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GENERALIZED SOBOLEV TYPE SPACES INVOLVING THE WEINSTEIN TRANSFORM

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Abstract. In this paper, the space $G_\omega^{p,s}(\mathbb{R}_+^{n+1})$ is considered, and many properties, including completeness and inclusion, are discussed by using the theory of the Weinstein transform. It is shown that the space $S_\omega(\mathbb{R}_+^{n+1})$, is dense in space $G_\omega^{p,s}(\mathbb{R}_+^{n+1})$. The generalized Hankel potential $\mathcal{H}^k$ associated with the Weinstein transform is introduced, and its properties are examined. The L^p-space of all such Hankel potential, denoted by $W_\omega^{m,p}(\mathbb{R}_+^{n+1})$, is defined and it is proven that the generalized Hankel potential $\mathcal{H}^t$ is an isometry of $W_\omega^{m,p}(\mathbb{R}_+^{n+1})$ onto $W_\omega^{m+t,p}(\mathbb{R}_+^{n+1})$.

1. Introduction

Sobolev space was introduced by S. L. Sobolev in the 20th century. This space is a useful and powerful tool that is frequently used in nonlinear analysis, differential geometry, physics, and other areas of the mathematical sciences. This theory is useful for solving the problems of partial differential equations. Pahk and Kang [10], extended the concept of Sobolev space to the generalized distribution spaces of Beurling-Björck type [1, 2], and investigated the Sobolev imbedding theorem, the Rellich's compactness theorem and others, exploiting the weight function ω. Pathak and Pandey [12], introduced the Sobolev space of type $G_\mu^{p,s}$ and discussed the properties including completeness and inclusion by using the theory of distributional Hankel transform. The aforesaid authors introduced the Hankel potential $\mathcal{H}_\mu^s$ and showed that the Hankel potential $\mathcal{H}_\mu^s$ is a continuous linear mapping of the Zemanian space $\mathcal{H}_\mu$ into itself. Other properties of Hankel potential were also found in this work. Later on, Pathak and Shrestha [13], defind the space of type $G_{\omega,\mu}^{p,s}$ and discussed many results. Pathak [11], considered the generalized Sobolev space $H_\omega^w(\mathbb{R}^n)$ which is a generalization of the Sobolev space $H^s(\mathbb{R}^n)$ and developed a multiresolution analysis for the generalised Sobolev space.

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For the Weinstein transform concern, many properties regarding the Sobolev type spaces were studied. In this connection, Salem and Dachraoui [15], studied the Sobolev type spaces $G^{p,s}_{\alpha,\beta}$ associated with Jacobi differential operators $\Delta_{\alpha,\beta}$, and examined that the Jacobi potential $\mathcal{H}^s_{\alpha,\beta}$, is a pseudodifferential operator with a precise symbol. Using the afforsaid results they extended the operator $\mathcal{H}^s_{\alpha,\beta}$, to the distributional space [2]. Hassen and Belgacem [7], introduced the Sobolev spaces of exponential type associated with the Weinstein operator, and found many results. Mehrez, [3], Mejjaoli, et al. [4, 6, 5], Salem [16], Saoudi [18, 17], and Trimèche, [20], observed useful results regarding the Weinstein transform. Our main objective of the present paper is to expose the properties of generalized sobolev space $G^{p,s}_{\omega}(\mathbb{R}^{n+1})$ and to obtain other results by the exploiting the theory of Weinstein transform. The present manuscript is organized in the following manner:
Section 1, is introductory. It introduces the Sobolev space, the Hankel transform, and Weinstein transform, and describes motivations for the generalised Sobolev space involving the Weinstein transform. In section 2, we present auxiliary results of the Weinstein transform that are useful for our investigations. In section 3, the generalized Sobolev space $G^{p,\,s}_{\omega}$ is defined and some related results are proven. In section 4, the generalized Hankel potential associated with the Weinstein transform is defined and many properties on space $W^{m,p}_{\omega}$ are studied by exploiting the theory of Weinstein transform.

2. Preliminaries

In this section, we discuss some properties related to the Weinstein transform that are useful for our research. From [8, 9] and [20], standard notations, definitions, and properties are given below:
- $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$.
- $x = (x',\, x_{n+1}) = (x_1,\, x_2,\, x_3 ... x_n,\, x_{n+1}) \in \mathbb{R}^{n+1}_0$.
- $-x = (-x',\, x_{n+1}) = (-x_1,\, -x_2,\, -x_3 ... -x_n,\, x_{n+1}) \in \mathbb{R}^{n+1}_0$.
- $\langle x,\, \xi\rangle = \sum_{k=1}^{n+1} x_k \xi_k$.
- $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} .... D_n^{\alpha_n} D_{n+1}^{\alpha_{n+1}}$ and $\alpha \in \mathbb{N}^{n+1}_0$.
- $C_*\left(\mathbb{R}^{n+1}_+\right)$, represents the space of continuous functions over $\mathbb{R}^{n+1}_+$, even with respect to the last variable.
- $C^\infty_*\left(\mathbb{R}^{n+1}_+\right)$, represents the space of infinitely differentiable function over $\mathbb{R}^{n+1}_+$, even with respect to the last variable.
- $S_*\left(\mathbb{R}^{n+1}_+\right)$, represents Schwartz space of rapidly decreasing functions over $\mathbb{R}^{n+1}_+$, even with respect to the last variable.
- L^p_β represents the space of measurable functions ϕ over $\mathbb{R}^{n+1}_+$ such that

$$||\phi||_{\beta,p} = \left(\int_{\mathbb{R}^{n+1}_+} |\phi(y)|^p \, d\mu_\beta(y) \right)^{\frac{1}{p}} < \infty, \;\; 1 \le p < \infty,$$

$$||\phi||_{\beta,\infty} = ess \sup_{y\,\in\,\mathbb{R}^{n+1}_+} |\phi(y)| < \infty,$$

where $d\mu_\beta(y)$ is the measure over $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$ given by

$$d\mu_\beta(y) = d\mu_\beta(y',\, y_{n+1}) = \frac{y_{n+1}^{2\beta+1}}{(2\pi)^{\frac{n}{2}}2^\beta\Gamma(\beta+1)}dy.$$

The Weinstein operator $\Delta^n_{W,\beta}$ over $\mathbb{R}^{n+1}_+$ is given by

$$\Delta^n_{W,\beta} = \sum_{j=1}^{n+1}\frac{\partial^2}{\partial x_j^2} + \frac{2\beta+1}{x_{n+1}}\frac{\partial}{\partial x_{n+1}} = \Delta^n_W + L_\beta,\ \ \beta > -1/2\,,$$

where Δ^n_W is the Laplacian operator for the first n variables and given by

$$\Delta^n_W = \sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}$$

and L_β is the Bessel operator for the last variable defined on $(0,\infty)$ by

$$L_\beta\phi = \frac{\partial^2}{\partial x_{j+1}^2}\phi + \frac{2\beta+1}{x_{n+1}}\frac{\partial}{\partial x_{n+1}}\phi\,.$$

The Weinstein transform :

Definition 2.1. Let $\phi \in L^1_\beta(\mathbb{R}^{n+1}_+)$. The Weinstein transform of ϕ is defined by

$$(\mathcal{F}_w\phi)(\xi) = \int_{\mathbb{R}^{n+1}_+} e^{-i\langle\xi',\,y'\rangle} j_\beta(\xi_{n+1}y_{n+1})\phi(y)\,d\mu_\beta(y),\ \ \xi\in\mathbb{R}^{n+1}_+, \tag{2.1}$$

where j_β is the normlized Bessel function of index β, which is given by

$$j_\beta(y) = \Gamma(\beta+1)\sum_{m=0}^{\infty}\frac{(-1)^m(y)^{2m}}{2^m m!\Gamma(\beta+m+1)}\ .$$

Inversion formula: Let $\phi \in L^1_\beta(\mathbb{R}^{n+1}_+)$. If $F_w\phi \in L^1_\beta(\mathbb{R}^{n+1}_+)$ then,

$$\phi(y) = \int_{\mathbb{R}^{n+1}_+} e^{i\langle y',\,\xi'\rangle} j_\beta(y_{n+1}\xi_{n+1})(\mathcal{F}_w\phi)(\xi)\,d\mu_\beta(\xi),\ \ a.e. \tag{2.2}$$

- For $f \in S_*(\mathbb{R}^{n+1}_+)$, we have
$$\mathcal{F}_w((\Delta^n_{W,\beta})^\alpha f)(\xi) = (-\|\xi\|^2)^\alpha(\mathcal{F}_w f)(\xi)\ (\forall\ \xi\ \in\ \mathbb{R}^{n+1}_+). \tag{2.3}$$
- Let $\phi \in S_*\big(\mathbb{R}^{n+1}_+\big)$, then the inverse Weinstein transform is defined by
$$\mathcal{F}_w^{-1}\phi(y) = \mathcal{F}_w\phi(-y)\ \ \forall y\in\mathbb{R}^{n+1}_+. \tag{2.4}$$

Definition 2.2. Let $x \in \mathbb{R}^{n+1}_+$, The translation operator τ^β_x, associated with the Weinstein operator $\Delta^n_{W,\beta}$ is defined by

$$\tau^\beta_x\phi(y) = \frac{\Gamma(\beta+1)}{\sqrt{\pi}\Gamma(\beta+\frac{1}{2})}\int_{\mathbb{R}^{n+1}_+}\phi\left(y'+x',\sqrt{y_{n+1}^2+x_{n+1}^2+2y_{n+1}x_{n+1}cos\theta}\right)(\sin\theta)^{2\beta}\,d\theta. \tag{2.5}$$

Convolution product $\phi\#_\beta\psi$ associated with translation, of the functions $\phi,\psi \in L^1(\mathbb{R}^{n+1}_+)$ is given by:

$$(\phi\#_\beta\psi)(x) = \int_{\mathbb{R}^{n+1}_+}\phi(y)(\tau^\beta_x\psi)(y)\,d\mu_\beta(y). \tag{2.6}$$

Proposition 2.3. The Weinstein transform $\mathcal{F}_w$ is a topological isomorphism from $S_*\big(\mathbb{R}_+^{n+1}\big)$ onto itself and we have

$$\int_{\mathbb{R}_+^{n+1}} |f(x)|^2\, d\mu_\beta(x) = \int_{\mathbb{R}_+^{n+1}} |(\mathcal{F}_w f)(\xi)|^2\, d\mu_\beta(\xi),\ \forall\ f\in S_*\big(\mathbb{R}_+^{n+1}\big). \tag{2.7}$$

Proposition 2.4. For all $f,g\in L^1(\mathbb{R}_+^{n+1}), f\#_\beta g$ belongs to $L^1(\mathbb{R}_+^{n+1})$ and

$$\mathcal{F}_w(f\#_\beta g) = \mathcal{F}_w(f)\mathcal{F}_w(g). \tag{2.8}$$

Proposition 2.5. Let $1\leq p,q,r\leq\infty$ such that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$. If $f\in L^p(\mathbb{R}_+^{n+1})$ and $g\in L^q(\mathbb{R}_+^{n+1})$, then $(f\#_\beta g)\in L^r(\mathbb{R}_+^{n+1})$ and

$$||f\#_\beta g||_{L^r(\mathbb{R}_+^{n+1})} \leq ||f||_{L^p(\mathbb{R}_+^{n+1})}||g||_{L^q(\mathbb{R}_+^{n+1})}. \tag{2.9}$$

3. Generalized Sobolev type space $G_\omega^{p,\,s}$

In this section, the generalized Sobolev space $G_\omega^{p,\,s}$ is defined and various properties including completeness and inclusion are obtained by exploiting the theory of the Weinstein transform.

Definition 3.1. Let ω be the real-valued continuous function defined on $\mathbb{R}_+^{n+1}$ satisfying the following condition:

(1) $0=\omega(0)=\lim\limits_{x\to 0}\omega(x)\leq\omega(x+y)\leq\omega(x)+\omega(y), \forall\, x,\, y\in\mathbb{R}_+^{n+1}$.

(2) $J_n(\omega)=\displaystyle\int_{|y|\geq 1}\frac{\omega(y)}{|\ y\ |^{n+1}}\,d\mu_\beta(y)<\infty$.

(3) $a+b\log(1+\mid y\mid)\leq\omega(y), \forall\, y\in\mathbb{R}_+^{n+1},\ a\in\mathbb{R},\ \textit{and}\ b>0$.

The class of all ω satisfying $1,2$ and 3 will be denoted by M.

Definition 3.2. Let $\omega\in M$ and $\lambda\geq 0$. Then for each multi-index α the space $S_\omega(\mathbb{R}_+^{n+1})$ is defined to be the set of all complex-valued infinitely differentiable function ϕ on $\mathbb{R}_+^{n+1}$ such that

$$p_{\alpha,\lambda}(\phi)=\sup_{x\in\mathbb{R}_+^{n+1}} e^{\lambda\omega(x)}\mid(\Delta_{W,\beta}^n)^\alpha\phi(x)\mid<\infty \tag{3.1}$$

and

$$\pi_{\alpha,\lambda}(\phi)=\sup_{\xi\in\mathbb{R}_+^{n+1}} e^{\lambda\omega(\xi)}\mid(\Delta_{W,\beta}^n)^\alpha(\mathcal{F}_w\phi)(\xi)\mid<\infty. \tag{3.2}$$

We denote $S'_\omega(\mathbb{R}_+^{n+1})$ as the dual of $S_\omega(\mathbb{R}_+^{n+1})$.

Theorem 3.3. *The Weinstein transform $\mathcal{F}_w f$ is an automorphism on $S_\omega(\mathbb{R}_+^{n+1})$.*

Proof: Let $\phi\in S_\omega(\mathbb{R}_+^{n+1})$, then first we show that $\mathcal{F}_w(\phi)\in S_\omega(\mathbb{R}_+^{n+1})$. From (3.1), we have

$$\begin{aligned} &p_{\alpha,\lambda}[\mathcal{F}_w\phi(\xi)]\\ &=\sup_{\xi\in\mathbb{R}_+^{n+1}} e^{\lambda\omega(\xi)}\mid(\Delta_{W,\beta}^n)_\xi^\alpha(\mathcal{F}_w\phi)(\xi)\mid\\ &=\sup_{\xi\in\mathbb{R}_+^{n+1}} e^{\lambda\omega(\xi)}\mid(\Delta_{W,\beta}^n)_\xi^\alpha\int_{\mathbb{R}_+^{n+1}} e^{-i\langle\xi',\,y'\rangle}j_\beta(\xi_{n+1}y_{n+1})\phi(y)\,d\mu_\beta(y)\mid. \end{aligned}$$

Using the derivative properties of the Weinstein transform (2.3), we get

$$p_{\alpha,\lambda}[\mathcal{F}_w\phi(\xi)] = \sup_{\xi\in\mathbb{R}^{n+1}_+} e^{\lambda\omega(\xi)} \mid \int_{\mathbb{R}^{n+1}_+} e^{-i\langle\xi',\,y'\rangle} j_\beta(\xi_{n+1}y_{n+1})(-\|y\|^2)^\alpha)\phi(y)\,d\mu_\beta(y) \mid .$$

Now, we can write $g(y) = (-\|y\|^2)^\alpha\phi(y)$ therefore above expression yeilds

$$\begin{aligned} p_{\alpha,\lambda}[\mathcal{F}_w\phi(\xi)] \quad &= \sup_{\xi\in\mathbb{R}^{n+1}_+} e^{\lambda\omega(\xi)} \mid \int_{\mathbb{R}^{n+1}_+} e^{-i\langle\xi',\,y'\rangle} j_\beta(\xi_{n+1}y_{n+1})g(y)\,d\mu_\beta(y) \mid \\ &\leq \sup_{\xi\in\mathbb{R}^{n+1}_+} e^{\lambda|\omega(\xi)|} \mid \int_{\mathbb{R}^{n+1}_+} e^{-i\langle\xi',\,y'\rangle} j_\beta(\xi_{n+1}y_{n+1})g(y)\,d\mu_\beta(y) \mid . \end{aligned}$$

Form [14] for every $\epsilon > 0$ there exist $C(\epsilon)$ such that $|\omega(\xi)| \leq \epsilon\|\xi\|^2 + C(\epsilon)$. Therefore, we have

$$\begin{aligned} &p_{\alpha,\lambda}[\mathcal{F}_w\phi(\xi)] \\ &\leq \sup_{\xi\in\mathbb{R}^{n+1}_+} e^{\lambda(\epsilon\|\xi\|^2+C(\epsilon))} \mid \int_{\mathbb{R}^{n+1}_+} e^{-i\langle\xi',\,y'\rangle} j_\beta(\xi_{n+1}y_{n+1})g(y)\,d\mu_\beta(y) \mid \\ &\leq e^{\lambda C(\epsilon)} \sup_{\xi\in\mathbb{R}^{n+1}_+} e^{\lambda\epsilon\|\xi\|^2} \mid \int_{\mathbb{R}^{n+1}_+} e^{-i\langle\xi',\,y'\rangle} j_\beta(\xi_{n+1}y_{n+1})g(y)\,d\mu_\beta(y) \mid \\ &\leq e^{\lambda C(\epsilon)} \sup_{\xi\in\mathbb{R}^{n+1}_+} \sum_{m=0}^{\infty} \frac{(\lambda\epsilon)^m}{m!}\|\xi\|^{2m} \mid \int_{\mathbb{R}^{n+1}_+} e^{-i\langle\xi',\,y'\rangle} j_\beta(\xi_{n+1}y_{n+1})g(y)\,d\mu_\beta(y) \mid \\ &= e^{\lambda C(\epsilon)} \sum_{m=0}^{\infty} \frac{(\lambda\epsilon)^m}{m!} \sup_{\xi\in\mathbb{R}^{n+1}_+} \mid \int_{\mathbb{R}^{n+1}_+} \|\xi\|^{2m} e^{-i\langle\xi',\,y'\rangle} j_\beta(\xi_{n+1}y_{n+1})g(y)\,d\mu_\beta(y) \mid \\ &= e^{\lambda C(\epsilon)} \sum_{m=0}^{\infty} \frac{(\lambda\epsilon)^m}{m!} \sup_{\xi\in\mathbb{R}^{n+1}_+} \mid \int_{\mathbb{R}^{n+1}_+} (\Delta^n_{W,\beta})^m_y (e^{-i\langle\xi',\,y'\rangle} j_\beta(\xi_{n+1}y_{n+1}))g(y)\,d\mu_\beta(y) \mid . \end{aligned}$$

By integrating by parts, we get

$$\begin{aligned} &p_{\alpha,\lambda}[\mathcal{F}_w\phi(\xi)] \\ &\leq e^{\lambda C(\epsilon)} \sum_{m=0}^{\infty} \frac{(\lambda\epsilon)^m}{m!} \sup_{\xi\in\mathbb{R}^{n+1}_+} \mid (-1)^m \int_{\mathbb{R}^{n+1}_+} e^{-i\langle\xi',\,y'\rangle} j_\beta(\xi_{n+1}y_{n+1})(\Delta^n_{W,\beta})^m_y g(y)\,d\mu_\beta(y) \mid \\ &\leq e^{\lambda C(\epsilon)} \sum_{m=0}^{\infty} \frac{(\lambda\epsilon)^m}{m!} \left(\sup_{\xi\in\mathbb{R}^{n+1}_+} \mid e^{-i\langle\xi',\,y'\rangle} j_\beta(\xi_{n+1}y_{n+1}) \mid \right) \mid \int_{\mathbb{R}^{n+1}_+} (\Delta^n_{W,\beta})^m_y g(y)\,d\mu_\beta(y) \mid \\ &\leq e^{\lambda C(\epsilon)} \sum_{m=0}^{\infty} \frac{(\lambda\epsilon)^m}{m!} \mid \int_{\mathbb{R}^{n+1}_+} e^{\lambda\omega(y)} (\Delta^n_{W,\beta})^m_y g(y) e^{-\lambda\omega(y)}\,d\mu_\beta(y) \mid \\ &\leq e^{\lambda C(\epsilon)} \sum_{m=0}^{\infty} \frac{(\lambda\epsilon)^m}{m!} \sup_{y\in\mathbb{R}^{n+1}_+} \mid e^{\lambda\omega(x)} (\Delta^n_{W,\beta})^m_y g(y) \|| \int_{\mathbb{R}^{n+1}_+} e^{-\lambda\omega(y)}\,d\mu_\beta(y) \mid \\ &= e^{\lambda C(\epsilon)} \sum_{m=0}^{\infty} \frac{(\lambda\epsilon)^m}{m!} \left(\sup_{y\in\mathbb{R}^{n+1}_+} e^{\lambda\omega(y)} \mid (\Delta^n_{W,\beta})^m_y g(y) \mid \right) \mid \int_{\mathbb{R}^{n+1}_+} e^{-\lambda\omega(y)}\,d\mu_\beta(y) \mid \end{aligned}$$

$$= e^{\lambda C(\epsilon)} \sum_{m=0}^{\infty} \frac{(\lambda\epsilon)^m}{m!} p_{m,\lambda}(g(y)) \mid \int_{\mathbb{R}_+^{n+1}} e^{-\lambda\omega(y)}\, d\mu_\beta(y) \mid < \infty.$$

This shows that $\mathcal{F}_w(\phi) \in S_\omega(\mathbb{R}_+^{n+1})$.
Since by the inversion formula for the Weinstein transform we have $\mathcal{F}_w^{-1}\phi(y) = \mathcal{F}_w\phi(-y)$. Therefore $\mathcal{F}_w$ is one-one. So that $\mathcal{F}_w$ is an automorphism.

Definition 3.4. The generalized Sobolev type space $G_\omega^{p,\,s}$ is the set of all ultradistributions $u \in S'_\omega(\mathbb{R}_+^{n+1})$, such that its the Weinstein transform $(F_w u)$ corresponds to a locally integrable function u over $\mathbb{R}_+^{n+1}$ for which

$$\|u\|_{G_\omega^{p,\,s}} = \|e^{s\omega(\xi)}(\mathcal{F}_w u)(\xi)\|_p < \infty. \tag{3.3}$$

Theorem 3.5. *The space $G_\omega^{p,\,s}$, $1 \le p < \infty$ is complete.*

Proof: Let $\{u_n\}$ be a Cauchy sequence in $G_\omega^{p,\,s}$. Then $e^{s\omega(\xi)}(\mathcal{F}_w u_n(\xi))$ is a Cauchy sequence in $L^p(\mathbb{R}_+^{n+1})$. Since $L^p(\mathbb{R}_+^{n+1})$ is complete, therefore there exists a function $f \in L^p(\mathbb{R}_+^{n+1})$ such that

$$e^{s\omega(\xi)}(\mathcal{F}_w u_n)(\xi) \to f(\xi) \in L^p(\mathbb{R}_+^{n+1}) \ as\ n \to \infty. \tag{3.4}$$

Let

$$g(\xi) = e^{-s\omega(\xi)} f(\xi),\ \ s > 0. \tag{3.5}$$

Then $f \in S'_\omega(\mathbb{R}_+^{n+1})$ and $g(\xi) = e^{-s\omega(\xi)} f(\xi)$ are in $S'_\omega(\mathbb{R}_+^{n+1})$. Therefore the inverse Weinstein transform of the above function exists. Using (3.5) we have

$$e^{s\omega(\xi)} g(\xi) = e^{s\omega(\xi)} \mathcal{F}_w(\mathcal{F}_w^{-1} g)(\xi) \in L^p(\mathbb{R}_+^{n+1}).$$

Therefore from (3.3), we get

$$\mathcal{F}_w^{-1} g \in G_\omega^{p,\,s}.$$

Let $\mathcal{F}_w^{-1} g = u$. Then

$$e^{s\omega(\xi)}(\mathcal{F}_w u)(\xi) = e^{s\omega(\xi)}(\mathcal{F}_w(\mathcal{F}_w^{-1} g))(\xi) = e^{s\omega(\xi)} g(\xi) = e^{s\omega(\xi)} e^{-s\omega(\xi)} f(\xi) = f(\xi).$$

This implies

$$e^{s\omega(\xi)}(\mathcal{F}_w u)(\xi) = f(\xi). \tag{3.6}$$

Using (3.4) and (3.6) we get $u_n \to u$ in $G_\omega^{p,\,s}$.
This shows that $G_\omega^{p,\,s}$ is complete.

Theorem 3.6. *For $t > r$, $G_\omega^{p,\,t} \subseteq G_\omega^{p,\,r}$, the inclusion map is continuous.*

Proof: Let $u \in G_\omega^{p,\,t}$ and $t > r$ then we take $t = r + \epsilon$, for some $\epsilon > 0$. Then

$$\begin{aligned}
\|u\|_{G_\omega^{p,\,t}} &= \|e^{t\omega(\xi)}(\mathcal{F}_w u)(\xi)\|_p = \left(\int_{\mathbb{R}_+^{n+1}} \mid e^{t\omega(\xi)}(\mathcal{F}_w u)(\xi) \mid^p \ d\mu_\beta(\xi)\right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{R}_+^{n+1}} \mid e^{(r+\epsilon)\omega(\xi)}(\mathcal{F}_w u)(\xi) \mid^p \ d\mu_\beta(\xi)\right)^{\frac{1}{p}} \\
&\ge \left(\int_{\mathbb{R}_+^{n+1}} \mid e^{r\omega(\xi)}(\mathcal{F}_w u)(\xi) \mid^p \ d\mu_\beta(\xi)\right)^{\frac{1}{p}} = \|e^{r\omega(\xi)}(\mathcal{F}_w u)(\xi)\|_p = \|u\|_{G_\omega^{p,\,r}}.
\end{aligned}$$

Therefore,

$$\|u\|_{G_\omega^{p,\,r}} \le \|u\|_{G_\omega^{p,\,t}}.$$

Hence

$$G^{p,\,t}_{\omega} \subseteq G^{p,\,r}_{\omega}.$$

Now let $\{u_n\} \to 0$ as $n \to \infty$ in $G^{p,\,t}_{\omega}$ then

$$\|u_n\|_{G^{p,\,t}_{\omega}} = \|e^{t\omega(\xi)}(\mathcal{F}_w u_n)(\xi)\|_p \to 0.$$

This implies that

$$\|u_n\|_{G^{p,\,r}_{\omega}} = \|e^{r\omega(\xi)}(F_w u_n)(\xi)\|_p \to 0 \ as\ n \to \infty.$$

This shows that $\{u_n\} \to 0$ in $G^{p,\,r}_{\omega}$ as $n \to \infty$.
This proves the continuity of the inclusion.

Theorem 3.7. *The space $S_\omega(\mathbb{R}^{n+1}_+)$ is dense in $G^{p,\,s}_{\omega}$.*

Proof: Let $u \in G^{p,\,s}_{\omega}$. Then $e^{s\omega(\xi)}(\mathcal{F}_w u)(\xi) \in L^p(\mathbb{R}^{n+1}_+)$. Since Schwartz space $S(\mathbb{R}^{n+1}_+)$ is dense in $L^p(\mathbb{R}^{n+1}_+)$, then there exists a sequence of functions $\{\phi_n\}$ belonging to $S(\mathbb{R}^{n+1}_+)$ such that

$$\phi_n(\xi) \to e^{s\omega(\xi)}(F_w u)(\xi)\ in\ L^p(\mathbb{R}^{n+1}_+). \tag{3.7}$$

Now, define

$$\psi_n(x) = \mathcal{F}_w^{-1}(e^{-s\omega(\xi)}\phi_n)(x). \tag{3.8}$$

Since $\phi_n \in S(\mathbb{R}^{n+1}_+)$, therefore $(e^{-s\omega(\xi)}\phi_n) \in S(\mathbb{R}^{n+1}_+)$; as $S(\mathbb{R}^{n+1}_+)$ is subspace of $S_\omega(\mathbb{R}^{n+1}_+), (e^{-s\omega(\xi)}\phi_n)$ also in $S_\omega(\mathbb{R}^{n+1}_+)$. Hence, its the inverse Weinstein transform exists, this implies that

$$\psi_n(x) = \mathcal{F}_w^{-1}(e^{-s\omega(\xi)}\phi_n)(x) \in H_\omega(\mathbb{R}^{n+1}_+).$$

From (3.3) we can get

$$\begin{aligned}\|\psi_n - u\|_{G^{p,\,s}_{\omega}} &= \|e^{s\omega(\xi)}\big(\mathcal{F}_w(\psi_n - u)(\xi)\big)\|_p\\ &= \|e^{s\omega(\xi)}(\mathcal{F}_w\psi_n)(\xi) - e^{s\omega(\xi)}(\mathcal{F}_w u)(\xi)\|_p.\end{aligned}$$

In view of (3.8), we have

$$\|\psi_n - u\|_{G^{p,\,s}_{\omega}} = \|\phi_n - e^{s\omega(\xi)}(\mathcal{F}_w u)(\xi)\|_p.$$

Using (3.7), we find

$$\|\psi_n - u\|_{G^{p,\,s}_{\omega}} = \|\phi_n - e^{s\omega(\xi)}(\mathcal{F}_w u)(\xi)\|_p \to 0\ as\ n \to \infty.$$

Hence

$$\psi_n \to u\ in\ G^{p,\,s}_{\omega}.$$

This shows that the space $S_\omega(\mathbb{R}^{n+1}_+)$ is dense in $G^{p,\,s}_{\omega}$.

Lemma 1. *Let s_1, s and s_2 be three real numbers with $s_1 < s < s_2$ then for every $\epsilon > 0$ there exists $C(\epsilon) > 0$ such that*

$$e^{s\omega(\xi)} \le \epsilon e^{s_2\omega(\xi)} + C(\epsilon)e^{s_1\omega(\xi)}. \tag{3.9}$$

Proof: The proof of above Lemma can be obtained from [13].

Theorem 3.8. *Let s_1, s and s_2 be three real numbers with $s_1 < s < s_2$ then for every $\epsilon > 0$ there exist $C(\epsilon) > 0$ such that*

$$\|u\|_{G^{p,\,s}_{\omega}} \le \epsilon\|u\|_{G^{p,\,s_2}_{\omega}} \le C(\epsilon)\|u\|_{G^{p,\,s_1}_{\omega}}\ \ \forall u \in \|u\|_{G^{p,\,s_2}_{\omega}}$$

Proof: Let $u \in G_\omega^{p,\, s_2}$. Then by Theorem (3.6), we have

$$u \in G_\omega^{p,\, s} \ and \ u \in G_\omega^{p,\, s_1}.$$

Then

$$\begin{aligned} \|u\|_{u\in G_\omega^{p,\, s}} &= \|e^{s\omega(\xi)}(\mathcal{F}_w u)(\xi)\|_p \\ &= \left(\int_{\mathbb{R}_+^{n+1}} \mid e^{s\omega(\xi)}(\mathcal{F}_w u)(\xi) \mid^p \ d\mu_\beta(\xi)\right)^{\frac{1}{p}}. \end{aligned}$$

By using (3.9), we get

$$\begin{aligned} \|u\|_{u\in G_\omega^{p,\, s}} &\le \left(\int_{\mathbb{R}_+^{n+1}} \mid \left(\epsilon e^{s_2\omega(\xi)} + C(\epsilon) e^{s_1\omega(\xi)}\right)(\mathcal{F}_w u)(\xi) \mid^p \ d\mu_\beta(\xi)\right)^{\frac{1}{p}} \\ &\le \left(\int_{\mathbb{R}_+^{n+1}} \mid \epsilon e^{s_2\omega(\xi)}(\mathcal{F}_w u)(\xi) \mid^p \ d\mu_\beta(\xi)\right)^{\frac{1}{p}} \\ &\quad + \left(\int_{\mathbb{R}_+^{n+1}} \mid C(\epsilon) e^{s_1\omega(\xi)}\mathcal{F}_w u(\xi) \mid^p \ d\mu_\beta(\xi)\right)^{\frac{1}{p}} \\ &\le \epsilon\left(\int_{\mathbb{R}_+^{n+1}} \mid e^{s_2\omega(\xi)}(\mathcal{F}_w u)(\xi) \mid^p \ d\mu_\beta(\xi)\right)^{\frac{1}{p}} \\ &\quad + C(\epsilon)\left(\int_{\mathbb{R}_+^{n+1}} \mid e^{s_1\omega(\xi)}\mathcal{F}_w u(\xi) \mid^p \ d\mu_\beta(\xi)\right)^{\frac{1}{p}} \\ &= \epsilon\|e^{s_2\omega(\xi)}(\mathcal{F}_w u)(\xi)\|_p + C(\epsilon)\|e^{s_1\omega(\xi)}(\mathcal{F}_w u)(\xi)\|_p \\ &= \epsilon\|u\|_{u\in G_\omega^{p,\, s_2}} + C(\epsilon)\|u\|_{u\in G_\omega^{p,\, s_1}}. \end{aligned}$$

This shows that

$$\|u\|_{G_\omega^{p,\, s}} \le \epsilon\|u\|_{G_\omega^{p,\, s_2}} \le C(\epsilon)\|u\|_{G_\omega^{p,\, s_1}}.$$

This proves the theorem.

Theorem 3.9. *Let $s < 0$, then $L^2(\mathbb{R}_+^{n+1}) \subset G_\omega^{2,\, s}$.*

Proof: Let $\phi \in L^2(\mathbb{R}_+^{n+1})$. Then from (2.7), we find

$$\int_{\mathbb{R}_+^{n+1}} |(\mathcal{F}_w\phi)(\xi)|^2 \, d\mu_\beta(\xi) = \int_{\mathbb{R}_+^{n+1}} |\phi(x)|^2 \, d\mu_\beta(x).$$

Therefore, above implies

$$\left(\int_{\mathbb{R}_+^{n+1}} |(\mathcal{F}_w\phi)(\xi)|^2 \, d\mu_\beta(\xi)\right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}_+^{n+1}} |\phi(x)|^2 \, d\mu_\beta(x)\right)^{\frac{1}{2}}.$$

Therefore, we have

$$\|\mathcal{F}_w\phi\|_{L^2(\mathbb{R}_+^{n+1})} = \|\phi\|_{L^2(\mathbb{R}_+^{n+1})}. \tag{3.10}$$

Hence for $s < 0$, we have

$$\|\phi\|_{G^{2,s}_\omega} = \|e^{s\omega(\xi)}(\mathcal{F}_w\phi)(\xi)\|_{L^2(\mathbb{R}^{n+1}_+)} = \left(\int_{\mathbb{R}^{n+1}_+} | \, e^{s\omega(\xi)}(\mathcal{F}_w\phi)(\xi) \, |^2 \ d\mu_\beta(\xi)\right)^{\frac{1}{2}}$$
$$\leq \left(\int_{\mathbb{R}^{n+1}_+} | \, (\mathcal{F}_w\phi)(\xi) \, |^2 \ d\mu_\beta(\xi)\right)^{\frac{1}{2}} = \|(\mathcal{F}_w\phi)(\xi)\|_{L^2(\mathbb{R}^{n+1}_+)}.$$

By using (3.10), we get

$$\|\phi\|_{G^{2,s}_\omega} \leq | \ (\mathcal{F}_w\phi)(\xi)\|_{L^2(\mathbb{R}^{n+1}_+)}.$$

Therefore

$$\|\phi\|_{G^{2,s}_\omega} < \infty.$$

Hence, above implies

$$\phi \in G^{2,s}_\omega.$$

This shows that $L^2(\mathbb{R}^{n+1}_+) \subset G^{2,s}_\omega$.

4. The generalized Hankel potential associated with the Weinstein transform

Let $a(x) \neq 0$ be a multiplier in $S_\omega(\mathbb{R}^{n+1}_+)$ such that

$$\mathcal{F}_w^{-1}(a^k)(x) \in L^1(\mathbb{R}^{n+1}_+) \ \forall \ k = 0, 1, 2, 3 \ ..... \tag{4.1}$$

Then, for $f \in S'_\omega(\mathbb{R}^{n+1}_+)$, we define the products fa^k and $fa^{-k} \in S'_\omega(\mathbb{R}^{n+1}_+)$ by the following relations.

$$\langle f \ a^k, \ \phi\rangle = \langle f, \ a^k \ \phi\rangle \ \forall \ \phi \in S_\omega(\mathbb{R}^{n+1}_+) \tag{4.2}$$

and

$$\langle f \ a^{-k}, \ a^k \ \phi\rangle = \langle f, \ \phi\rangle \ \forall \ \phi \in S_\omega(\mathbb{R}^{n+1}_+). \tag{4.3}$$

Now, the generalized Hankel potential associated with the Weinstein transform is defined by

$$\mathcal{H}^k u = \mathcal{F}_w^{-1}(a^{-k}(\mathcal{F}_w u)) \ \forall \ u \in \ S'_\omega(\mathbb{R}^{n+1}_+). \tag{4.4}$$

Since the generalized Weinstein transformation $\mathcal{F}'_w$ corresponds to a continuous linear mapping between $S'_\omega(\mathbb{R}^{n+1}_+)$ onto $S'_\omega(\mathbb{R}^{n+1}_+)$, the same holds for $(\mathcal{F}'_w)^{-1}$ as well . Therefore we conclude that the generalized Hankel potential associated with the Weinstein transform (4.5) is a continuous linear mapping of $S'_\omega(\mathbb{R}^{n+1}_+)$ onto $S'_\omega(\mathbb{R}^{n+1}_+)$. From (2.1) and (2.4) we have

$$(\mathcal{H}^k u)(x) = \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle} j_\beta(x_{n+1}\xi_{n+1}) a^{-k}(\xi)(\mathcal{F}_w u)(\xi) \, d\mu_\beta(\xi). \tag{4.5}$$

This shows that the generalized Hankel potential associated with the Weinstein transform is a pseudo differential operator with symbol $a^{-k}(\xi)$ (See [21, p.10]).

Theorem 4.1. *Let $u \in S'_\omega(\mathbb{R}^{n+1}_+)$. Then for any $m, l \in \mathbb{Z}$ we have*
(i) $\mathcal{H}^l\mathcal{H}^m u = \mathcal{H}^{l+m}u$.
(ii) $\mathcal{H}^0 u = u$.

Proof:(i): Let $u \in S'_\omega(\mathbb{R}^{n+1}_+)$ then from (4.4), we have

$$(\mathcal{H}^l\mathcal{H}^m u)(x) = \mathcal{F}_w^{-1}\bigg(a^{-l}(\mathcal{F}_w(\mathcal{H}^m u)\bigg)(x).$$

Using (2.1), we get

$$\begin{aligned}(\mathcal{H}^l\mathcal{H}^m u)(x) &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\,\xi'\rangle} j_\beta(x_{n+1}\xi_{n+1})(a^{-l}\mathcal{F}_w(\mathcal{H}^m u))(\xi)\,d\mu_\beta(\xi)\\ &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\,\xi'\rangle} j_\beta(x_{n+1}\xi_{n+1})a^{-l}(\xi)\mathcal{F}_w(\mathcal{H}^m u)(\xi)\,d\mu_\beta(\xi).\end{aligned}$$

In view of (4.4), we have

$$(\mathcal{H}^m u)(\xi) = (\mathcal{F}_w^{-1}(a^{-m}(\mathcal{F}_w u)))(\xi).$$

Then, by the Weinstein transform we find

$$\mathcal{F}_w(\mathcal{H}^m u)(\xi) = a^{-m}(\xi)((\mathcal{F}_w u))(\xi). \tag{4.6}$$

Using (4.6), we find

$$\begin{aligned}(\mathcal{H}^l\mathcal{H}^m u)(x) &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\,\xi'\rangle} j_\beta(x_{n+1}\xi_{n+1})a^{-l}(\xi)a^{-m}(\xi)(\mathcal{F}_w u)(\xi)\,d\mu_\beta(\xi)\\ &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\,\xi'\rangle} j_\beta(x_{n+1}\xi_{n+1})a^{-(l+m)}(\xi)(\mathcal{F}_w u)(\xi)\,d\mu_\beta(\xi)\\ &= \mathcal{F}_w^{-1}\bigg(a^{-(l+m)}(\xi)\mathcal{F}_w u(\xi)\bigg)(x)\\ &= (\mathcal{H}^{l+m}u)(x).\end{aligned}$$

Therefore,

$$\mathcal{H}^l\mathcal{H}^m u = \mathcal{H}^{l+m}u.$$

(ii): Now, we have

$$\begin{aligned}(\mathcal{H}^0 u)(x) &= \mathcal{F}_w^{-1}(a^0(\mathcal{F}_w u))(x)\\ &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\,\xi'\rangle} j_\beta(x_{n+1}\xi_{n+1})(a^{-0}(\xi))(\mathcal{F}_w(u))(\xi)\,d\mu_\beta(\xi)\\ &= \mathcal{F}_w^{-1}(\mathcal{F}_w u)(x)\\ &= u(x).\end{aligned}$$

This shows that

$$\mathcal{H}^0 u = u.$$

The Space $W^{m,p}_\omega(\mathbb{R}^{n+1}_+)$. In this section we introduce the L^p-space of all Hankel potential $\mathcal{H}^k$ which is represented by the $W^{m,p}_\omega(\mathbb{R}^{n+1}_+)$ and exploiting the theory of Weinstein transform, many properties of this space will be discussed.

Definition 4.2. Let $m \in \mathbb{Z}$ and $1 < p < \infty$, the space $W_\omega^{m,p}(\mathbb{R}_+^{n+1})$ is the set of all those ultradistributions $u \in S'_\omega(\mathbb{R}_+^{n+1})$ for which

$$\mathcal{H}^{-m}u \in L^p(\mathbb{R}_+^{n+1}). \tag{4.7}$$

The norm of this space is defind by

$$||u||_{m,p,\omega} = ||\mathcal{H}^{-m}u||_p = \left(\int_{\mathbb{R}_+^{n+1}} |(\mathcal{H}^{-m}u)(\xi)|^p \, d\mu_\beta(\xi) \right)^{\frac{1}{p}}. \tag{4.8}$$

In the next theorem we are going to prove that the space $W_\omega^{m,p}(\mathbb{R}_+^{n+1})$ is a Banach space.

Theorem 4.3. *The space $W_\omega^{m,p}(\mathbb{R}_+^{n+1})$ is a Banach space with respect to the norm $||\ ||_{m,p,\omega}$.*

Proof: Let $\{u_k\}$ be a Cauchy sequence in $W_\omega^{m,p}(\mathbb{R}_+^{n+1})$. Then $\{\mathcal{H}^{-m}u_k\}$ is a Cauchy sequence in $L^p(\mathbb{R}_+^{n+1})$. Since $L^p(\mathbb{R}_+^{n+1})$ is complete, therefore there exists a function $u \in L^p(\mathbb{R}_+^{n+1})$ such that

$$\mathcal{H}^{-m}u_k \to u \in L^p(\mathbb{R}_+^{n+1}) \ as \ k \to \infty. \tag{4.9}$$

Take $v = \mathcal{H}^m u$. Then from (4.5), we have

$$\begin{aligned} v(x) =& \int_{\mathbb{R}_+^{n+1}} e^{i\langle x', \xi' \rangle} j_\beta(x_{n+1}\xi_{n+1}) a^{-m}(\xi)(\mathcal{F}_w u)(\xi) \, d\mu_\beta(\xi) \\ =& \mathcal{F}_w^{-1}(a^{-m}(\mathcal{F}_w u))(x). \end{aligned}$$

Therefore, by the property of the Weinstein transform (2.4), we get

$$u(x) = \mathcal{F}_w^{-1}(a^m(\mathcal{F}_w v))(x) = (\mathcal{H}^{-m}v)(x) \in L^p(\mathbb{R}_+^{n+1}). \tag{4.10}$$

Hence, we find

$$v \in W_p^{m,p}.$$

From (4.9) and (4.10), it follows that

$$u_k \to v \in W_p^{m,p} \ as \ k \to \infty.$$

This proves that the space $W_\omega^{m,p}(\mathbb{R}_+^{n+1})$ is a Banach space.

Theorem 4.4. *The generalized Hankel potential $\mathcal{H}^t$ is an isometry of $W_\omega^{m,p}(\mathbb{R}_+^{n+1})$ onto $W_\omega^{m+t,p}(\mathbb{R}_+^{n+1})$; and we have*

$$||\mathcal{H}^t\phi||_{m+t,p,\omega} = ||\phi||_{m,p,\omega}, \ \forall \phi \in W_\omega^{m,p}(\mathbb{R}_+^{n+1}). \tag{4.11}$$

Proof: Let $\phi \in W_\omega^{m,p}$, then from (4.8) we have

$$||\mathcal{H}^t\phi||_{m+t,p,\omega} = ||\mathcal{H}^{-(m+t)}(\mathcal{H}^t\phi)||_p. \tag{4.12}$$

Now, we have

$$(\mathcal{H}^{-(m+t)}(\mathcal{H}^t\phi))(x) = \mathcal{F}_w^{-1}(a^{m+t}(x)\mathcal{F}_w(\mathcal{H}^t\phi))(x)$$

By inversion formula of the Weinstein transform (2.4), above get

$$(\mathcal{H}^{-(m+t)}(\mathcal{H}^t\phi))(x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x', \xi' \rangle} j_\beta(x_{n+1}\xi_{n+1}) a^{m+t}(\xi)\mathcal{F}_w(\mathcal{H}^t\phi)(\xi) \, d\mu_\beta(\xi).$$

Taking (4.6), we get

$$\begin{aligned}(\mathcal{H}^{-(m+t)}(\mathcal{H}^t\phi))(x) &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle} j_\beta(x_{n+1}\xi_{n+1}) a^{m+t}(\xi) a^{-t}(\xi)(\mathcal{F}_w\phi)(\xi)\, d\mu_\beta(\xi)\\ &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle} j_\beta(x_{n+1}\xi_{n+1}) a^{m}(\xi)(\mathcal{F}_w\phi)(\xi)\, d\mu_\beta(\xi)\\ &= (\mathcal{H}^{-m}\phi)(x).\end{aligned}$$

Therefore, from above expression we find that

$$(\mathcal{H}^{-(m+t)}(\mathcal{H}^t\phi)) = (\mathcal{H}^{-m}\phi).$$

In veiw of (4.12), we have

$$\begin{aligned}||\mathcal{H}^t\phi||_{m+t,p,\omega} &= ||(\mathcal{H}^{-m}\phi)||_p\\ &= ||\phi||_{m,p,\omega}.\end{aligned}$$

Let $\phi \in W^{m+t,p}_\omega(\mathbb{R}^{n+1}_+)$. Then from (4.7), we obtain

$$\mathcal{H}^{-(m+t)}\phi \in L^p(\mathbb{R}^{n+1}_+).$$

Again, we get

$$\begin{aligned}(\mathcal{H}^{-(m+t)}\phi)(x) &= \mathcal{F}_w^{-1}(a^{m+t}(x)(\mathcal{F}_w\phi))(x)\\ &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle} j_\beta(x_{n+1}\xi_{n+1}) a^{m+t}(\xi)(\mathcal{F}_w\phi)(\xi)\, d\mu_\beta(\xi)\\ &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle} j_\beta(x_{n+1}\xi_{n+1}) a^{m}(\xi) a^{t}(\xi)(\mathcal{F}_w\phi)(\xi)\, d\mu_\beta(\xi).\end{aligned}$$

Then applying (4.6), we find

$$\begin{aligned}(\mathcal{H}^{-(m+t)}\phi)(x) &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle} j_\beta(x_{n+1}\xi_{n+1}) a^{m}(\xi)\mathcal{F}_w(\mathcal{H}^{-t}\phi)(\xi)\, d\mu_\beta(\xi)\\ &= (\mathcal{H}^{-m}(\mathcal{H}^{-t}\phi))(x).\end{aligned}$$

This show that

$$\mathcal{H}^{-(m+t)}\phi = \mathcal{H}^{-m}(\mathcal{H}^{-t})\phi \in L^p(\mathbb{R}^{n+1}_+).$$

Hence,

$$(\mathcal{H}^{-t}\phi) \in W^{m,p}_\omega.$$

And, we have to obtain

$$\begin{aligned}\mathcal{H}^t(\mathcal{H}^{-t}\phi)(x) &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle} j_\beta(x_{n+1}\xi_{n+1}) a^{-t}(\xi)\mathcal{F}_w(\mathcal{H}^{-t}\phi)(\xi)\, d\mu_\beta(\xi)\\ &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle} j_\beta(x_{n+1}\xi_{n+1}) a^{-t}(\xi) b^{-t}(\xi)\mathcal{F}_w\phi)(\xi)\, d\mu_\beta(\xi)\\ &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle} j_\beta(x_{n+1}\xi_{n+1})(\mathcal{F}_w\phi)(\xi)\, d\mu_\beta(\xi)\\ &= \mathcal{F}^{-1}((\mathcal{F}_w\phi)(\xi))(x) = \phi(x).\end{aligned}$$

Therefore, for each $\phi \in W^{m,p}_{\omega}$, there exists an $\mathcal{H}^{-t}\phi \in W^{m+t,p}_{\omega}$ such that

$$\mathcal{H}^{t}\mathcal{H}^{-t}\phi = \phi.$$

Hence, $\mathcal{H}^{t}$ is onto.

Theorem 4.5. *Let $1 < p < \infty$ and $m = 0, 1, 2....$ then,*

$$||\mathcal{H}^{m}\phi|| \leq C||\phi||_{p} \ \forall \ \phi \in S_{\omega}(\mathbb{R}^{n+1}_{+}). \tag{4.13}$$

Proof: From (2.8), we have

$$\mathcal{F}_{w}(f\#_{\beta}\phi)(\xi) = (\mathcal{F}_{w}f)(\xi)(\mathcal{F}_{w}\phi)(\xi). \tag{4.14}$$

Take

$$a^{-m}(\xi) = \mathcal{F}_{w}(f(t))(\xi). \tag{4.15}$$

Then,

$$\begin{aligned} f(t) &= \mathcal{F}_{w}^{-1}(a^{-m}(\xi))(t) = \mathcal{F}_{w}(a^{-m}(\xi))(-t) \\ &= \int_{\mathbb{R}^{n+1}_{+}} e^{i\langle t', \xi' \rangle} j_{\beta}(t_{n+1}\xi_{n+1}) a^{-m}(\xi)\, d\mu_{\beta}(\xi) = \mathcal{F}_{w}^{-1}(a^{-m})(t). \end{aligned}$$

Utilising (4.1), it follows that

$$f(t) = \mathcal{F}_{w}^{-1}(a^{-m})(t) \in L^{1}(\mathbb{R}^{n+1}_{+}).$$

In view of (4.14), we have

$$\mathcal{F}_{w}(f\#_{\beta}\phi)(\xi) = \mathcal{F}_{w}(\mathcal{F}_{w}^{-1}(a^{-m}(t))(\xi)(\mathcal{F}_{w}\phi)(\xi) = a^{-m}(\xi)(\mathcal{F}_{w}\phi)(\xi).$$

Therefore,

$$\begin{aligned} (f\#_{\beta}\phi)(t) &= \mathcal{F}_{w}^{-1}(a^{-m}(\xi)(\mathcal{F}_{w}\phi)(\xi))(t) \\ &= \mathcal{F}_{w}^{-1}(a^{-m}\mathcal{F}_{w}\phi)(t) \\ &= \mathcal{H}^{m}\phi. \end{aligned}$$

Then

$$||\mathcal{H}^{m}\phi||_{p} = ||f\#_{\beta}\phi||_{p}.$$

Using (2.9), we obtain

$$||\mathcal{H}^{m}\phi||_{p} \leq ||f||_{1}||\phi||_{p} \leq C||\phi||_{p}.$$

This proves the theorem.

Theorem 4.6. *Let $1 < p < \infty$ and $m \leq l$. Then*

$$W^{l,p}_{\omega}(\mathbb{R}^{n+1}_{+}) \subseteq W^{m,p}_{\omega}(\mathbb{R}^{n+1}_{+}),$$

and

$$||\phi||_{m,p,\omega} \leq C||\phi||_{l,p,\omega}.$$

Proof: Let $\phi \in W^{l,p}_{\omega}(\mathbb{R}^{n+1}_{+})$. Then from (4.7), we have

$$\mathcal{H}^{-l}\phi \in L^{p}(\mathbb{R}^{n+1}_{+}).$$

Then

$$
\begin{aligned}
(\mathcal{H}^{-m}\phi)(x) &= \mathcal{F}_w^{-1}(a^m \mathcal{F}_w \phi)(x) \\
&= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\, \xi' \rangle} j_\beta(x_{n+1}\xi_{n+1}) a^m(\xi)(\mathcal{F}_w\phi)(\xi)\, d\mu_\beta(\xi) \\
&= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\, \xi' \rangle} j_\beta(x_{n+1}\xi_{n+1}) a^{-(l-m)}(\xi) a^l(\xi)(\mathcal{F}_w\phi)(\xi)\, d\mu_\beta(\xi).
\end{aligned}
$$

From (4.6), we obtain

$$
(\mathcal{H}^{-m}\phi)(x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\, \xi' \rangle} j_\beta(x_{n+1}\xi_{n+1}) a^{-(l-m)}(\xi)\mathcal{F}_w(\mathcal{H}^l\phi)(\xi)\, d\mu_\beta(\xi).
$$

From (4.4), we can find above expression

$$
(\mathcal{H}^{-m}\phi)(x) \ = \mathcal{H}^{l-m}(\mathcal{H}^{-l}\phi)(x).
$$

Therefore, in view of (4.7), we have

$$
||\phi||_{m,p,\omega} = ||\mathcal{H}^{-m}\phi||_p = ||\mathcal{H}^{l-m}(\mathcal{H}^{-l}\phi)||_p.
$$

Using (4.13), we get

$$
||\phi||_{m,p,\omega} \leq C||\mathcal{H}^{-l}\phi||_p = ||\phi||_{l,p,\omega}.
$$

Hence,

$$
||\phi||_{m,p,\omega} \leq ||\phi||_{l,p,\omega}.
$$

This proves the theorem.

Theorem 4.7. *Let $1 < p < \infty$ and for any symbol $a^m, m \in \mathbb{Z}$ define the pseudo differential operator*

$$
(A(D)u)(x) = \mathcal{F}_w^{-1}(a^m \mathcal{F}_w u)(x).
$$

Then

$$
A(D) : W_\omega^{m,p}(\mathbb{R}_+^{n+1}) \longrightarrow W_\omega^{0,p}(\mathbb{R}_+^{n+1})
$$

is a bounded linear operator.

Proof: Consider the following operators;

$$
\begin{aligned}
\mathcal{H}^{-l} &: W_\omega^{l,p}(\mathbb{R}_+^{n+1}) \longrightarrow W_\omega^{0,p}(\mathbb{R}_+^{n+1}), \\
A(D)\mathcal{H}^m &: W_\omega^{0,p}(\mathbb{R}_+^{n+1}) \longrightarrow W_\omega^{0,p}(\mathbb{R}_+^{n+1}) \\
\mathcal{H}^{l-m} &: W_\omega^{0,p}(\mathbb{R}_+^{n+1}) \longrightarrow W_\omega^{l-m,p}(\mathbb{R}_+^{n+1}).
\end{aligned}
$$

We have to show that operator $\mathcal{H}^{-l}$ is bounded. Now we take $\phi \in W_\omega^{l,p}$, then

$$
||\phi||_{l,p,\omega} = ||\mathcal{H}^{-l}\phi||_p < \infty, \tag{4.16}
$$

and

$$
||\mathcal{H}^{-l}\phi||_{0,p,\omega} = ||\mathcal{H}^0(\mathcal{H}^{-l}\phi)||_p. \tag{4.17}
$$

Now,

$$
(\mathcal{H}^0(\mathcal{H}^{-l}\phi))(x) = \mathcal{F}_w^{-1}(a^0 \mathcal{F}_w(\mathcal{H}^{-l}\phi))(x).
$$

By the inverse Weinstein transform (2.4), we get

$$\begin{aligned}(\mathcal{H}^0(\mathcal{H}^{-l}\phi))(x) &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x'\ \xi'\rangle} j_\beta(x_{n+1}\xi_{n+1}) a^0(\xi)\mathcal{F}_w(\mathcal{H}^{-l}\phi)(\xi)\, d\mu_\beta(\xi)\\ &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x'\ \xi'\rangle} j_\beta(x_{n+1}\xi_{n+1})\mathcal{F}_w(\mathcal{H}^{-l}\phi)(\xi)\, d\mu_\beta(\xi)\\ &= \mathcal{F}_w^{-1}(\mathcal{F}_w(\mathcal{H}^{-l}\phi))(x) = (\mathcal{H}^{-l}\phi)(x).\end{aligned}$$

Therefore,

$$\mathcal{H}^0(\mathcal{H}^{-l}\phi) = (\mathcal{H}^{-l}\phi). \tag{4.18}$$

In veiw of (4.17), we get

$$||\mathcal{H}^{-l}\phi||_{0,p,\omega} = ||(\mathcal{H}^{-l}\phi)||_p < \infty.$$

We have to show that the second operator is bounded. Let $\phi \in W^{0,p}_\omega$. Then

$$||\phi||_{0,p,\omega} = ||\mathcal{H}^0\phi||_p = ||\phi||_p < \infty, \tag{4.19}$$

and

$$||A(D)\mathcal{H}^m\phi||_{0,p,\omega} = ||A(D)\mathcal{H}^m\phi||_p. \tag{4.20}$$

Now, we have to calculate

$$\begin{aligned}(A(D)\mathcal{H}^m\phi)(x) &= \mathcal{F}_w^{-1}(a^m(\xi)\mathcal{F}_w(\mathcal{H}^m\phi))(x)\\ &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\,\xi'\rangle} j_\beta(x_{n+1}\xi_{n+1}) a^m(\xi)\mathcal{F}_w(\mathcal{H}^m\phi)(\xi)\, d\mu_\beta(\xi).\end{aligned}$$

Therefore, using (4.6), we can obtain

$$\begin{aligned}(A(D)\mathcal{H}^m\phi)(x) &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\,\xi'\rangle} j_\beta(x_{n+1}\xi_{n+1}) a^m(\xi)a^{-m}(\xi)(\mathcal{F}_w\phi)(\xi)\, d\mu_\beta(\xi)\\ &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\,\xi'\rangle} j_\beta(x_{n+1}\xi_{n+1})(\mathcal{F}_w\phi)(\xi)\, d\mu_\beta(\xi)\\ &= \mathcal{F}_w^{-1}(\mathcal{F}_w\phi)(x) = \phi(x).\end{aligned}$$

Using (4.19) and (4.20) we get

$$||A(D)\mathcal{H}^m\phi||_{0,p,\omega} = ||\phi||_p < \infty.$$

Third operator $\mathcal{H}^{l-m}$, is bounded by the following way:

$$||\mathcal{H}^{l-m}\phi||_{l-m,p,\omega} = ||\mathcal{H}^{-(l-m)}\mathcal{H}^{l-m}\phi||_p = ||\phi||_p < \infty.$$

Hence, the product $\mathcal{H}^{l-m}A(D)\mathcal{H}^{m-l}$ is a bounded linear operator from $W^{l,p}_\omega(\mathbb{R}^{n+1}_+)$ into $W^{l-m,p}_\omega(\mathbb{R}^{n+1}_+)$.

Theorem 4.8. *For $m \in \mathbb{Z}$ and symbol a^m, the pseudo differential operator*

$$A(D) = \mathcal{F}_w^{-1}a^m\mathcal{F}_w : W^{l,p}_\omega(\mathbb{R}^{n+1}_+) \longrightarrow W^{l-m,p}_\omega(\mathbb{R}^{n+1}_+)$$

is a bounded linear operator.

Proof: Let $\phi \in W^{l,p}_{\omega}(\mathbb{R}^{n+1}_+)$. Then

$$||\phi||_{l,p,\omega} = ||\mathcal{H}^{-l}\phi||_p < \infty,$$

and

$$||A(D)\phi||_{l-m,p,\omega} = ||\mathcal{H}^{m-l}(A(D)\phi)||_p < \infty. \tag{4.21}$$

Therefore, we have

$$\begin{aligned}
\mathcal{H}^{m-l}(A(D)\phi)(x) &= \mathcal{F}_w^{-1}(a^{-(m-l)}\mathcal{F}_w A(D)\phi)(x) \\
&= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\,\xi'\rangle} j_\beta(x_{n+1}\xi_{n+1}) a^{-(m-l)}(\xi)\mathcal{F}_w(A(D)\phi)(\xi)\, d\mu_\beta(\xi) \\
&= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\,\xi'\rangle} j_\beta(x_{n+1}\xi_{n+1}) a^{-(m-l)}(\xi)\mathcal{F}_w(\mathcal{F}_w^{-1} a^m \mathcal{F}_w\phi)(\xi)\, d\mu_\beta(\xi) \\
&= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\,\xi'\rangle} j_\beta(x_{n+1}\xi_{n+1}) a^{-(m-l)}(\xi) a^m(\xi)(\mathcal{F}_w\phi)(\xi)\, d\mu_\beta(\xi) \\
&= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\,\xi'\rangle} j_\beta(x_{n+1}\xi_{n+1}) a^{l}(\xi)(\mathcal{F}_w\phi)(\xi)\, d\mu_\beta(\xi)
\end{aligned}$$

By (2.4), we have

$$\begin{aligned}
\mathcal{H}^{m-l}(A(D)\phi)(x) &= \mathcal{F}_w^{-1}(a^l\mathcal{F}_w\phi)(x) \\
&= \mathcal{H}^{-l}\phi(x).
\end{aligned}$$

Therefore, we have

$$\mathcal{H}^{m-l}(A(D)\phi)(x) = \mathcal{H}^{-l}\phi(x). \tag{4.22}$$

Using (4.21) and (4.22) the above expression yeilds

$$||A(D)\phi||_{l-m,p,\omega} = ||\mathcal{H}^{-l}\phi)||_p < \infty.$$

Above shows that the mapping

$$A(D) : W^{l,p}_{\omega}(\mathbb{R}^{n+1}_+) \longrightarrow W^{l-m,p}_{\omega}(\mathbb{R}^{n+1}_+)$$

is a bounded linear operator.

Conclusion

The Weinstein transform has a rich calculus and involves many problems in the mathematical sciences and their applications. Utilizing this theory, many researchers obtained observations in the theory of Sobolev-type spaces.
The Sobolev space is heavily used in partial differential equations and has wide applications in wavelets, quantum calculus, and other areas of mathematics. The authors made a proper framework for the generalised Sobolev space with the help of definitions and properties. In the present paper, the generalized Sobolev space of type $G^{p,\,s}_{\omega}$ is introduced and with the help of the Weinstein transform various properties are obtained. The L^p-space of Hankel potential, $W^{m,p}_{\omega}(\mathbb{R}^{n+1}_+)$ is cosidered and many results related to the Hankel potential $\mathcal{H}^k$ is studied by exploiting the theory of Weinstein transform. Various updated research works related to the generalized distributions and generalized Sobolev space are given in form of references.

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