta_j\}$, $\zeta_j \to \zeta_0$ such that (for sufficiently large j) $H_j(z_j + \rho_j \zeta_j) = 1$. The assumption H and $H^{(k)}$ share 1 CM implies that $H_j^{(k)}(z_j + \rho_j \zeta_j) = 1$. Now from (4.18) we see that

$$g^{(k)}(\zeta_0) = \big(h^n P(h)\big)^{(k)}(\zeta_0) = \lim_{j\to\infty} \rho_j^k H_j^{(k)}(z_j + \rho_j \zeta_j) = \lim_{j\to\infty} \rho_j^k = 0.$$

Thus $g = 1 \Rightarrow g^{(k)} = 0$.

Finally we prove that $g^{(k)} = 0 \Rightarrow g = 1$. Now from (4.18) we see that

$$\rho_j^k \left(H_j^{(k)}(z_j + \rho_j \zeta) - 1 \right) \to g^{(k)}(\zeta) = \big(h^n(\zeta) P(h(\zeta))\big)^{(k)}. \tag{4.19}$$

Let $g^{(k)}(\xi_0) = 0$. Then by (4.19) and Hurwitz's theorem, there exists a sequence $\{\xi_j\}$, $\xi_j \to \xi_0$ such that (for sufficiently large j) $H_j^{(k)}(z_j + \rho_j \xi_j) = 1$. The assumption H and $H^{(k)}$ share 1 CM implies that $H_j(z_j + \rho_j \xi_j) = 1$. Therefore from (4.17) we have

$$g(\xi_0) = h^n(\xi_0) P(h(\xi_0)) = \lim_{j\to\infty} H_j(z_j + \rho_j \xi_j) = 1.$$

Thus $g^{(k)} = 0 \Rightarrow g = 1$. As a result we have $g = 1 \Leftrightarrow g^{(k)} = 0$. Note that $h = 0 \Rightarrow g = 0$. Since $n \geq k+1$, it follows that $h = 0 \Rightarrow g^{(k)} = 0$. Therefore one can easily deduce that

$h \neq 0$. Since h is a non-constant entire function of order at most 1, then there exist $c, \lambda \in \mathbb{C} \setminus \{0\}$ such that $h(\zeta) = ce^{\lambda\zeta}$.

Now we divide the following sub-cases.

Sub-case 2.1. Suppose $k = 1$. Since $g = 1 \Leftrightarrow g' = 0$, one can easily conclude that zeros of $g - 1$ are of multiplicity at least 2. Also by Lemma 3.6, we have $T(r,g) = (n+m)\,T(r,h) + O(1)$ and so $S(r,g) = S(r,h)$. Therefore by the second fundamental theorem we have

$$\begin{aligned}(n+m)\,T(r,h) &= T(r,g) + O(1)\\ &\leq \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \overline{N}(r,1;g) + S(r,g)\\ &\leq \overline{N}(r,0;P(h)) + \frac{1}{2}\,N(r,1;g) + S(r,g)\\ &\leq m\,T(r,h) + \frac{1}{2}\,T(r,g) + S(r,g)\\ &= m\,T(r,h) + \frac{n+m}{2}\,T(r,h) + S(r,h),\end{aligned}$$

which is a contradiction as $n > m$.

Sub-case 2.2. Suppose $k \geq 2$. In this case we concentrate our attention on the nature of the polynomial P. Therefore we have to consider the following sub-cases.

Sub-case 2.2.1. Suppose $P \equiv d_0 \in \mathbb{C} \setminus \{0\}$. Then $g = d_0h^n$ and so $g^{(k)} = d_0(h^n)^{(k)}$. Note that $g \neq 0$ and $g^{(k)} \neq 0$. Since $g = 1 \Leftrightarrow g^{(k)} = 0$, it follows that $g \neq 1$. Therefore 0 and 1 are the Picard exceptional values of g and so g is a constant. Since $T(r,g) = nT(r,h) + O(1)$, we conclude that h is a constant, which is impossible.

Sub-case 2.2.2. Suppose $\deg(P) = 1$. Clearly $P(\zeta) = a_1\zeta + a_0$. If $a_0 = 0$, then we conclude that $g \neq 0$ and $g^{(k)} \neq 0$. In this case also we arrive at a contradiction. Hence $a_0 \neq 0$. Note that

$$g(\zeta) = h^n(\zeta)P(h(\zeta)) = c^{n+1}a_1e^{(n+1)\lambda\zeta} + c^na_0e^{n\lambda\zeta} \tag{4.20}$$

and so

$$(g(\zeta))^{(k)} = \lambda^kc^ne^{n\lambda z}\left(c(n+1)^ka_1e^{\lambda\zeta} + a_0n^k\right). \tag{4.21}$$

Let

$$P_{n+1}(\zeta) = a_1\zeta^{n+1} + a_0\zeta^n - 1 \tag{4.22}$$

and

$$P_1(\zeta) = (n+1)^ka_1\zeta + a_0n^k. \tag{4.23}$$

Now from (4.20) and (4.22), we see that

$$g(\zeta) - 1 = P_{n+1}\left(ce^{\lambda\zeta}\right). \tag{4.24}$$

Also from (4.21) and (4.23), we see that

$$(g(\zeta))^{(k)} = \lambda^kc^ne^{n\lambda z}P_2\left(ce^{\lambda\zeta}\right). \tag{4.25}$$

Since $g = 1 \Leftrightarrow g^{(k)} = 0$, from (4.24) and (4.25), we conclude that $P_{n+1}\left(ce^{\lambda\zeta}\right) = 0 \Leftrightarrow P_1\left(ce^{\lambda\zeta}\right) = 0$ and so $\zeta = -\frac{a_0n^k}{(n+1)^ka_1}$ is the only zeros of P_1. Therefore from (4.22), we have

$$P_{n+1}(\zeta) = a_1\left(\zeta + \frac{a_0n^k}{(n+1)^ka_1}\right)^{n+1}$$

and so from (4.24), we get

$$g(\zeta)-1=a_1\left(ce^{\lambda\zeta}+\frac{a_0n^k}{(n+1)^ka_1}\right)^{n+1}.$$

This shows that all the zeros of $g-1$ have multiplicity at least $n+1$. On the other hand by Lemma 3.6, we see that $T(r,g)=(n+1)\;T(r,h)+O(1)$ and so $S(r,g)=S(r,h)$. Now by using the second fundamental theorem, we have

$$\begin{aligned}(n+1)\;T(r,h) &= T(r,g)+O(1)\\ &\le \overline{N}(r,0;g)+\overline{N}(r,\infty;g)+\overline{N}(r,1;g)+S(r,g)\\ &\le \overline{N}(r,0;P(h))+\frac{1}{n+1}\;N(r,1;g)+S(r,g)\\ &\le T(r,h)+\frac{1}{n+1}\;T(r,g)+S(r,g)\\ &= T(r,h)+\frac{n+1}{n+1}\;T(r,h)+S(r,h)=2\;T(r,h)+S(r,h),\end{aligned}$$

which is a contradiction since $n\ge k+1$.

Sub-case 2.2.3. Suppose $\deg(P)=m\ge 2$. If $P(\zeta)\equiv a_m\zeta^m$, then we conclude that $g\ne 0$ and $g^{(k)}\ne 0$. In this case also we arrive at a contradiction. Hence $P(\zeta)\not\equiv a_m\zeta^m$. For the sake of simplicity, we may assume that $a_m,a_0\ne 0$. Note that

$$\begin{aligned}g(\zeta) &= h^n(\zeta)P(h(\zeta)) \\ &= c^{n+m}a_me^{(n+m)\lambda\zeta}+c^{n+m-1}a_{m-1}e^{(n+m-1)\lambda\zeta}+\ldots+c^na_0e^{n\lambda\zeta}\end{aligned}\tag{4.26}$$

and so

$$\begin{aligned}&(g(\zeta))^{(k)}\\ =\;& \lambda^kc^ne^{n\lambda z}\{c^m(n+m)^ka_me^{m\lambda\zeta}+c^{m-1}(n+m-1)^ka_{m-1}e^{(m-1)\lambda\zeta}+\ldots+a_0n^k\}.\end{aligned}\tag{4.27}$$

Let

$$P_{n+m}(\zeta)=a_m\zeta^{n+m}+a_{m-1}\zeta^{n+m-1}+\ldots+a_0\zeta^n-1\tag{4.28}$$

and

$$P_m(\zeta)=(n+m)^ka_m\zeta^m+(n+m-1)^ka_{m-1}\zeta^{m-1}+\ldots+a_0n^k.\tag{4.29}$$

Clearly $\deg(P_{n+m})=n+m$ and $\deg(P_m)=m$. Therefore from (4.29), we may assume that

$$P_m(\zeta)=(n+m)^ka_m(\zeta-b_1)^{m_1}(\zeta-b_2)^{m_2}\ldots(\zeta-b_s)^{m_s},\tag{4.30}$$

where $b_1,b_2,\ldots,b_s\in\mathbb{C}\setminus\{0\}$ such that $b_1\ne b_2\ne\ldots\ne b_s$ and $m_1+m_2+\ldots+m_s=m$.

Now from (4.26) and (4.28), we see that

$$g(\zeta)-1=P_{n+m}\left(ce^{\lambda\zeta}\right).\tag{4.31}$$

Also from (4.27) and (4.30), we see that

$$\begin{aligned}(g(\zeta))^{(k)} &= \lambda^kc^ne^{n\lambda z}P_m\left(ce^{\lambda\zeta}\right)\\ &= \lambda^kc^ne^{n\lambda z}(ce^{\lambda\zeta}-b_1)^{m_1}(ce^{\lambda\zeta}-b_2)^{m_2}\ldots(ce^{\lambda\zeta}-b_s)^{m_s}.\end{aligned}\tag{4.32}$$

Since $g = 1 \Leftrightarrow g^{(k)} = 0$, from (4.31) and (4.32), we conclude that $P_{n+m}\left(ce^{\lambda\zeta}\right) = 0 \Leftrightarrow P_m\left(ce^{\lambda\zeta}\right) = 0$and so $b_1, b_2, \ldots, b_s$ are the only zeros of P_{n+m}. Therefore we may assume that

$$P_{n+m}(\zeta) = a_m(\zeta - b_1)^{l_1}(\zeta - b_2)^{l_2} \ldots (\zeta - b_s)^{l_s}, \tag{4.33}$$

where $l_1 + l_2 + \ldots + l_s = m + n$.

We now want to prove that P_{n+m} has only one zero. If possible suppose that P_{n+m} has more than one zeros. For the sake of simplicity we may assume that P_{n+m} has two zeros, say b_1 and b_2. Therefore

$$P_{n+m}(\zeta) = a_m(\zeta - b_1)^{l_1}(\zeta - b_2)^{l_2}, \tag{4.34}$$

where $l_1 + l_2 = m + n$. Therefore from (4.31) and (4.34), we see that

$$g(\zeta) - 1 = a_m(ce^{\lambda\zeta} - b_1)^{l_1}(ce^{\lambda\zeta} - b_2)^{l_2}. \tag{4.35}$$

On the other hand from (4.30), we have

$$P_m(\zeta) = (n + m)^k a_m(\zeta - b_1)^{m_1}(\zeta - b_2)^{m_2}, \tag{4.36}$$

where $m_1 + m_2 = m$ and so from (4.32), we have

$$(g(\zeta))^{(k)} = \lambda^k c^n e^{n\lambda z}(ce^{\lambda\zeta} - b_1)^{m_1}(ce^{\lambda\zeta} - b_2)^{m_2}. \tag{4.37}$$

Now from (4.34), we have

$$\begin{aligned} P'_{n+m}(\zeta) &= \zeta^{n-1}\left((n + m)a_m\zeta^m + (n + m - 1)a_{m-1}\zeta^{m-1} + \ldots + na_0\right) \quad (4.38) \\ &= a_m(\zeta - b_1)^{l_1-1}(\zeta - b_2)^{l_2-1}\left((l_1^* + l_2^*)\zeta - (l_2^* b_1 + l_1^* b_2)\right), \end{aligned}$$

where

$$l^* = \begin{cases} 1, & \text{if } l = 1, \\ l - 1, & \text{if } l \geq 2. \end{cases}$$

Clearly $(l_1^* + l_2^*)b_1 - (l_2^* b_1 + l_1^* b_2) \neq 0$ and $(l_1^* + l_2^*)b_2 - (l_2^* b_1 + l_1^* b_2) \neq 0$.

Note that $n \geq k + 1 \geq 3$ and so $n - 1 \geq 2$. Since $a_0 \neq 0$, from (4.38) one can easily arrive at a contradiction.

Hence P_{n+m} has only one zero, say b_1. Consequently $P_{n+m}(\zeta) = a_m(\zeta - b_1)^{n+m}$. Therefore from (4.31), we see that

$$g(\zeta) - 1 = a_m\left(ce^{\lambda\zeta} - b_1\right)^{n+m}.$$

This shows that all the zeros of $g - 1$ have multiplicity at least $n + m$.

Now from Lemma 3.6, we have $T(r, g) = (n + m)\, T(r, h) + O(1)$ and so $S(r, g) = S(r, h)$. Therefore by using the second fundamental theorem we have

$$\begin{aligned} (n + m)\, T(r, h) &= T(r, g) + O(1) \\ &\leq \overline{N}(r, 0; g) + \overline{N}(r, \infty; g) + \overline{N}(r, 1; g) + S(r, g) \\ &\leq \overline{N}(r, 0; P(h)) + \frac{1}{n + m}\, N(r, 1; g) + S(r, g) \\ &\leq m\, T(r, h) + \frac{1}{n + m}\, T(r, g) + S(r, g) \\ &= m\, T(r, h) + \frac{n + m}{n + m}\, T(r, h) + S(r, h) \\ &= (m + 1)\, T(r, h) + S(r, h), \end{aligned}$$

which is a contradiction since $n \geq 3$. This completes the proof. $\square$

References

[1] R. Brück, On entire functions which share one value CM with their first derivative, Results Math., 30 (1996), 21–24.

[2] T. B. Cao, On the Brück conjecture, Bull. Aust. Math. Soc., 93 (2016), 248–259.

[3] J. M. Chang and L. Zalcman, Meromorphic functions that share a set with their derivatives, J. Math. Anal. Appl., 338 (2008), 1191–1205.

[4] Z. X. Chen and K. H. Shon, On conjecture of R. Brück concerning the entire function sharing one value CM with its derivative, Taiwanese J. Math., 8 (2004), no. 2, 235–244.

[5] G. G. Gundersen, Meromorphic functions that share finite values with their derivative, J. Math. Anal. Appl., 75 (1980), 441-446; Errata, 86 (1982), 307.

[6] G. G. Gundersen and L. Z. Yang, Entire functions that share one value with one or two of their derivatives, J. Math. Anal. Appl., 223 (1998), 88–95.

[7] W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.

[8] I. Laine, *Nevanlinna theory and complex differential equations*, Walter de Gruyter & Co., Berlin, 1993.

[9] E. Mues and N. Steinmetz, Meromorphe Funktionen, die mit ihrer Ableitung Werte teilen, Manuscripta Math., 29 (1979), 195–206.

[10] L. A. Rubel and C. C. Yang, Values shared by an entire function and its derivative, Complex analysis (Proc. Conf., Univ. Kentucky, Lexington, Ky., 1976), pp. 101-103, Lecture Notes in Math., Vol. 599, Springer-Verlag, Berlin-New York, 1977.

[11] J. L. Schiff, *Normal families*, Springer-Verlag, New York, 1993.

[12] C. C. Yang, On deficiencies of differential polynomials, Math. Z., 116 (1970), 197–204.

[13] C. C. Yang and H.-X. Yi, *Uniqueness theory of meromorphic functions*, Math. Appl., 557, Kluwer Academic Publishers Group, Dordrecht, 2003.

[14] L. Z. Yang and J. L. Zhang, Non-existence of meromorphic solutions of Fermat type functional equation, Aequations Math., 76 (1-2) (2008), 140–150.

[15] L. Zalcman, Normal families: new perspectives, Bull. Amer. Math. Soc., 35 (1998), 215–230.

[16] J. L. Zhang and L. Z. Yang, A power of an entire function sharing one value with its derivative, Comput. Math. Appl., 60 (2010), 2153–2160.

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