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VOLTERRA PROPERTIES IN GENERALIZED TOPOLOGICAL SPACES

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Abstract. In this paper, we study Volterra properties in generalized topological spaces and also those Volterra properties in subspaces by using compactness and separation properties.

1. Introduction

In 1881, Vito Volterra [19] introduced an interesting and important class of spaces, now called after his name, *Volterra spaces* about which he obtained several important and interesting results. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that both the sets of points where f is continuous and its complement are dense. Then there is no function $g : \mathbb{R} \to \mathbb{R}$ such that the set of points where g is continuous is precisely the set of points where f is discontinuous. In 1993, the concept of Volterra spaces was restated by Gauld et al. [[6], Definition 15], and proved that a space X is Volterra if for each $f, g\colon \mathbb{X} \to \mathbb{R}$ for which $C(f)$ and $C(g)$ are dense in X, the set $C(f)\cap C(g)$ is also dense in X where $C(f)$ denotes the set of points of X at which f is continuous. In [7], they also proved that X is strongly Volterra (respectively, Volterra) if $C(f) \cap C(g)$ is dense in X (respectively, non-empty) whenever $f, g\colon X \to \mathbb{R}$ are two functions for which $C(f)$ and $C(g)$ are dense in X. In [8], the concepts of Volterra and weakly Volterra were stabilized and the authors proved that a topological space X is Volterra (respectively, weakly Volterra) if for every pair G and H of dense G_δ subsets of X, the set $G \cap H$ is dense (respectively, non-empty). A subset of X is called G_δ if it is the intersection of countably many open subsets of X. In 2005, Cao et al.[1] revisited the papers of Gauld et al.[7, 8] and also studied Volterraness in homogeneous spaces. Spadaro [17] established the relation between P-spaces and the Volterra property. Milan Matejdes (see [14] and [15]) studied the basic properties of weak ε-Volterra and ε-Volterra spaces which correspond to the known results of the Volterra

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and weakly Volterra spaces. Vadakasi [18] obtained several interesting and important results on generalized topologies. Peng and Yang [16] provided some conditions under which a Baire space is equivalent to a Volterra space. In [9] and [2], the authors studied relationships between Baire and Volterra properties in topological spaces. A topological space X is called a *Baire space* if the intersections of countably many dense open subsets of X are also dense.

Császár [5] initiated discussion on the theory of generalized topological spaces. In what follows, we denote a generalized topological space by GTS and generalized topological spaces by $GTSs$. The definition of GTS is recalled below. Further, he studied various basic operators related to $GTSs$ and discovered many important families of sets in $GTSs$ [3, 4]. Li and Lin[13] defined g-dense, g-nowhere dense and g-residual in X and also studied Baire properties in $GTSs$. We [10] introduced the concepts of g-Volterra space and g-first countable space and studied their relationship. In [11], we studied the relation between g-developable spaces and g-Volterra spaces in $GTSs$. Further, we [12] introduced the concept of weakly g-Volterra spaces in $GTSs$ and studied the relation between g-Volterra spaces and weakly g-Volterra spaces in $GTSs$. In this paper, we study Volterra properties in $GTSs$.

Next, we recall some basic concepts and the known results which are useful for the present study. Let X be a non-empty set and $\mathcal{P}$(X) be the power set of X. A family of sets $g \subset \mathcal{P}$(X) is called a *generalized topology* on X, if (i) $\emptyset \in g$ (ii) The union of elements of g belongs to g. The pair (X, g) is called a GTS. The elements of g are called *g-open* subsets of X and the complements are called *g-closed* subsets of X.
Let (X, g) be a GTS. Let $x \in X$ and $A \subset X$. The closure of A and interior of A in (X, g) are defined as $c_g(A) = \bigcap\{F \supseteq A : X\backslash F \in g\}$ and $i_g(A) = \bigcup\{U : U \in g \text{ and } U \subset A\}$. A point $x \in X$ is called a *g-cluster point* of A if $U \cap (A \setminus \{x\}) \neq \emptyset$ for each U in g with $x \in U$.

Definition 1.1. [13] A subset A of X is called

i) *g-dense* in X if $c_g(A) = X$.
ii) *g-nowhere dense* in X if $i_g c_g(A) = \emptyset$.
iii) *g-preopen* if $A \subset i_g(c_g(A))$.

The family of all g-dense(resp. g-nowhere dense) subsets of (X, g) is denoted by $D(g)$(resp. $N(g)$). The family of all g-preopen sets in (X, g) is denoted by $\pi(g)$ or $\pi(g_X)$, the union of all elements of g will be denoted by M_g.
Also, (i) $c_g(A) = X \setminus i_g(X \setminus A)$ (ii) $i_g(A) = X \setminus c_g(X \setminus A)$.

Let (X, g) be a GTS and $A \subset X$. Then $A \in N(g)$ if and only if $c_g(A)$ is g-codense in X, that is $X \setminus c_g(A) \in D(g)$. A GTS (X, g) is called strong [4] if $M_g = X$. We denote it by $sGTS$.

Definition 1.2. [13] Let (X, g) be a GTS. Then a subset A of X is called

(i) of *g-first category* in X if there exists a sequence $\{A_n\}$ consisting of g-nowhere dense subsets of X such that $A = \bigcup\limits_{n \in N} A_n$.
(ii) of *g-second category* in X, if A is not g-first category in X.
(iii) of *g-residual* in X, if $X \setminus A$ is g-first category in X.

The family of all g-first category subsets of (X, g) is denoted by $M(g)$.

Definition 1.3. [13] Let (X, g) be a GTS and $A \subset S \subset X$. Then (S, g_S), where $g_S = \{U \cap S : U \in g\}$ is the relative generalized topology on S, is a GTS which is called a subspace of (X, g). We denote the closure of A and the interior of A in the subspace (S, g_S) by $c_{g_S}(A)$ and $i_{g_S}(A)$ respectively.

Definition 1.4. [10] A GTS (X, g) is said to be *g-Volterra space* if for each pair G and H of g-dense G_δ subsets of X, $G \cap H$ is g-dense in X.

Example 1.5. Let X be the set of all natural numbers. Consider $A = \{1, 3, 5, ...\}$ and $g = \{\phi\} \cup \{A \cup B : \phi \neq B \subset \{2, 4, 6, ...\}\}$. Then (X, g) is a $sGTS$ but g is not a topology on X. Clearly A is g-dense in X. Here $U \cap A \neq \phi$ for any $U \in g - \{\phi\}$. By Proposition 3.3 in [13], $A \in D(g)$. Therefore, $g - \{\phi\} \subset D(g)$. Let G and H be g-dense G_δ subsets of X. Also, $A \subset G \cap H$, $G \cap H \in D(g)$. Hence (X, g) is g-Volterra.

Definition 1.6. [11] Let (X, g) be a GTS. For every pair of g-dense G_δ subsets G and H of X. If $G \cap H$ is non-empty then X is a *weakly g-Volterra space.*

Example 1.7. Consider X to be the set of all natural numbers and $g = \{\emptyset, N - \{a\}\}$ where $a \in N$. Then (X, g) is a GT space but (X, g) is not a topology on X. Let G and $H \in g - \{\emptyset\}$. For every pair G and H of g-dense G_δ subsets of X, $G \cap H$ is non-empty. Hence, (X, g) is weakly g-Volterra.

Definition 1.8. [10] Let (X, g) be a GTS and $x \in X$. A *g-local base* of x denoted by $\mathbf{C}_x$ is a collection of g-open neighbourhoods of x such that for all $U \in g$ with $x \in U$ there exists $C \in \mathbf{C}_x$ such that $x \in C \subset U$.

Definition 1.9. [20] Let X be a GTS. By g_x we mean $g_x = \{U : x \in U \in g\}$.

Definition 1.10. [20] Let (X, g) be a GTS. Then X is called a *gT_2-space* (*gT_3-space* or *g-regular space*, *gT_4-space* respectively) if X satisfies the following gT_2-separation (gT_3-separation, gT_4-separation) conditions:

(i) If $x, y \in X$ and $x \neq y$, then there are $U_x \in g_x$ and $U_y \in g_y$ such that $U_x \cap U_y = \emptyset$.
(ii) gT_3-separation: If $x \notin A$ with A being g-closed, then there are $U_x \in g_x$ and $U_A \in g$ such that $A \subset U_A$ and $U_x \cap U_A = \emptyset$.
(iii) gT_4-separation: If A_1 and A_2 are g-closed and $A_1 \cap A_2 = \emptyset$, then there are U_1, $U_2 \in g$ with $A_1 \subset U_1$ and $A_2 \subset U_2$ such that $U_1 \cap U_2 = \emptyset$.

Lemma 1.11. [13] *Let (X, g) be a sGTS. The following conditions are equivalent.*

(i) *X is a gT_3-space.*
(ii) *For any $x \in X$ and every g-closed set F with $x \notin F$, there exists g-open set U such that $c_g(U) \cap F = \emptyset$.*
(iii) *For any $x \in X$ and $U_x \in g_x$, there exists $V_x \in g_x$ such that $x \in V_x \subset c_g(V_x) \subset U_x$.*

2. Volterra Properties in $GTSs$

Definition 2.1. A subset η of $\mathcal{P}$(X) is said to satisfy the *finite intersection property* if for every finite subcollection $\{F_1, F_2, ..., F_n\}$ of η, the intersection $F_1 \cap F_2 \cap ... \cap F_n$ is non-empty.

Definition 2.2. Let (X, g) be a *sGTS*. Then X is called *g-compact* if every g-open cover of X has a finite subcover.

Lemma 2.3. *Let (X, g) be a GTS. Then X is g-compact if and only if for every collection of g-closed sets of X which satisfy the finite intersection condition, the intersection of all the g-closed sets in the collection is non-empty.*

Proof. Suppose ζ is a collection of subsets of X. Let $\xi = \{X \setminus G : G \in \zeta\}$ be the collection of their complements. Then we have the following equivalent statements:

(i) ζ is a collection of g-open sets if and only if ξ is a collection of g-closed sets in X.

(ii) The collection ζ is a g-open cover of X if and only if the intersection of all the g-closed elements in ξ is empty.

(iii) The finite subcollection $\{G_1, G_2, ..., G_n\}$ of ζ is a g-open cover of X if and only if the interrsection of the corresponding g-closed elements of ξ is non- empty.

Now X is g-compact if and only if ζ is a collection of g-open sets, if ζ is a g-open cover of X, then there is a finite g-open subcollection of ζ covers X if and only if the given collection ζ of g-open sets, if there is no finite g-open subcollection of ζ covers X, then ζ is not a g-open cover of X if and only if given any collection ξ of g-closed sets, if every finite intersection of g-closed sets in ξ is non-empty, then the intersection of all the g-closed sets in ξ is non-empty. □

Theorem 2.4. *Let (X, g) be a sGTS. If X is a g-compact gT_3-space, then X is g-Volterra.*

Proof. Let $\{G_n\}$ be any sequence of g-dense G_δ subsets of X. Put $F_n = X \setminus G_n$, $n \in N$. Then $\{F_n\}$ be any sequence of g-closed subsets of X. Now, $i_g(F_n) = i_g(X \setminus G_n) = X \setminus c_g(G_n) = \emptyset$. In order to prove that X is g-Volterra, let U be any non–empty g-open subset of X. Now, $i_g(F_1) = \emptyset$, we have $U \not\subset F_1$. It follows that there exists $x_1 \in U$ such that $x_1 \notin F_1$. Since X is a gT_3-space and g-closed set, there exists g-open set V_1 such that $c_g(V_1) \subset U$ and $c_g(V_1) \cap F_1 = \emptyset$. Similarly we have $V_1 \not\subset F_2$, there exists $x_2 \in V_1$ such that $x_2 \notin F_2$, By Lemma 1.11 , there exists g-open set V_2 such that $c_g(V_2) \subset V_1$ and $c_g(V_2) \cap F_2 = \emptyset$. By induction process, there exists a non-empty g-open set V_n such that $c_g(V_n) \subset V_{n-1}$ and $c_g(V_n) \cap F_n = \emptyset$. Thus, $\{c_g(V_n)\}$ is a decreasing sequence of non-empty g-closed subsets of X which satisfy the finite intersection property. Since X is g-compact and by Lemma 2.3, $\cap c_g(V_n)$ is non-empty. Choose an element $z \in \bigcap c_g(V_n)$ such that $c_g(V_n) \cap F_n = \emptyset$ and $\cap c_g(V_n) \subset c_g(V_n) \subset U$. We have $z \in U$ and $z \notin F_n$, $n \in N$. This implies that $U \not\subset F_n$ and $U \not\subset \cup F_n$. Thus, $i_g(\cup F_n) = \emptyset$. Also $\emptyset = i_g(\cup F_n) = i_g(\cup(X \setminus G_n)) = i_g(X \setminus \cap G_n) = X \setminus c_g(\cap G_n)$. It follows that $\cap G_n \in D(g)$. Hence X is g-Volterra. □

Theorem 2.5. *Let X be a g-compact gT_3- space. If there exists a countable collection $\{F_n\}$ of g-closed sets of X with $i_g(F_n) = \emptyset$, then $i_g(\cup F_n) = \emptyset$.*

Proof. Suppose $\{F_n\}$ is a countable collection of g-closed sets in X such that $i_g(F_n) = \emptyset$. In order to show that $i_g(\cup F_n) = \emptyset$. Let U_0 be any non–empty g-open subset of X, we have to find a point x of U_0 that does not lie in any of the sets F_n. Consider the set F_1. Since $i_g(F_1) = \emptyset$, F_1 does not contain U_0. Therefore, we choose a point y of U_0 that is not in F_1. Since F_1 is g-closed and $y \notin F_1$ and by using the gT_3 condition there exists g-open set U_1 such that $y \in U_1$, $c_g(U_1) \subset U_0$ and $c_g(U_1) \cap F_1 = \emptyset$. In general, there exists a non-empty g-open set U_n such that $c_g(U_n) \subset U_{n-1}$ and $c_g(U_n) \cap F_n = \emptyset$. Thus, $\{c_g(U_n)\}$ is a nested sequence of non-empty g-closed sets which satisfy the finite intersection property. Since X is g-compact and by Lemma 2.3, $\cap c_g(U_n)$ is non-empty. Choose $x \in \bigcap c_g(U_n)$. We have $x \in U_0$ because $c_g(U_1) \subset U_0$. The point $x \notin F_n$, for each n. Thus, $i_g(\cup F_n) = \emptyset$. □

Theorem 2.6. *Let (X, g) be gT_2-space and $x \in X$. If A is a g-compact subset of X and $x \notin A$, then there exist a g-open set B such that $A \subset B$.*

Proof. Suppose A is a g-compact subset of X and $x \in X$, $x \notin A$. For each $a \in A$ and by using the gT_2- condition, there exist g-open sets U_x and V_a such that $x \in U_x$ and $a \in V_a$ and $U_x \cap V_a = \emptyset$. The collection $\{V_a : a \in A\}$ is a g-open covering of A. Since X is g-compact, there is a finite subcollection $\{V_{\alpha_1}, V_{\alpha_2}, ..., V_{\alpha_n}\}$ of g-open sets covering A. Let $B = \bigcup V_n$. Clearly B is g-open and $A \subset B$. □

Lemma 2.7. *Let X be g-compact gT_2 space. If every point of X is a g-cluster point of X, and $x \in X$ and A is a non-empty g-open set of X, then there exists a non-empty set B contained in A such that $c_g(B)$ does not contain x.*

Proof. Suppose $x \in A$. Since x is a g-cluster point of X, $A \cap (X \setminus \{x\}) \neq \emptyset$. Then we choose a point $y \in A$. Suppose $x \notin A$, we choose a point $y \in A$, since A is non-empty. By using the gT_2 condition, there exists g-open sets $U_x \in g_x$ and $U_y \in g_y$ such that $x \in U_x$ and $y \in U_y$ and $U_x \cap U_y = \emptyset$. Let $B = A \cap U_y$. Then $B \subset A$ and also by the definition of $c_g(B)$, $c_g(B)$ does not contain x. □

Theorem 2.8. *Suppose X is a non-empty g-compact gT_2-space. Let every point of X be a g-cluster point of X. If $(A_n)_{n \in J}$, where $J \subseteq \mathbb{Z}_+$, is a non-empty nested collection of g-open sets then X is uncountable.*

Proof. Suppose that we define the function $f\colon \mathbb{Z}_+ \to X$ by $f(x) = x_n$. Suppose that there exists a non-empty nested collection $A_0 \supset A_1 \supset A_2 \supset ...$ and take $A_0 = X$. Since A_0 is a non-empty g-open set and $x_1 \in X$ then by Lemma 2.7 there exists a set $V_1 \subset A_0$ such that $c_g(V_1)$ does not contain x_1. In general, there exists a non-empty g-open set A_{n-1} such that $V_n \subset A_{n-1}$ and $c_g(V_n)$ does not contain x_n. The collection $\{c_g(V_1), c_g(V_2), ...\}$ is a nested collection of g-closed sets which satisfy the finite intersection condition. Since X is g-compact and by Lemma 2.3 then the intersection $\cap c_g(V_i) \neq \emptyset$. Let $x \in \cap c_g(V_i)$. This implies that $x \in c_g(V_i)$ for every i, then $x \neq x_n$ for any n. Since $x \in c_g(V_i)$ for all i, $x_n \notin c_g(V_n)$ for any n. So X is uncountable. □

If (X, g) is a gT_2- space and Y is a g-compact subset of X, then Y need not be g-closed.

Example 2.9. Let $X = \{a, b, c, d, e\}$ and $g = \{\emptyset, \{a\}, \{b\}, \{c\}, \{c, e\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, c, e\}, \{b, c, e\}, \{a, b, c, d\}, \{a, b, c, e\}, \{a, b, d, e\}, \{a, c, d, e\}, \{b, c, d, e\}, X\}$. Then (X, g) be $sGTS$. Also g is a gT_2 space. Let $Y = \{a, b, c\}$. Clearly Y is g-compact. Since $X \setminus Y$ is not g-open, Y is not g-closed.

Remark 2.10. Every g-compact gT_2-space is not gT_3. Every g-compact gT_2-space is not gT_4.

Theorem 2.11. *Let (X, g) be a g-compact gT_3 space. Then the following conditions are equivalent.*

(i) *X is g-Volterra.*

(ii) *There exists a countable collection $\{F_n\}$ of g-closed sets of X with $i_g(F_n) = \emptyset$ then $i_g(\cup F_n) = \emptyset$.*

Proof. $(ii) \to (i)$ Let X be a g-compact gT_3 space and $\{G_n\}$ be any sequence of g-dense G_δ subsets of X. By Theorem 2.4, $\cap G_n \in D(g)$. Therefore, X is g-Volterra.
$(i) \to (ii)$ Suppose that X is g-Volterra. Let $\{F_n\}$ be any sequence of g-closed sets in X with $i_g(F_n) = \emptyset$. By Theorem 2.5, $i_g(\cup F_n) = \emptyset$. □

3. Volterra properties of subspaces of $GTSs$

In this section, we study Volterra properties of subspaces of $GTSs$.

Lemma 3.1. [13] *Let (X, g) be a GTS and $A \subset S \subset X$. Then $c_{g_s}(A) = c_g(A) \cap S$.*

Theorem 3.2. *Let (X, g) be a $sGTS$ and S be a G_δ subset of X. If X is a g-compact gT_3-space, then S is g-Volterra.*

Proof. Put $S = \cap_{n \in \mathbb{N}} \mathcal{S}_n$ where $S_n \in g \setminus \{\emptyset\}$ for each $n \in N$. Let $\{G_n\}$ be any sequence of g-dense G_δ subsets of S. Putting $F_n = X \setminus G_n$, $n \in N$ and $\{F_n\}$ be any sequence of g_s-closed subsets of S. In order to prove that S is g-Volterra, let U be any non–empty g-open subset of S. Take $U = U_0 \cap S$ for some $U_0 \in g \setminus \{\emptyset\}$. Then we have $U \not\subset F_1$. Also there exists $x_1 \in U \subset S$ such that $x_1 \notin F_1$. By Lemma 3.1, $F_1 = c_g(F_1) = c_g(F_1) \cap S$ and hence $x_1 \notin c_g(F_1)$. Since X is a g-regular space and $c_g(F_1)$ is a g-closed subset of X then by Lemma 1.11, we have $c_g(V_1) \subset U_0$, $c_g(V_1) \subset S_1$ and $c_g(V_1) \subset c_g(F_1) = \emptyset$. Similarly we have $c_g(V_2) \subset V_1$, $c_g(V_2) \subset S_2$ and $c_g(V_2) \subset c_g(F_2) = \emptyset$ for some non-empty g-open set V_2 with $x_2 \in V_1$ and $x_2 \notin c_g(F_2)$. By the induction process, there exists a non-empty g-open set V_n which satisfies the following conditions:

(i) $c_g(V_n) \subset V_{n-1}$

(ii) $c_g(V_n) \subset S_n$

(iii) $c_g(V_n) \subset c_g(F_n) = \emptyset$

Thus, $\{c_g(V_n)\}$ is a decreasing sequence of non-empty g-closed subsets of X which satisfies the finite intersection property. Since X is g-compact and by Lemma 2.3, $\cap c_g(V_n)$ is non-empty. Choose $z \in \bigcap c_g(V_n)$. Since $\cap c_g(V_n) \subset c_g(V_n) \subset U_0$ and $\cap c_g(V_n) \subset \cap S_n = S$. We have $z \in U_0 \cap S = U$. Also $c_g(V_n) \cap c_g(F_n) = \emptyset$, then $z \notin F_n$, $n \in N$. This implies that $U \not\subset F_n$ and then $U \not\subset \cup F_n$. Thus, $i_{g_s}(\cup F_n) = \emptyset$. Also $\emptyset = i_{g_s}(\cup F_n) = i_{g_s}(\cup(X \setminus G_n)) = i_{g_s}(X \setminus \cap G_n) = X \setminus c_{g_s}(\cap G_n)$. Therefore, $\cap G_n \in D(g_s)$. Hence S is g-Volterra.

□

Definition 3.3. A GTS (X, g) is called *locally g-compact at a point p* of X if there exist a g-open set U of X and a g-compact set K of X such that $p \in U \subset K$.

Definition 3.4. A GTS (X, g) is called *locally g-compact* if it is locally g-compact at each point of X.

Theorem 3.5. *Let (X, g) be a locally g-compact gT_2-space. If $\{D_n\}$ is a countable collection of g-dense g-open subsets of X then X is g-Volterra.*

Proof. Let $D_1, D_2, ..., D_n, ...$ be g-dense g-open subsets of X. Let U be any non-empty g-open subset of X. We have to show that $U \cap \bigcap_{i=1}^{\infty} D_i \neq \emptyset$. Since D_1 is g-dense in X then we have $U \cap D_1 \neq \emptyset$. By the definition of locally g-compact, there exists a non-empty g-compact g-open subset B_1 such that $c_g(B_1) \subset U \cap D_1$. For B_1 and D_2, there exists a non-empty g-compact g-open subset B_2 such that $c_g(B_2) \subset B_1 \cap D_2$. Proceeding by the induction process, we obtain a sequence $\{B_n\}$ of non-empty g-open sets such that $c_g(B_n) \subset B_{n-1} \cap D_n$ for all n. The sets $c_g(B_n)$ are g-closed in the g-compact set $c_g(B_1)$ which satisfies the finite intersection property. Hence, $\bigcap_{n=1}^{\infty} c_g(B_n) \neq \emptyset$. Since $c_g(B_1) \subset U \cap D_1$ and $c_g(B_n) \subset D_n$ for all n, then we have $\bigcap_{n=1}^{\infty} c_g(B_n) \subset U \cap \bigcap_{n=1}^{\infty} D_n$. Thus, $U \cap \bigcap_{n=1}^{\infty} D_n \neq \emptyset$. Therefore, $\bigcap_{n=1}^{\infty} D_n$ is g-dense in X. Hence X is g-Volterra. □

Definition 3.6. [13] Let (X, g) be a GTS and $\mathcal{F} \subset \mathcal{P}(\mathrm{X})$. If $\cup\mathcal{F} \in D(g)$ then $\mathcal{F}$ is called a *pseudo cover*.

Theorem 3.7. *Let (X, g) be a GTS and $\{A_n : n \in \mathbb{N}\}$ be a countable g-closed cover of X. If $\{i_g(A_n) : n \in \mathbb{N}\}$ is a pseudo cover of X then X is g-Volterra.*

Proof. Let $\{G_n\}$ be any sequence of g-dense G_δ subsets of X. Put $F_n = X \setminus G_n$, $n \in N$. Since each $X \setminus c_g(F_n) = i_g(G_n) = G_n \in D(g)$, by [[12], Proposition 3.4], each $F_n \in N(g)$. Then $\bigcup_{n\in\mathbb{N}} i_g(F_n) = \bigcup_{n\in\mathbb{N}} i_g(c_g(F_n)) = \emptyset$. Moreover, we have $X = c_g(\bigcap_{n\in\mathbb{N}} G_n) \cup (X \setminus c_g(\bigcap_{n\in\mathbb{N}} G_n)) = c_g(\bigcap_{n\in\mathbb{N}} (X \setminus F_n)) \cup (X \setminus c_g(\bigcap_{n\in\mathbb{N}} (X \setminus F_n)))$ $\subset c_g(\bigcap_{n\in\mathbb{N}} (X \setminus F_n)) \cup (X \setminus \bigcap_{n\in\mathbb{N}} (X \setminus F_n)) = c_g(\bigcap_{n\in\mathbb{N}} (X \setminus F_n)) \cup (\bigcup_{n\in\mathbb{N}} F_n)$. Put $A_1 = c_g(\bigcap_{n\in\mathbb{N}} (X \setminus F_n))$, $A_{n+1} = F_n (n \in \mathbb{N})$. Then $\{A_n : n \in \mathbb{N}\}$ is a countable g-closed cover of X. Let $\{i_g(A_n) : n \in \mathbb{N}\}$ be a pseudo cover of X. Then $\bigcup_{n\in\mathbb{N}} i_g(A_n) \in D(g)$. Also, $i_g(A_1) \cup (\bigcup_{n\in\mathbb{N}} i_g(A_{n+1})) = i_g(c_g(\bigcap_{n\in\mathbb{N}} (X \setminus F_n))) \cup \emptyset = i_g(c_g(\bigcap_{n\in\mathbb{N}} (X \setminus F_n)))$. We have $X = c_g(i_g(c_g(\bigcap_{n\in\mathbb{N}} (X \setminus F_n)))) = c_g(i_g(c_g(\bigcap_{n\in\mathbb{N}} G_n))) \subset c_g(c_g(\bigcap_{n\in\mathbb{N}} G_n)) = c_g(\bigcap_{n\in\mathbb{N}} G_n)$. Hence X is g-Volterra. □

Remark 3.8. For a given gT_2-space X, let $\mathcal{P}(\mathrm{X})$ denote the collection of non-empty g-closed subsets of X. For any finite family $\mathcal{U} = \{U_1, U_2, ..., U_n\}$ of subsets of X, we define $(\mathcal{U}) \subset \mathcal{P}(X)$ by $(\mathcal{U}) = \{F \in \mathcal{P}(X) : F \subset \cup\{U_i : 1 \leq i \leq n\}\ and\ F \cap U_i \neq \emptyset\ for\ all\ i = 1, 2, ..., n\}$. The family of all subsets of the form $(\mathcal{U})$ as a base where $\mathcal{U}$ runs through all finite families of g-open subsets of X.
Let $\chi(X)$ be the subspace of $\mathcal{P}(\mathrm{X})$ consist of all non-empty g-compact subsets of X with generalized topology.

Lemma 3.9. *Let (X, g) be a gT_2-space, If $\chi(X)$ is g-Volterra then X is g-Volterra.*

Proof. For any family $\{B_\alpha : \alpha \in A\}$ of subsets of X

(i) $(\cap\{B_\alpha : \alpha \in A\}) = \cap\{(B_\alpha) : \alpha \in A\}$ and
(ii) for any $\alpha \in A$, B_α is g-dense in X if and only if (B_α) is g-dense in $\chi(X)$.

Suppose that $\chi(X)$ is g-Volterra. Let U and V be g-dense G_δ sets in X. Then (U) and (V) are g-dense G_δ sets in $\chi(X)$. Since $\chi(X)$ is g-Volterra then $(U) \cap (V)$ is g-dense in $\chi(X)$. By (i) and (ii) we have $U \cap V$ is g-dense in X. Therefore, X is g-Volterra. $\square$

Remark 3.10. The same argument is true if we replace $U \cap V$ is g-dense by $U \cap V$ is non-empty then we get X is weakly g-Volterra.

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References

[1] J. Cao and D. Gauld, Volterra spaces revisited, J. Aust. Math. Soc., 79 (2005), 61–76.
[2] J. Cao and H. J. K. Junnila, When is a Volterra space Baire?, Topology Appl., 154 (2007), 527–532.
[3] Á. Császár, Generalized open sets, Acta Math. Hungar., 75 (1997), 65–87.
[4] Á. Császár, Generalized open sets in generalized topologies, Acta Math. Hungar., 106 (2005), 53–66.
[5] Á. Császár, Generalized topology and generalized continuity, Acta Math. Hungar., 96 (2002), 351–357.
[6] D. Gauld and Z. Piotrowski, On Volterra spaces, Far East J. Math. Sci., 1 (1993), 209–214.
[7] D. Gauld and Z. Piotrowski, On Volterra spaces II, Ann. N. Y. Acad. Sci., 806 (1996), 169–173.
[8] D. Gauld, S. Greenwood and Z. Piotrowski, On Volterra spaces III, Topological operations, Topology Proc., 23 (1998), 167–182.
[9] G. Gruenhage and D. Lutzer, Baire and Volterra spaces, Proc. Amer. Math. Soc., 128 (2000), 3115–3124.
[10] P. Jeyanthi and G. Geetha, Generalized Volterra spaces, An. Univ. Oradea, Fasc. Mat., XXVII (2020), no. 1, 43–46.
[11] P. Jeyanthi and G. Geetha, Generalized developable spaces, An. Univ. Oradea, Fasc. Mat., XXIX (2022), no. 1, 93–102.
[12] P. Jeyanthi and G. Geetha, Weakly g-Volterra spaces in generalized topological spaces, An. Univ. Oradea, Fasc. Mat., XXIX (2022), no. 2, 81–87.
[13] Z. Li and F. Lin, Baireness on generalized topological spaces, Acta Math. Hungar., 139 (2013), no. 4, 320–336.
[14] M. Matejdes, Generalized Volterra spaces, Int. J. Pure Appl. Math., 85 (2013), no. 5, 955–963.
[15] M. Matejdes, Characterization of Volterra spaces, Int. J. Pure Appl. Math., 88 (2013), no. 3, 437–442.
[16] L. X. Peng and C. Yang, A note on Volterra and Baire spaces, Bull. Iranina Math. Soc., 41 (2015), no. 6, 1445–1452.
[17] S. Spadaro, P-spaces and the Volterra property, Bull. Australian Math. Soc., 87 (2013), no. 2, 339–345.
[18] S. Vadakasi, Some properties of generalized topologies in $GTSs$, Comm. Adv. Math. Sci., III (2020), 162–172.
[19] A. Volterra, Alcune osservasioni sulle funzioni punteggiate discontinue, Giorn. Mat., 19 (1881), 76–86.
[20] GE. Xun and GE Ying, μ-separations in generalized topological spaces, Appl. Math. J. Chinese Univ., 2 (2013), 339–345.

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