Poincare Journal of Analysis & Applications
Vol. 10, No. 3 (2023), Special issue, 165-170
©*Poincare Publishers*
DOI: 10.46753/pjaa.2023.v010i03.010

ON $\mathcal{H}$-FRAME INDUCED MONOMORPHISMS FOR A REDUCING SUBSPACE $\mathcal{H}$

SWATI SRIVASTAVA † AND RESHMA SINGH

Date of Receiving	:	03. 04. 2023
Date of Revision	:	01. 02. 2024
Date of Acceptance	:	23. 02. 2024

Abstract. In this paper, we characterize the $\mathcal{H}$-frame induced monomorphism for a normalized tight $\mathcal{H}$-frame wavelet set in a reducing subspace $\mathcal{H}$. This leads to a construction of normalized tight $\mathcal{H}$-frame wavelet sets. Also, we study symmetric normalized tight $\mathcal{H}$-frame wavelet sets as a special case. Further, fixed point sets of these maps are studied. Different examples are considered for illustration.

1. Introduction

A function $\psi \in L^2(\mathbb{R})$ is said to be an *orthonormal wavelet* if the system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$, where $\psi_{j,k} \equiv 2^{j/2}\psi(2^j \cdot -k)$, $j, k \in \mathbb{Z}$, forms an orthonormal basis for $L^2(\mathbb{R})$. If $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ forms (tight, normalized tight) frame for $L^2(\mathbb{R})$, then the function ψ is called a (tight, normalized tight) *frame wavelet* for $L^2(\mathbb{R})$. If the modulus of the Fourier transform of a wavelet is a characteristic function on a measurable set W in $\mathbb{R}$, then the set W is called a *wavelet set*. The Lebesgue measure of a wavelet set is 2π. If the characteristic function on a measurable set is modulus of Fourier transform of a normalized tight frame wavelet, then we call the measurable set as a *frame wavelet set*. The Lebesgue measure of a frame wavelet set is less than or equal to 2π. Thus, a wavelet is particularly a normalized tight frame wavelet and a frame wavelet set is a wavelet set if its Lebesgue measure is 2π. A study of frame wavelet sets can be found in Ref. [1, 2, 3, 4, 5, 6] and [12].

Let $\mathcal{H}$ be a closed subspace of $L^2(\mathbb{R})$. We say that $\mathcal{H}$ is a *reducing subspace* if $f(2^m \cdot -l) \in \mathcal{H}$ for all $f \in \mathcal{H}$ and $m, l \in \mathbb{Z}$. A characterization of reducing subspaces is given as follows:

Theorem 1.1. [8] *Let $\mathcal{H}$ be a closed subspace of $L^2(\mathbb{R})$. Then $\mathcal{H}$ is a reducing subspace if and only if $\hat{\mathcal{H}} = L^2(\mathbb{R}) \cdot \chi_{\mathbb{E}}$ for some measurable set $\mathbb{E}$ of $\mathbb{R}$ with the property $\mathbb{E} = 2\mathbb{E}$.*

2010 *Mathematics Subject Classification.* 42C40, 54H25.
Key words and phrases. wavelet set, frame wavelet set, wavelet induced isomorphism, frame induced monomorphism, fixed point set.
Communicated by: Firdous Ahmad Shah
† *Corresponding author*

Note that $\hat{\mathcal{H}}$ is the collection of the Fourier transforms of all $f \in \mathcal{H}$. For $\mathbb{E} = [0, \infty)$, the reducing subspace is called the *Hardy space*, and for $\mathbb{E} = \mathbb{R}$, the reducing space is the $L^2(\mathbb{R})$ itself. Different examples of reducing subspaces can be found in Ref. [2], [5], [8] and [10].

A function $\psi \in \mathcal{H}$ is said to be an *$\mathcal{H}$-frame wavelet* if the system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ forms a normalized tight frame for $\mathcal{H}$. If $\left|\hat{\psi}\right| = \chi_F$ for a measurable set F of $\mathbb{E}$, then F is called an *$\mathcal{H}$-frame wavelet set*. In particular, if ψ is an $\mathcal{H}$-wavelet, then F is an $\mathcal{H}$-wavelet set.

The existence of an $\mathcal{H}$-wavelet set is shown in [8] while the existence of an $\mathcal{H}$-frame wavelet set is shown in [2].

Let E and F be two measurable subsets of $\mathbb{R}$. For $\alpha > 0$, E is said to be *α-translation congruent* to F, if there exists a measurable partition $\{E_n : \ n \in \mathbb{Z}\}$ of E such that $\{E_n + n\alpha : \ n \in \mathbb{Z}\}$ is a measurable partition of F. If $\alpha = 1$, then E is said to be *$\mathbb{Z}$-translation congruent* to F. The set E is said to be *dilation congruent* to F, if there exists a measurable partition $\{E_n : \ n \in \mathbb{Z}\}$ of E such that $\{2^n E_n : \ n \in \mathbb{Z}\}$ is a measurable partition of F. Also, a measurable set E is said to be *α-translation generator* of $\mathbb{R}$ if $\{E_n + n\alpha : \ n \in \mathbb{Z}\}$ is a measurable partition of $\mathbb{R}$. If $\alpha = 1$, then E is said to be a *$\mathbb{Z}$-translation generator* of $\mathbb{R}$.

A measurable set W is a wavelet set in $\mathbb{R}$ if and only if it is both 2π-translation congruent and dilation congruent to $[-2\pi, -\pi) \cup [\pi, 2\pi)$[7].

A characterization of an $\mathcal{H}$-frame wavelet set is given as following

Theorem 1.2. [8] *A measurable set $F \subset \mathbb{E}$ is an $\mathcal{H}$-frame wavelet set if and only if*

1. *F is 2π-translation congruent to a subset of $[0, 2\pi)$, and*
2. *$\{2^m F : m \in \mathbb{Z}\}$ is a measurable partition of $\mathbb{E}$.*

In particular, if F is 2π-translation congruent to $[0, 2\pi)$, then F is an $\mathcal{H}$-wavelet set.

Let $\tau : \mathbb{R} \to E$ and $\delta : \mathbb{R} \setminus \{0\} \to E$ be maps defined, respectively, by $\tau(x) = x + 2k\pi$ and $\delta(x) = 2^j x$, where $E = [-2\pi, -\pi) \cup [\pi, 2\pi)$ is the Shannon wavelet set and k and j are the unique integers such that $x + 2k\pi$ and $2^j x$ are in E.

Suppose F is a frame wavelet set and E the Shannon wavelet set. Then $\delta|_F : F \to E$ is a measurable bijection, while $\tau|_F : F \to E$ is a measurable injection [6]. Thus, the map

$$h_F = \tau|_F \circ \delta|_F^{-1} : E \to E$$

is a measurable injection. Consider the bijective map $\xi : E \to [0, 1)$ defined by

$$\xi(x) = \begin{cases} \frac{x}{2\pi} & \text{if } x \in [\pi, 2\pi), \\ \frac{x}{2\pi} + 1 & \text{if } x \in [-2\pi, -\pi). \end{cases}$$

Then the map $\tilde{h}_F : [0, 1) \to [0, 1)$ defined by

$$\tilde{h}_F(x) = \xi \circ h_F \circ \xi^{-1}(x), \ x \in [0, 1),$$

is a measurable injection. We call this map to be the *frame induced monomorphism* for F. We have the following characterization for frame induced monomorphisms [12].

Theorem 1.3. *Let F be a frame wavelet set. Then there exist measurable partitions $\{A_k\}_{k\in\mathbb{Z}}$ of $\left[\frac{1}{2}, 1\right)$ and $\{B_k\}_{k\in\mathbb{Z}}$ of $\left[0, \frac{1}{2}\right)$, such that the frame induced monomorphism*

$\tilde{h}_F$ for F has the form

$$\tilde{h}_F(x) = \begin{cases} \lfloor 2^k x \rfloor & \text{if } x \in A_k, \\ \lfloor 2^k (x-1) \rfloor & \text{if } x \in B_k, \end{cases} \tag{1.1}$$

where $\lfloor x \rfloor$ denotes the fractional part of a real number x.

Moreover, if $h : [0,1) \to [0,1)$ is a measurable injection defined likewise, then there is a frame wavelet set F such that $h = \tilde{h}_F$.

2. $\mathcal{H}$-Frame Induced Monomorphism for an $\mathcal{H}$-frame wavelet set

Let E be a fixed $\mathcal{H}$-wavelet set in reducing subspace $\mathcal{H}$. Let F be an $\mathcal{H}$-frame wavelet set. From Theorem 1.2, we conclude that the map $\delta|_F : F \to E$ is a measurable bijection, while the map $\tau|_F : F \to E$ is a measurable injection. Define the map $h_F = \tau|_F \circ \delta|_F^{-1}$. Thus, $h_F : E \to E$ is an injective map.

The measurable set E, being an $\mathcal{H}$-wavelet set, is a 2π- translation generator of $\mathbb{R}$. Therefore, the set $\frac{E}{2\pi}$ is a $\mathbb{Z}$-translation generator of $\mathbb{R}$. Observe that the set $[0, 2\pi)$ is also a $\mathbb{Z}$-translation generator of $\mathbb{R}$. Therefore, we can define the map

$$\phi : \frac{E}{2\pi} \to [0,1)$$

given by

$$\phi(x) = x + l$$

for some $l \in \mathbb{Z}$. Hence, we have a bijective map

$$\xi : E \to [0,1)$$

defined by

$$\xi(x) = \frac{x}{2\pi} + l$$

for $l \in \mathbb{Z}$. Thus, we get measurable partitions $\{E_i\}_{i\in\mathbb{Z}}$ of E and $\{I_i\}_{i\in\mathbb{Z}}$ of $[0,1)$ for which

$$\xi(x) = \frac{x}{2\pi} + i \text{ for } x \in E_i,\ i \in \mathbb{Z}.$$

With the help of the map ξ, we define $\tilde{h}_F = \xi \circ h_F \circ \xi^{-1}$. Thus,

$$\tilde{h}_F : [0,1) \to [0,1)$$

is an injective map. We call this map as *$\mathcal{H}$-frame induced monomorphism* for the $\mathcal{H}$-frame wavelet set F.

We fix the notation $\{I_i\}_{i\in\mathbb{Z}}$ for the measurable partition of $[0,1)$ corresponding to the fixed $\mathcal{H}$-wavelet set E.

Theorem 2.1. *Let F be an $\mathcal{H}$-frame wavelet set. Then there is a measurable partition $\left\{I_i^k\right\}_{k\in\mathbb{Z}}$ of each $\{I_i\}, i \in \mathbb{Z}$ such that the $\mathcal{H}$-frame induced monomorphism $\tilde{h}_F$ is given by*

$$\tilde{h}_F(x) = \left\lfloor 2^k (x-i) \right\rfloor,\ x \in I_i^k. \tag{2.1}$$

Moreover, if $h : [0,1) \to [0,1)$ is an injective map defined in the same manner, then there exists an $\mathcal{H}$-frame wavelet set F such that $h = \tilde{h}_F$ a.e.

Proof. Let $x \in I_i$, $i \in \mathbb{Z}$. Then $\xi^{-1}(x) = 2\pi(x-i) \in E_i$. Hence $\delta|_F^{-1} \circ \xi^{-1}(x) = 2^k \cdot 2\pi(x-i) \in F$ for some $k \in \mathbb{Z}$. Collect all $x \in I_i$ with the same k and name the set I_i^k. Thus, for $x \in I_i^k$,

$$\tau|_F \circ \delta|_F^{-1} \circ \xi^{-1}(x) = 2^k \cdot 2\pi(x-i) + 2\pi l \in E$$

for some $l \in \mathbb{Z}$. Hence,

$$\xi \circ \tau|_F \circ \delta|_F^{-1} \circ \xi^{-1}(x) = 2^k(x-i) + l'$$

for $l' \in \mathbb{Z}$. Therefore,

$$\tilde{h}_F(x) = \lfloor 2^k(x-i) \rfloor \in [0,1).$$

Next, suppose $h : [0,1) \to [0,1)$ is an injective map defined likewise, i.e.

$$h(x) = \lfloor 2^k(x-i) \rfloor, \ x \in I_i^k$$

for measurable partition $\{I_i^k\}_{k\in\mathbb{Z}}$ of each $\{I_i\}, i \in \mathbb{Z}$. Define a map $h_1 : E \to E$ by

$$h_1(x) = \xi^{-1} \circ h \circ \xi(x), \ x \in E.$$

Then h_1 is an injective map and

$$h_1(x) = 2^{k(x)}x + 2\pi l(x)$$

for $k(x), l(x) \in \mathbb{Z}$ for $x \in E$. Define a map $\psi : E \to \mathbb{E}$ by

$$\psi(x) = 2^{k(x)}x$$

and set $F = \psi(E)$. Thus $\psi : E \to F$ is a measurable bijection. Now define $\phi : F \to E$ by

$$\phi(x) = x + 2\pi l(\psi^{-1}(x)), \ x \in F.$$

Therefore, we have

$$\phi \circ \psi(x) = 2^{k(x)}x + 2\pi l(x) = h_1(x), \ x \in E.$$

Note that $\psi = \delta|_F^{-1}$ and $\phi = \tau|_F$. Hence, we have that F is an $\mathcal{H}$-frame wavelet set and $h = \tilde{h}_F$.

The above theorem generalizes the results found in [11] and [12] and provides a construction of $\mathcal{H}$-frame wavelet sets with help of injective maps.

Let $\mathbb{E} = \pm 2\mathbb{E}$. We say that a set A is symmetric if $A = -A$. Following is the result regarding symmetric $\mathcal{H}$-frame wavelet set. The proof is similar as that given in [11].

Theorem 2.2. *An $\mathcal{H}$-frame induced monomorphism h is defined on measurable partitions $\{I_i^k\}, k \in \mathbb{Z}$ of $[\frac{1}{2}, 1)$ and $\{J_i^k\}, k \in \mathbb{Z}$ of $[0, \frac{1}{2})$ by*

$$h(x) = \begin{cases} \lfloor 2^k(x-i) \rfloor & \text{if } x \in I_i^k, \\ \lfloor 2^k(x+i-1) \rfloor & \text{if } x \in J_i^k, \end{cases},$$

where

$$J_i^k = 1 - I_i^k$$

if and only if the map h is associated with a symmetric $\mathcal{H}$-frame wavelet set.

Theorem 2.3. *Let $Fix\,\tilde{h}_F$ denotes the fixed point set of the $\mathcal{H}$-frame induced monomorphism $\tilde{h}_F$ for the $\mathcal{H}$-frame wavelet set F. Then we have*

$$Fix\,\tilde{h}_F = \xi(F \cap E).$$

Proof. For $x \in I_i^k$, $\tilde{h}_F$ is given by

$$\tilde{h}_F = \left\lfloor 2^k(x-i) \right\rfloor, \ k \in \mathbb{Z}.$$

Then for $u = \xi^{-1}(x)$,

$$\delta^{-1}(u) = 2^k u, \text{ where } k \in \mathbb{Z}.$$

Therefore,

$$\tilde{h}_F(x) = x \text{ if and only if } k = 0.$$

Equivalently,

$$x \in \text{Fix } \tilde{h}_F \text{ if and only if } x \in \xi(F \cap E).$$

3. Illustrations

1. Consider the Hardy space H^2 for which $\mathbb{E} = [0, \infty)$. Fix the H^2-wavelet set $E = [2\pi, 4\pi)$. Then the map $\xi : E \to [0,1)$ is given by

$$\xi(x) = \frac{x}{2\pi} - 1.$$

For an H^2-frame wavelet F, the H^2-frame induced monomorphism is given by

$$\tilde{h}_F = \left\lfloor 2^k(x+1) \right\rfloor, \ x \in I^k,$$

where $\left\{I^k\right\}_{k\in\mathbb{Z}}$ is a measurable partition of $[0,1)$.

For example, $F = \left[2^{-n}\pi, 2^{-n+1}\pi\right)$, $n \in \mathbb{N}$, is an H^2-frame wavelet set. The H^2-frame induced monomorphism $\tilde{h}_F$ is found to be

$$\tilde{h}_F(x) = 2^{-n-1}(x+1), \ x \in [0,1).$$

If an injective map $h : [0,1) \to [0,1)$ is given by

$$h = \begin{cases} \left\lfloor 2^{-4}(x+1) \right\rfloor & \text{if } x \in [0, \frac{1}{2}), \\ \left\lfloor 2^{-2}(x+1) \right\rfloor & \text{if } x \in [\frac{1}{2}, 1), \end{cases}$$

then the H^2-frame wavelet set is given by $\left[\frac{\pi}{8}, \frac{3\pi}{16}\right) \cup \left[\frac{3\pi}{4}, \pi\right)$.

Also, since for a.e. $x \in [0,1), h(x) \neq x$, the measure of the set Fix h is zero.

2. Consider the reducing subspace $\mathcal{H}$ for which $\mathbb{E} = \bigcup_{j\in\mathbb{Z}} 2^j([-3\pi, -2\pi) \cup [2\pi, 3\pi))$. Fix the $\mathcal{H}$-wavelet set $E = [-3\pi, -2\pi) \cup [2\pi, 3\pi)$. Then $\xi : E \to [0,1)$ is given by

$$\xi(x) = \begin{cases} \frac{x}{2\pi} - 1 & \text{if } x \in [2\pi, 3\pi), \\ \frac{x}{2\pi} + 2 & \text{if } x \in [-3\pi, -2\pi) \end{cases}$$

and the partition of $[0,1)$ is given by $I_{-1} = \left[0, \frac{1}{2}\right)$ and $I_2 = \left[\frac{1}{2}, 1\right)$. Let $F = \left[\frac{-3\pi}{2}, -\pi\right) \cup \left[\pi, \frac{3\pi}{2}\right)$. Then F is translation congruent to $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right)$, a subset of $[0, 2\pi)$, and $\cup_{m\in\mathbb{Z}} 2^m F$ is a measurable partition of $\mathbb{E}$. Thus, F is an $\mathcal{H}$-frame wavelet set. The $\mathcal{H}$-frame induced monomorphism is given by

$$\tilde{h}_F = \begin{cases} \left\lfloor 2^{-1}(x+1) \right\rfloor & \text{if } x \in [0, \frac{1}{2}), \\ \left\lfloor 2^{-1}(x-2) \right\rfloor & \text{if } x \in [\frac{1}{2}, 1), \end{cases}$$

or

$$\tilde{h}_F = \begin{cases} \frac{x+1}{2} & \text{if } x \in [0, \frac{1}{2}), \\ \frac{x}{2} & \text{if } x \in [\frac{1}{2}, 1), \end{cases}$$

Further, note that in this case $F \cap E$ has measure zero, and hence Fix $\tilde{h}_F$ has also measure zero.

3. In the above example of reducing subspace $\mathcal{H}$ with the same fixed $\mathcal{H}$-wavelet set E, consider $F = [-3\pi, -2\pi) \cup [2\pi, 3\pi)$. Then $F \cap E = E$. Hence, Fix $\tilde{h}_F = E$, or equivalently, Fix $\tilde{h}_F$ has measure 2π.

References

[1] X. Dai, Y. Diao and Q. Gu, Frame wavelet sets in $\mathbb{R}$, Proc. Amer. Math. Soc., 132 (2004), 2567–2575.

[2] X. Dai, Y. Diao and Q. Gu, Subspaces with normalized tight frame wavelets in $\mathbb{R}$, Proc. Amer. Math. Soc., 30 (2001), no. 6, 1661–1667.

[3] X. Dai, Y. Diao, Q. Gu and D. Han, Frame wavelet sets in $\mathbb{R}^d$, J. Comput. Appl. Math., 155 (2003), no. 1, 69–82.

[4] X. Dai, Y. Diao, Q. Gu and D. Han, Frame wavelets in subspaces of $L^2(\mathbb{R}^d)$, Proc. Amer. Math. Soc., 130 (2002), no. 11, 3259–3267.

[5] X. Dai, Y. Diao, Q. Gu and D. Han, The existence of subspace wavelet sets, J. Comput. Appl. Math., 155 (2003), 83–90.

[6] D. Han and D. R. Larson, Frames, bases and group representation, Mem. Amer. Math. Soc., 147 (2000), no. 697, x+94 pp.

[7] X. Dai and D. R. Larson, Wandering vectors for unitary systems and orthogonal wavelets, Mem. Amer. Math. Soc., 134 (1998), no. 640, viii+68 pp.

[8] X. Dai and S. Lu, Wavelets in subspaces, Michigan Math. J., 43 (1996), 81–98.

[9] E. J. Ionascu, A new construction of wavelet sets, Real Anal. Exchange, 28 (2002/2003), no. 2, 593–609.

[10] N. K. Shukla and R. Dubey, Joint dilation scaling sets on the reducing subspaces, Adv. Pure Appl. Math., 3 (2012), no. 3, 329–349.

[11] S. Srivastava and G. C. S. Yadav, On wavelet induced isomorphisms for reducing subspaces, Int. J. Wavelets Multiresolut. Inf. Process., 14 (2016), no. 3, Art. ID 1650009, 12 pp.

[12] S. Srivastava and G. C. S. Yadav,On fixed point sets of wavelet induced isomorphisms and frame induced monomorphisms, Int. J. Wavelets Multiresolut. Inf. Process., 14 (2016), no. 3, Art. ID 1650016, 9 pp.

(SWATI SRIVASTAVA) Department of Mathematics, Shyama Prasad Mukherjee Government Degree College, University of Allahabad, Prayagraj, India

E-mail address: `swatisrivastava276@gmail.com`

(RESHMA SINGH) Department of Mathematics, Rambriksh Benipuri Mahila College, Babasaheb Bhimrao Ambedkar Bihar University, Muzaffarpur, Bihar, India

E-mail address: `reshma.gzp@gmail.com`