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$P^{\maltese}$-PROPERTY AND S-CYCLIC MAPPINGS IN GEODESIC SPACES

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Abstract. We investigate the geometric notions of P and $P^{\maltese}$-properties. This way, we prove a best proximity point existence theorem for, the newly introduced, S-cyclic relatively nonexpansive mappings in the setting of geodesic metric spaces. As a consequence, we get both a best proximity point and a fixed point result for cyclic relatively nonexpansive mappings.

1. Introduction

Since the beginning of the current century, best proximity points of cyclic mappings fulfilling certain contractive conditions have been significantly investigated. Let A, B be nonempty subsets of a metric space X, a mapping $T : A\cup B \longrightarrow A\cup B$ is said to be cyclic (*resp.* noncyclic) if $T(A) \subseteq B$ and $T(B) \subseteq A$ (*resp.* $T(A) \subseteq A$ and $T(B) \subseteq B$). If the distance between a point in $A \cup B$ and its image is the least of all distances between points of A and B, then it's called a best proximity point. Studying the existence of such point, the authors of [6] introduced a new class of mappings and proved, under the condition of proximal normal structure, the existence of best proximty point thereof.

Definition 1.1. ([6]) Retaining the same notations, T is said to be relatively nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x \in A$ and $y \in B$.

Theorem 1.2. ([6]) *Let (A, B) be a nonempty, weakly compact convex pair in a Banach space such that (A, B) has proximal normal structure, then all cyclic relatively nonexpansive mappings on $A \cup B$ have a pair of best proximity points.*

Afterwards, the research on the subject took many different paths. Widening up the framework of the study, introducing more general contractive conditions, weaker conditions on the sets and also defining new algorithms converging to the best proximity point.

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Very recently, the current authors introduced the class of tricyclic mappings and best proximity points thereof. A mapping $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ is said to be tricyclic provided that $T(A) \subseteq B$, $T(B) \subseteq C$ and $T(C) \subseteq A$, where A, B, C are nonempty subsets of a metric space (X, d), and a best proximity point of T is a point $x \in A \cup B \cup C$ such that $D\left(x, Tx, T^2x\right) = D(A, B, C)$ where the mapping $D : X \times X \times X \longrightarrow [0, +\infty)$ is defined by $D(x, y, z) := d(x, y) + d(y, z) + d(z, x)$ and $D(A, B, C) := \inf\{D(x, y, z) : x \in A,\ y \in B \text{ and } z \in C\}$. For detailed informations, we refer to [12, 13].

In this work, we take a deeper look at the geometric notions of P and $P^{\maltese}$-properties for both pairs and triads of $CAT(0)$ spaces. This way, we are enabled to prove a best proximity point existence theorem for S-cyclic relatively nonexpansive mappings in the setting of reflexive and Busemann convex metric spaces. As a consequence, we get a new best proximity point result for cyclic relatively nonexpansive mappings, but with a fixed point conclusion as well.

2. Preliminaries

In this section, we assemble all the concepts and results needed for conducting our study. We start by recalling the spaces where we will mainly be working, geodesic spaces. A metric space (X, d) is called a (*uniquely*) geodesic space if two arbitrary points x and y of X can be joined by a (*unique*) geodesic, *i.e.* an isometry $c : [0, l] \subseteq \mathbb{R} \longrightarrow X$ such that $c(0) = x$ and $c(l) = y$. The image $c([0, l])$ is called a geodesic segment and is denoted $[x, y]$. A convex subset of X contains all the geodesic segments joining its points. A point z is in the segment $[x, y]$ if and only if there exists $t \in [0, 1]$ such that $d(x, z) = td(x, y)$ and $d(y, z) = (1 - t) d(x, y)$, then, z is simply written as $(1 - t) x \oplus ty$. A geodesic metric space X is said to be reflexive if for every decreasing chain $\{C_i\} \subset X$ such that C_i is nonempty, bounded, closed and convex for all i we have $\cap_i C_i$ is nonempty. Obviously, reflexive metric spaces generalize the notion of reflexivity from Banach to metric spaces. A geodesic triangle consists of three pointd of X, called vertices, and three geodesic segments joining each pair of vertices, called edges. To learn more about geodesic spaces, we refer [2, 3].

In a uniquely geodesic space (X, d), the metric $d : X \times X \longrightarrow \mathbb{R}$ is said to be convex if for all x, y, z of X,

$$d(x, (1 - t) y \oplus tz) \leq (1 - t) d(x, y) + td(x, z).$$

A geodesic space (X, d) is said to be convex in the sense of Busemann [1] if for all pairs of geodesics $c_1 : [0, l_1] \longrightarrow X$ and $c_2 : [0, l_2] \longrightarrow X$,

$$d(c_1(tl_1), c_2(tl_2)) \leq (1 - t) d(c_1(0), c_2(0)) + td(c_1(l_1), c_2(l_2)), \text{ for all } t \in [0, 1].$$

Equivalently, the geodesic metric space (X, d) is convex in the sense of Busemann provided that

$$d((1 - t) x \oplus ty, (1 - t) z \oplus tw) \leq (1 - t) d(x, z) + td(y, w),$$

for all $x, y, z, w \in X$ and $t \in [0, 1]$.

It was established that Busemann convex spaces are uniquely geodesic and of convex metric [7].

A very substantial subclass of Busemann convex spaces is $CAT(0)$ spaces, spaces of nonpositive curvature in the sense of Gromov. A brief description of these spaces is given below.

Let (X,d) be a uniquely geodesic metric space. A comparison triangle for a geodesic triangle $\Delta(x_1,x_2,x_3)$ in X is a triangle in the Euclidean plane $\Delta(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ such that $d(\bar{x}_i,\bar{x}_j)=d(x_i,x_j)$ for all $i,j\in\{1,2,3\}$. Comparison triangles exist and are unique up to an isometry. A point $\bar{p}\in[\bar{x}_1,\bar{x}_2]$ is said to be a comparison point of $p\in[x_1,x_2]$ if $d(\bar{x}_1,\bar{p})=d(x_1,p)$ and $d(\bar{x}_2,\bar{p})=d(x_2,p)$.The geodesic metric space X is said to be a CAT(0) space if and only if every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane. That is, for every $p,q\in\Delta(x_1,x_2,x_3)$, we have $d(p,q)\leq d(\bar{p},\bar{q})$ where $\bar{p}$ and $\bar{q}$ are their comparison points.

Hadamard spaces are referred to complete $CAT(0)$ spaces.

Orthogonal projections onto closed and convex subsets of *Hadamard* spaces have, in certain sense, the same behaviour as in Hilbert spaces.

Proposition 2.1. ([3], *Proposition* 2.4, *p.*176) Let X be a complete $CAT(0)$ space and C a nonempty, closed and convex subset of X. Then

(1) for every $x\in X$, there exists a unique point $\pi(x)\in X$ such that $d(x,\pi(x))=d(x,C):=\inf_{y\in C}d(x,y)$;
(2) if x' belongs to the geodesic segment $[x,\pi(x)]$, then $\pi(x')=\pi(x)$;
(3) given $x\notin C$ and $y\in C$, if $y\neq\pi(x)$ then $\angle_{\pi(x)}(x,y)\geq\frac{\pi}{2}$.

For a scrutinization of $CAT(0)$ spaces and other Metric Geometry topics, see [3,9].

Let's now set some notations and definitions that will subsequently be used. Let (A,B,C) be a triad of nonempty subsets of a metric space (X,d), we set:

$$\begin{aligned} D(x,y,z) &= d(x,y)+d(y,z)+d(z,x), \\ D(A,B,C) &= \inf\{D(x,y,z): x\in A,\ y\in B \text{ and } z\in C\}. \end{aligned}$$

The proximal pair (A_0,B_0) of (A,B) is given by:

$$\begin{aligned} A_0 &:= \{x\in A: d(x,y')=dist(A,B) \text{ for some } y'\in B\}, \\ B_0 &:= \{y\in B: d(x',y)=dist(A,B) \text{ for some } x'\in A\}. \end{aligned}$$

While the proximal triad (A_{00},B_{00},C_{00}) of (A,B,C) is defined by:

$$\begin{aligned} A_{00} &:= \{x\in A: D(x,y',z'')=D(A,B,C) \text{ for some } y'\in B \text{ and } z''\in C\}, \\ B_{00} &:= \{y\in B: D(x'',y,z')=D(A,B,C) \text{ for some } x''\in A \text{ and } z'\in C\}, \\ C_{00} &:= \{z\in C: D(x',y'',z)=D(A,B,C) \text{ for some } x'\in A \text{ and } y''\in B\}. \end{aligned}$$

Note that if (A,B,C) is a triad of nonempty, compact and convex subsets of a Busemann convex space, then so is the triad (A_{00},B_{00},C_{00}), and that $D(A,B,C)=D(A_{00},B_{00},C_{00})$. Indeed, let $x_1,x_2\in A_{00}$, that is, $D(x_1,y_1,z_1)=D(x_2,y_2,z_2)=D(A,B,C)$ for some $y_1,y_2\in B$ and $z_1,z_2\in C$. A direct usage of the convexity in the sense of Busemann implies that, for all $t\in[0,1]$

$$D((1-t)x_1\oplus tx_2,(1-t)y_1\oplus ty_2,(1-t)z_1\oplus tz_2)=D(A,B,C).$$

We also point out that in the special case where $C=A$, we have $A_{00}=A_0$ and $B_{00}=B_0$.

Besides ordinary cyclic and noncyclic mappings, the new class of S-cyclic mappings was introduced in [5].

Definition 2.2. ([5]) Let (A, B, C) be a triad of nonempty subsets of a metric space (X, d). A mapping $T : A \cup B \longrightarrow B \cup C$ is said to be

- S-cyclic if, $T(A) \subseteq B$ and $T(B) \subseteq C$.
- S-cyclic relatively nonexpansive if T is S-cyclic and $d(Tx, Ty) \leq d(x, y)$ for all $x \in A$ and $y \in B$.

It is clear than in the special case where $A = C$, we return to a cyclic mapping. Hence, from every result established for S-cyclic mappings, a result for cyclic mappings can be deduced.

In [6], Eldred *et al*, introduced the notion of proximal normal structure to study relatively nonexpansive mappings. Their concept is a semblable to the famous normal structrue in the sense of Brodski and Milman, which is used for nonexpansive mappings.

Definition 2.3. ([6]) A convex pair (K_1, K_2) in a geodesic space is said to have proximal normal structure if for all closed, bounded and convex proximal pairs $(H_1, H_2) \subseteq (K_1, K_2)$ for which $dist(H_1, H_2) = dist(K_1, K_2)$ and $\delta(H_1, H_2) > dist(H_1, H_2)$, there exists $(x_1, x_2) \in H_1 \times H_2$ such that

$$\delta(x_1, H_2) < \delta(H_1, H_2) \text{ and } \delta(H_1, x_2) < \delta(H_1, H_2).$$

In the linear case, the pair (K, K) has proximal normal structure if and only if K has normal structure in the sense of [4].

3. Main Results

We start our main results by recalling the P and $P^{\maltese}$-properties for pairs and triads and provide sufficient conditions so they are fulfilled in $CAT(0)$ spaces. The two geometric properties are the cornerstone of the proof of our main result.

In [11], the geometric notion of P-property was introduced for pairs and it was extended for triads in [5].

Definition 3.1. Let (A, B, C) be a triad of nonempty subsets of a metric space (X, d). The pair (A, B) is said to have the the P-property if and only if

$$\left.\begin{array}{l} d(x_1, y_1) = dist(A, B) \\ d(x_2, y_2) = dist(A, B) \end{array}\right\} \Longrightarrow d(x_1, x_2) = d(y_1, y_2).$$

And, we shall say that (A, B, C) has the P-property if and only if

$$\left.\begin{array}{l} D(x_1, y_1, z_1) = D(A, B, C) \\ D(x_2, y_2, z_2) = D(A, B, C) \end{array}\right\} \Longrightarrow d(x_1, x_2) = d(y_1, y_2) = d(z_1, z_2).$$

For all $x_1, x_2 \in A$, $y_1, y_2 \in B$ and $z_1, z_2 \in C$.

The new notion of $P^{\maltese}$-property for both pairs and triads was presented as follows.

Definition 3.2. ([5]) Let (A, B, C) be a triad of nonempty subsets of a metric space (X, d). We say that (A, B) has the $P^{\maltese}$-property if

$$\left.\begin{array}{l} d(x_1, y_1) = dist(A, B) \\ d(x_2, y_2) = dist(A, B) \end{array}\right\} \Longrightarrow d(x_1, y_2) = d(x_2, y_1).$$

We also shall say that (A, B, C) has the $P^{\maltese}$-property provided that

$$\left.\begin{array}{l} D(x_1, y_1, z_1) = D(A, B, C) \\ D(x_2, y_2, z_2) = D(A, B, C) \end{array}\right\} \Longrightarrow d(x_1, y_2) = d(y_1, z_2) \text{ and } d(x_2, y_1) = d(y_2, z_1).$$

For all $x_1, x_2 \in A$, $y_1, y_2 \in B$ and $z_1, z_2 \in C$.

The P and $P^{\maltese}$-properties for pairs, can, particularly, be sufficient conditions for triads to have those properties as well.

Proposition 3.3. Let (A, B, C) be a triad of nonempty subsets of a metric space (X, d) such that $D(A, B, C) = dist(A, B) + dist(B, C) + dist(C, A)$. Then (A, B, C) has the P-property if (A, B) and (B, C) do. Moreover, in the special case where $A = C$, the traid (A, B, C) has the $P^{\maltese}$-property if the pair (A, B) does.

Proof. Suppose $D(x_1, y_1, z_1) = D(x_2, y_2, z_2) = D(A, B, C)$, for some $x_1, x_2 \in A$, $y_1, y_2 \in B$ and $z_1, z_2 \in C$. It is immediate that

$$\left\{\begin{array}{l} d(x_1, y_1) = dist(A, B) \\ d(x_2, y_2) = dist(A, B) \end{array}\right. \text{ and } \left\{\begin{array}{l} d(y_1, z_1) = dist(B, C) \\ d(y_2, z_2) = dist(B, C) \end{array}\right.$$

Now, if (A, B) has the $P^{\maltese}$-property and $A = C$, we have $D(A, B, C) = D(A, B, A) = 2dist(A, B)$. Then we necessarily have

$$x_1 = z_1, \ x_2 = z_2 \text{ and } d(x_1, y_1) = d(x_2, y_2) = dist(A, B).$$

Hence, (A, B, A) has the $P^{\maltese}$-property. □

Now, we prove the $P^{\maltese}$-property for a particular triad in a Hadamard space.

Proposition 3.4. Let (A, B, C) be a triad of nonempty, closed and convex subsets of a complete $CAT(0)$ space X such that $D(A, B, C) = dist(A, B) + dist(B, C) + dist(C, A)$ and $dist(A, B) = dist(B, C)$. Then (A, B, C) has $P^{\maltese}$-property.

Proof. Let $x_1, x_2 \in A$, $y_1, y_2 \in B$ and $z_1, z_2 \in C$, such that $D(x_1, y_1, z_1) = D(x_2, y_2, z_2) = D(A, B, C)$, then

$$d(x_1, y_1) = d(x_2, y_2) = dist(A, B) \text{ and } d(y_1, z_1) = d(y_2, z_2) = dist(B, C).$$

Which implies,

$$y_1 = \pi_{[y_1, y_2]}(x_1),$$

where $\pi_{[y_1, y_2]}$ is the projection on the convex set $[y_1, y_2]$. According to ([3], Proposition 2.4, p.176), the Alexandrov angle between x_1 and y_2 at y_1, $\angle_{y_1}(x_1, y_2)$, is at least $\frac{\pi}{2}$. The same goes, of course, for $\angle_{y_2}(y_1, x_2)$, $\angle_{x_2}(y_2, x_1)$ and $\angle_{x_1}(x_2, y_1)$.

Now, we consider the quadrilateral in X which is the union of geodesic segments $[x_1, x_2], [x_2, y_2], [y_2, y_1]$ and $[y_1, x_1]$. Since all of the angles of this quadrilateral are at least $\frac{\pi}{2}$, we conclude, by The Flat Quadrilateral Theorem (see [3, *p*.181]) that the convex hull of $\{x_1, x_2, y_2, y_1\}$ is isometric to a Euclidean rectangle, say $\{a_1, a_2, b_2, b_1\}$. The exact same arguement shows that convex hull of $\{y_1, y_2, z_2, z_1\}$ is also isometric to a Euclidean rectangle, say $\{c_1, c_2, d_2, d_1\}$. Since $dist(A, B) = dist(B, C)$,

$$d(a_1, b_1) = d(c_1, d_1) \text{ and } d(a_2, b_2) = d(c_2, d_2).$$

And, by (*Lemma* 4.3 of [10]), we get,

$$d(x_1, x_2) = d(y_1, y_2) = d(z_1, z_2),$$

which means,

$$d(a_1,a_2)=d(b_1,b_2)=d(c_1,c_2)=d(d_1,d_2),$$

Thus, $\{a_1,a_2,b_2,b_1\}$ and $\{c_1,c_2,d_2,d_1\}$ are identical. Therefore,

$$\begin{aligned} d(x_1,y_2) &= d(a_1,b_2)=d(c_1,d_2)=d(y_1,z_2), \\ d(x_2,y_1) &= d(a_2,b_1)=d(c_2,d_1)=d(y_2,z_1). \end{aligned}$$

That is the $P^{\maltese}$-property for (A,B,C). □

As a ready consequence (*when* $C=A$), we get the following result for pairs.

Corollary 3.5. Let (A,B) be a pair of nonempty, closed and convex subsets of a Hadamard space. Then (A,B) has $P^{\maltese}$-property.

When the subsets A,B and C are mentioned with no additional specifications on their nature, they are nonempty, closed and convex. Now, we turn to the proximal pair (A_0,B_0) and establish some basic properties.

In the Euclidean plane, elementary geometry shows that A_0 and B_0 are either singletons or segments. However, that's not usually the case nor it is the case for a linear space whose dimension is greater than two either (In $\mathbb{R}^3$, consider A and B two "parallel" cubes). Nevertheless, we can learn a couple of things about A_0 and B_0 in the setting of $CAT(0)$ spaces.

Proposition 3.6. If A_0 contains more than one element then so does B_0.

Proof. Let x and y be two different points of A_0. Suppose B_0 contains only one element, say z, and consider the isosceles, comparaison triangle $\bar{\Delta}(\bar{x},\bar{y},\bar{z})$ in the Euclidean plane E^2. Let m be the midpoint if the geodesic segment $[x,y]$, then

$$d(m,z)\le d_{E^2}(\bar{m},\bar{z})<d_{E^2}(\bar{x},\bar{z})=d(x,z)=dist(A,B).$$

And, that means m is outside of A, which contradicts the fact that A is convex. □

Before stating the next property, we recall the following important theorem.

Theorem 3.7. *([3], Theorem 6.8, p. 231) Let X be a $CAT(0)$ space.*

(1) An isometry γ of X is hyperbolic if and only if there exists a geodesic line $c:\mathbb{R}\longrightarrow X$ which is translated non-trivially by γ, namely $\gamma.c(t)=c(t+a)$, for some $a>0$. The set $c(\mathbb{R})$ is called an axis of γ. For any such axis, the number a is actually equal to $|\gamma|$.
(2) The axes of γ are parallel to each other.

For a detailed study of isometries of $CAT(0)$ spaces, we refer the reader to ([3], Chapter II.6).

Proposition 3.8. Suppose A_0 is not a singleton. Any two different points of A_0 are from the same geodesic line that is parallel to the geodesic line that passes through their correspondant two points in B_0.

Proof. Let x_1,x_2 be two different points of A_0 and y_1,y_2 be their correspondant points in B_0, *i.e.*

$$d(x_1,y_1)=d(x_2,y_2)=dist(A,B).$$

And, by ([10], Lemma 4.3), we get

$$d(x_1,x_2)=d(y_1,y_2).$$

Let γ be a hyperbolic isometry such that $\gamma(x_1)=x_2$, $\gamma(y_1)=y_2$ and of $|\gamma|=d(x_1,x_2)$. Since γ is hyperbolic, we may consider two geodesic lines $c,c\prime:\mathbb{R}\longrightarrow X$ such that

$$\gamma.c(t)=c(t+|\gamma|) \text{ and } \gamma.c'(t)=c'(t+|\gamma|).$$

This gives

$$\gamma.c(0)=c(|\gamma|) \text{ and } \gamma.c'(0)=c'(|\gamma|).$$

We may assume, by reparameterizing if necessary, that

$$c(0)=x_1,\ c(d(x_1,x_2))=x_2 \text{ and } c'(0)=y_1,\ c'(d(y_1,y_2))=y_2.$$

And, since c and $c\prime$ are axes of γ, they are parallel by the previous theorem. □

Unlike cyclic relatively nonexpansive mappings, the inclusions $T(A_{00})\subseteq B_{00}$ and $T(B_{00})\subseteq C_{00}$ are not always granted for $S-cyclic$ relatively nonexpansive mappings.

Example 3.9. In the Euclidean plane, take $A=\{0\}\times[0,1]$, $B=\{1\}\times\left[\frac{3}{2},2\right]$ and $C=\left\{\frac{3}{2}\right\}\times\left[\frac{3}{2},2\right]$. It is clear that $A_{00}=\{(0,1)\}$, $B_{00}=\left\{\left(1,\frac{3}{2}\right)\right\}$ and $C_{00}=\left\{\left(\frac{3}{2},\frac{3}{2}\right)\right\}$. Since $dist(A,B)>\delta(B,C)$, then any $S-cyclic$ mapping defined on $A\cup B$ is relatively nonexpansive. It suffices to consider a mapping T that satisfies $T(0,1)=(1,2)$ and $T\left(1,\frac{3}{2}\right)=\left(\frac{3}{2},2\right)$ to see that we don't have neither $T(A_{00})\subseteq B_{00}$ nor $T(B_{00})\subseteq C_{00}$.

The next lemma has a decisive role in proving our main result.

Lemma 3.10. ([5]) *Let (A,B,C) be a triad of nonempty, closed subsets of a metric space (X,d) such that A_{00} is nonempty. Assume that the triad (A,B,C) has both the (P) property and $\left(P^{\maltese}\right)$ property. Then there exists a bijective, S-cyclic isometry $g:A_{00}\cup B_{00}\longrightarrow B_{00}\cup C_{00}$ such that $D\left(x,gx,g^2x\right)=D\left(g^{-1}y,y,gy\right)=D(A,B,C)$ for all $x\in A_{00}$ and $y\in B_{00}$.*

Now, that all the necessary tools are furnished, we state our main theorem.

Theorem 3.11. *Let X be a reflexive and Busemann convex metric space X and let (A,B,C) be a nonempty, closed and convex triad of X such that A_{00} is bounded and (A_{00},B_{00}) has proximal normal structure. Suppose (A,B,C) has the $\left(P^{\maltese}\right)$ property and verifies $D(A,B,C)=dist(A,B)+dist(B,C)+dist(C,A)$. Let $T:A\cup B\longrightarrow B\cup C$ be a $S-cyclic$ relatively nonexpansive mapping such that $T(A_{00})\subseteq B_{00}$, $T(B_{00})\subseteq C_{00}$. Then, T has a best proximity point.*

Proof. Proposition 9, says that (A,B,C) has the (P) property if (A,B) and (B,C) do, which is the case as stated in ([10], Lemma 4.3). Consider the bijective, $S-cyclic$ isometry $g:A_{00}\cup B_{00}\longrightarrow B_{00}\cup C_{00}$ defined in the previous lemma. As $T(A_{00})\subseteq B_{00}$, $T(B_{00})\subseteq C_{00}$, the mapping $g^{-1}T:A_{00}\cup B_{00}\longrightarrow A_{00}\cup B_{00}$ is noncyclic. We also have

$$d\left(g^{-1}Tx,g^{-1}Ty\right)=d(Tx,Ty)\leq d(x,y),$$

for all $x\in A_{00}$ and $y\in B_{00}$. Thus $g^{-1}T$ is a noncyclic relatively nonexpansive mapping. By ([8], Theorem 3.4), $g^{-1}T$ has a best proximity pair, say $(x,y)\in A_{00}\times B_{00}$, i.e.

$g^{-1}Tx = x$, $g^{-1}Ty = y$ and $d(x,y) = dist(A_{00}, B_{00}) = dist(A,B)$. On the other hand we have

$$\begin{aligned} d(x,gx) + d\left(gx, g^2x\right) + d\left(g^2x, x\right) &= D\left(x, gx, g^2x\right) \\ &= D(A,B,C) \\ &= dist(A,B) + dist(B,C) + dist(C,A). \end{aligned}$$

This implies that $d(x,gx) = dist(A,B)$. Since the pair (A,B) has the (P) property, we obtain $gx = y$. And since $Tx = gx$ and $Ty = gy$, we have $T^2x = g^2x$. Then

$$D\left(x, Tx, T^2x\right) = D\left(x, gx, g^2x\right) = D(A,B,C).$$

And that finishes the proof. □

Apart from the $\left(P^{\maltese}\right)$, we get a result which has the same suppositions as $([8], \text{Theorem } 3.3)$ but with a fixed point conclusion too and proximal normal structure only satisfied only by (A_0, B_0).

Theorem 3.12. *Let (A,B) be a nonempty, closed and convex pair of a reflexive and Busemann convex metric space X. Suppose A_0 is bounded, (A,B) has the $\left(P^{\maltese}\right)$ property and (A_0, B_0) has proximal normal structure. Let $T : A \cup B \longrightarrow A \cup B$ be a cyclic relatively nonexpansive mapping. Then T has best proximity point which is also a fixed point of T^2.*

Proof. The triad (A,B,A) is nonempty, closed and convex, it has both the (P) and $\left(P^{\maltese}\right)$ properties and verifies $D(A,B,A) = 2dist(A,B)$. Moreover, $T : A \cup B \longrightarrow B \cup A$ is a $S - cyclic$ relatively nonexpansive mapping such that $T(A_0) \subseteq B_0$, $T(B_0) \subseteq A_0$. Now, the previous theorem guarantees the existence of a point $x \in A$ such that

$$D\left(x, Tx, T^2x\right) = 2dist(A,B).$$

Therefore, we necessarily get

$$x = T^2x \text{ and } d(x,Tx) = dist(A,B).$$

And that's the desired result. □

Corollary 3.13. Let (A,B) be a nonempty, closed and convex pair of a Hadamard space X. Suppose A_0 is bounded and (A_0, B_0) has proximal normal structure. Let $T : A \cup B \longrightarrow A \cup B$ be a cyclic relatively nonexpansive mapping. Then T has best proximity point which is also a fixed point of T^2.

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