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APPROXIMATION OF FUNCTIONS BELONGING TO $C^{M,\alpha}[0,1)$ CLASS AND SOLUTION OF CHANDRASEKHAR'S WHITE DWARFS AND PANTOGRAPH DIFFERENTIAL EQUATION BY GENOCCHI WAVELETS

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Abstract. In this paper, the approximation of the solution function f for Chandrasekhar's white dwarfs and the Pantograph differential equation of class $C^{M,\alpha}[0,1)$ by the $(2^{k-1}, M)^{th}$ partial sums of their Genocchi wavelet expansion in the interval $[0,1)$ has been estimated. The Genocchi wavelet technique has been employed to determine the solution of Chandrasekhar's white dwarfs and the Pantograph differential equation. The solution obtained through the Genocchi wavelet method approaches their exact solution and is compared to the Chebyshev wavelet method, Legendre wavelet method, and ODE-45 method. This represents an achievement of wavelet analysis in this research article.

1. Introduction

Wavelet theory stands out as a versatile tool widely utilized across diverse fields such as engineering, biotechnology, physics, viscoelastic materials, experimental physics, biosciences, and statistical mechanics [3], [9], [12], [10], [21], [26], [29], [30]. Its applications extend to solving real-world challenges, including the detection of submarines and airplanes. This theory is an integral part of the expanding realm of numerical studies, where functions are systematically broken down into smaller versions of a basic wavelet for in-depth analysis [32], [6]. In tandem with wavelet theory, Fourier analysis plays a crucial role in the approximation process. Fourier analysis employs trigonometric polynomials and commonly complements the methodology discussed earlier (Debnath [7], Meyer [19], Zygmund [32]). Together, these analytical tools

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contribute to a comprehensive understanding and solution to complex problems in various scientific and engineering disciplines.

This research delves into the application of wavelet theory, particularly focusing on Genocchi wavelets [18], to approximate the solution function of two significant equations: Chandrasekhar's white dwarfs [5] and the Pantograph differential equation [23]. The solution of these equations belong to the $C^{M,\alpha}[0,1)$ class of order α, where $\alpha \in (0,1]$. The motivation for selecting Genocchi wavelets lies in their demonstrated efficiency and accuracy in approximating a variety of functions [2], [18]. These wavelets offer a robust solution for capturing the intricate behavior of functions, making them particularly suitable for approximating the solution function f in Chandrasekhar's white dwarfs and the Pantograph differential equation. The overarching advantage of using wavelets, in general, lies in their adaptability to local features and their ability to effectively represent sharp changes or singularities [8], [14], [15], [25].

The choice of Chandrasekhar's white dwarfs and the Pantograph differential equation as the primary focus stems from their profound significance in astrophysics and mathematical modeling. Chandrasekhar's white dwarf equation [5] provides insights into the equilibrium state of a degenerate gas of electrons, contributing to our understanding of the structure and properties of white dwarfs stars. On the other hand, the Pantograph differential equation [23], a specialized type of delay differential equation, finds applications in diverse fields, including economics and engineering. Remarkably, there is a notable gap in existing literature concerning the approximations of the solution function f of Chandrasekhar's white dwarfs and the Pantograph differential equation belonging to the $C^{M,\alpha}[0,1)$ class using the Genocchi wavelet technique [12], [14]), [11], [16], [22], [17].

This research advances the analysis by considering the solution of Chandrasekhar's white dwarfs and the Pantograph differential equation using Genocchi wavelets in the interval [0,1). Specifically, within the $C^{M,\alpha}[0,1)$ class, the convergence analysis of the solution function f has been thoroughly investigated through Genocchi wavelet series. Noteworthy contributions from researchers, including Adem et al. [11], Lee et al. [16], Hussain et al. [8], Li et al. [17], Yadav et al. [28], Lee et al. [16] and Singh et al. [25], in the realm of solving differential and integral equations, provide valuable context. To facilitate an in-depth analysis, this research paper outlines its objectives: (i) To define the Genocchi polynomial, Genocchi wavelet, and $C^{M,\alpha}[0,1)$ class in the interval [0,1). (ii) To derive the approximation of the solution function f of Chandrasekhar's white dwarf and the Pantograph differential equation belonging to class $C^{M,\alpha}[0,1)$. (iii) To introduce the process for computing the numerical solution of the differential equation and illustrate the effectiveness of this process through examples. (iv) To compare the exact solution and the solution of Chandrasekhar's white dwarfs and the Pantograph differential equation obtained by the Genocchi wavelet with solutions derived by Chebyshev wavelet, Legendre wavelet, and ODE-45 methods.

The subsequent sections of this article are organized as follows, offering a systematic exploration of the research findings. Section 2 provides basic definitions and properties of the class $C^{M,\alpha}[0,1)$, the Genocchi wavelet, and the orthonormality of the Genocchi wavelet. In Section 3, the convergence analysis of the solution function f of Chandrasekhar's white dwarfs and the Pantograph differential equation through Genocchi wavelet series is thoroughly investigated. Section 4 delves into the definition of Genocchi wavelet approximation and estimator in the class $C^{M,\alpha}[0,1)$ by the Genocchi wavelet. Section 5 presents the algorithm for solving linear and nonlinear differential equations in the interval [0,1) and applies it to obtain the solution of Chandrasekhar's white dwarfs and the Pantograph differential equation. In Section 6, the solution of Chandrasekhar's white dwarfs and the Pantograph differential equation using the Genocchi wavelet method is compared with solutions obtained through the Chebyshev wavelet method, Legendre wavelet method, and ODE-45 method. Finally, Section 7 concludes this research paper, summarizing the key findings and outlining potential avenues for future exploration.

2. Definitions and Preliminaries

2.1. Function of $C^{M,\alpha}[0,1)$ class. A function $f \in C^{M,\alpha}[0,1)$, (Titchmarsh [27]), $\alpha \in$ (0,1] if

$$f^{(M)}(x) - f^{(M)}(y) = O(|x-y|^{\alpha}), \forall\ x, y \in [0,1).$$

2.2. Genocchi Polynomial. The function $G_m(t)$ be the Genocchi Polynomial of degree m

$$\frac{2t}{e^t+1}e^{xt} \quad = \quad e^{G(x)t} = \sum_{m=0}^{\infty} G_m(x)\frac{t^m}{m!} \tag{2.1}$$

and which has values $G_0(t) = 0$, $G_1(t) = 1$, $G_2(t) = 2t-1$, $G_3(t) = 3t^2-3t$, $G_4(t) = 4t^3-6t^2+1$, $G_5(t) = 5t^4-10t^3+5t$.
Some of the properties of Genocchi polynomials are (Seog, Joung and Joohee [24]):

$$\int_0^1 G_m(t)G_{m'}(t)dt = \frac{(-1)^m(2m)!m'!}{(m+m')!}G_{m+m'},\ m, m' \geq 1 \tag{2.2}$$

$$\int_0^1 G_m(t)dt = -\frac{2G_{m+1}}{m+1},\ m \geq 1 \tag{2.3}$$

$$\frac{d}{dt}G_m(t) = mG_{m-1}(t),\ m \geq 1 \tag{2.4}$$

$$G_m(1) + G_m(0) = 0,\ m > 1. \tag{2.5}$$

2.3. Genocchi Wavelets. The Genocchi wavelets [18] $\psi_{n,m}(t)$ are describe over the interval [0,1) in the following way:

$$\psi_{n,m}(t) = \begin{cases} \frac{2^{\frac{k-1}{2}}}{\sqrt{R(m)}}G_m(2^k t - n + 1), & \text{if } t \in \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right); \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

Let k, M be the positive integers and $R(m)$ (Abdulnasir and Chang [2]) is defined as

$$R(m) = \frac{(-1)^m 2(m!)^2}{(2m!)}G_{2m} \tag{2.7}$$

where $n = 1, 2, 3, ..., 2^{k-1}$, $m = 1, 2, ..., M$ and G_{2m} is the well-known Genocchi number.

2.4. Orthonormality of Genocchi Wavelets. $\{\psi_{n,m}(t)\}$ forms an orthonormal set for $n = 1, 2, ..., 2^{k-1}, m = 1, 2, 3...$, i.e.

$$\langle \psi_{n,m}, \psi_{n',m'} \rangle = \begin{cases} 1, & \text{if } \ n = n', m = m', \\ 0, & \text{otherwise.} \end{cases}$$

3. Convergence of Genocchi Wavelet Series

In this section, the convergence analysis of the solution function $f \in L^2[0,1)$ of Chandrasekhar's white dwarf and the Pantograph differential equation by Genocchi wavelet expansion has been discussed.

Theorem 3.1. *If $f(t)$ is the exact solution of the Pantograph differential equation then its Genocchi wavelet series $\sum_{n=1}^{2^{k-1}} \sum_{m=1}^{\infty} c_{n,m}\psi_{n,m}(t)$ converges uniformly to $f(t)$ in the interval [0,1).*

Proof.

$$\begin{aligned} \text{Let } \ f(t) &= \sum_{n=1}^{2^{k-1}} \sum_{m=1}^{\infty} c_{n,m}\psi_{n,m}(t). \\ \text{Then } \langle f, f \rangle &= \left\langle \sum_{n=1}^{2^{k-1}} \sum_{m=1}^{\infty} c_{n,m}\psi_{n,m}(t), \sum_{n'=1}^{2^{k-1}} \sum_{m'=1}^{\infty} c_{n',m'}\psi_{n',m'}(t) \right\rangle \\ &= \sum_{n=1}^{2^{k-1}} \sum_{m=1}^{\infty} \sum_{n'=1}^{2^{k-1}} \sum_{m'=1}^{\infty} c_{n,m}\overline{c_{n',m'}} < \psi_{n,m}, \psi_{n',m'} > \\ &= \sum_{n=1}^{2^{k-1}} \sum_{m=1}^{\infty} |c_{n,m}|^2 ||\psi_{n,m}||_2^2 \\ &= \sum_{n=1}^{2^{k-1}} \sum_{m=1}^{\infty} |c_{n,m}|^2, \ (\{\psi_{n,m}\} \text{ is an orthonormal basis of } L^2[0,1) \\ & \quad \ \& \ ||\psi_{n,m}||_2 = 1.) \\ \text{Thus, } \sum_{n=1}^{2^{k-1}} \sum_{m=1}^{\infty} |c_{n,m}|^2 &= < f, f > = \int_0^1 |f(t)|^2 dt < \infty, \ f \in L^2[0,1). \end{aligned}$$

Therefore the wavelet series $\sum_{n=1}^{2^{k-1}} \sum_{m=1}^{\infty} c_{n,m}\psi_{n,m}(t)$ is convergent and by Bessel's inequality,
$\sum_{n=1}^{2^{k-1}} \sum_{m=1}^{M} |c_{n,m}|^2 \leq ||f||_2^2 < \infty, \ \forall \ M \geq 1$.

$$\begin{aligned} \text{Let } (S_{2^{k-1},M}f)(t) &= \sum_{n=1}^{2^{k-1}} \sum_{m=1}^{M} c_{n,m}\psi_{n,m}(t). \\ \text{For } N > M, \quad & ||(S_{2^{k-1},N}f) - (S_{2^{k-1},M}f)||_2^2 \\ &= \left\| \sum_{n=1}^{2^{k-1}} \sum_{m=1}^{N} c_{n,m}\psi_{n,m}(t) - \sum_{n=1}^{2^{k-1}} \sum_{m=1}^{M} c_{n,m}\psi_{n,m}(t) \right\|_2^2 \end{aligned}$$

$$= \left\|\sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{N} c_{n,m}\psi_{n,m}(t)\right\|_2^2 = \sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{N}|c_{n,m}|^2 \to 0 \text{ as } M\to\infty,\ N\to\infty.$$

Therefore, $\quad ||(S_{2^{k-1},N}f)-(S_{2^{k-1},M}f)||_2^2 \to 0$ as $M\to\infty,\ N\to\infty$.

Hence $(S_{2^{k-1},N}f)_{N=0}^{\infty}$ is a Cauchy sequence in $L^2[0,1)$. Since $L^2[0,1)$ is a Banach space, therefore the Cauchy sequence $(S_{2^{k-1},N}f)_{N=0}^{\infty}$ converges to a function $g(t)$, say.
Here, $g(t)=\lim_{N\to\infty}(S_{2^{k-1},N}f)=\lim_{N\to\infty}\sum_{n=1}^{2^{k-1}}\sum_{m=1}^{N}c_{n,m}\psi_{n,m}(t)$.
Now we need to show that $g(t)=f(t)$, for this

$$\begin{aligned}\langle g(t)-f(t),\psi_{n,m}(t)\rangle &= \langle g(t),\psi_{n,m}(t)\rangle-\langle f(t),\psi_{n,m}(t)\rangle\\ &= \lim_{N\to\infty}\langle (S_{2^{k-1},N}f),\psi_{n,m}(t)\rangle - c_{n,m}\\ &= c_{n,m}-c_{n,m}=0\\ \text{Therefore, } g(t) &= f(t),\ \forall\ t\in[0,1). \qquad (3.1)\end{aligned}$$

Hence the wavelet series $\sum_{n=1}^{2^{k-1}}\sum_{m=1}^{N}c_{n,m}\psi_{n,m}(t)$ converges uniformly to $f(t)$ as $N\to\infty$. □

4. Genocchi Wavelet Approximation

Since $\{\psi_{n,m}(t)\}$ forms an orthonormal basis for $L^2[0,1)$, therefore a function $f\in L^2[0,1)$ can be expressed into Genocchi wavelet series as:

$$f(t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}c_{n,m}\psi_{n,m}(t),\ \text{where } c_{n,m}=\langle f,\psi_{n,m}\rangle. \qquad (4.1)$$

The $(2^{k-1},M)^{\text{th}}$ partial sum $(S_{2^{k-1},M}f)(t)$ of Genocchi wavelet series equation (4.1) is given by

$$(S_{2^{k-1},M}f)(t)=\sum_{n=1}^{2^{k-1}}\sum_{m=1}^{M}c_{n,m}\psi_{n,m}(t)=C^T\psi(t). \qquad (4.2)$$

where C and $\psi(t)$ are given by
$C=[c_{1,1},...,c_{1,M},c_{2,0},...,c_{2,M},...,c_{2^{k-1},1},...,c_{2^{k-1},M}]^T$ and
$\psi(t)=[\psi_{1,1}(t),...,\psi_{1,M}(t),\psi_{2,1}(t),...,\psi_{2,M}(t),...,\psi_{2^{k-1},1}(t),...,\psi_{2^{k-1},M}(t)]^T$.

The Genocchi wavelet approximation $E_{2^{k-1},M}(f)$ of f by $(2^{k-1},M)^{\text{th}}$ partial sum $(S_{2^{k-1},M}f)$ of Genocchi wavelet series equation (4.1) is defined by

$$E_{2^{k-1},M}(f)=\min_{(S_{2^{k-1},M}f)}||f-(S_{2^{k-1},M}f)||_2. \qquad (4.3)$$

$E_{2^{k-1},M}(f)$ is regarded as the most accurate approximation of the function f by $(S_{2^{k-1},M}f)$ if $E_{2^{k-1},M}(f)\to 0$ as $k\to\infty$, $M\to\infty$ (Zygmund [32]).

Theorem 4.1. *If the solution function f of Pantograph differential equation belongs to $C^{M,\alpha}[0,1)$ class and its Genocchi wavelet expansion be*

$$f(t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}c_{n,m}\psi_{n,m}(t) \qquad (4.4)$$

with $(2^{k-1}, M)^{th}$ *partial sums*

$$(S_{2^{k-1},M}f)(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=1}^{M} c_{n,m}\psi_{n,m}(t) \tag{4.5}$$

then the Genocchi wavelet approximation of f *by* $(S_{2^{k-1},M}f)$ *under* $||\cdot||_2$ *is given by*

$$\begin{aligned} E_{2^{k-1},M}(f) &= min\left\|f - \sum_{n=1}^{2^{k-1}} \sum_{m=1}^{M} c_{n,m}\psi^{(G)}_{n,m}(t)\right\|_2 \\ &= O\left(\frac{1}{2^{M(k-1)+\alpha}(M!)\sqrt{2M+2\alpha+1}}\right),\ M \geq 1. \end{aligned}$$

Proof. Following Eqns. (4.4) and (4.5),

$$\begin{aligned} & f(t)\chi_{\left[\frac{n-1}{2^{k-1}},\frac{n}{2^{k-1}}\right)} - \sum_{m=1}^{M} c_{n,m}\psi_{n,m}(t) \\ =\ & f(t)\chi_{\left[\frac{n-1}{2^{k-1}},\frac{n}{2^{k-1}}\right)} - \sum_{m=1}^{M} c_{n,m}\frac{2^{\frac{k-1}{2}}G_m(2^kt-n+1)}{\sqrt{R(m)}} \\ =\ & f\left(\frac{n-1}{2^{k-1}} + \frac{v}{2^{k-1}}\right) - \sum_{m=1}^{M} c_{n,m}\frac{2^{\frac{k-1}{2}}}{\sqrt{R(m)}}G_m(v), \\ & \qquad v = 2^kt - n + 1 \\ =\ & f\left(\frac{n-1}{2^{k-1}}\right) + f^{(1)}\left(\frac{n-1}{2^{k-1}}\right)\left(\frac{v}{2^{k-1}}\right) \\ +\ & f^{(2)}\left(\frac{n-1}{2^{k-1}}\right)\frac{\left(\frac{v}{2^{k-1}}\right)^2}{2!} + ... + f^{(M-1)}\left(\frac{n-1}{2^{k-1}}\right)\times \\ & \frac{\left(\frac{v}{2^{k-1}}\right)^{M-1}}{(M-1)!} + f^{(M)}\left(\frac{n-1}{2^{k-1}} + \frac{\theta v}{2^{k-1}}\right)\frac{\left(\frac{v}{2^{k-1}}\right)^{M}}{M!} \\ -\ & \sum_{m=1}^{M} c_{n,m}\frac{2^{\frac{k-1}{2}}}{\sqrt{R(m)}}G_m(v),\ 0 < \theta \leq 1. \end{aligned}$$

$$\begin{aligned} \text{Now,}\qquad & \left|f(t)\chi_{\left[\frac{n-1}{2^{k-1}},\frac{n}{2^{k-1}}\right)} - \sum_{m=1}^{M} c_{n,m}\psi_{n,m}(t)\right| \\ \leq\ & \frac{\left(\frac{v}{2^{k-1}}\right)^{M}}{M!}\left|f^{(M)}\left(\frac{n-1}{2^{k-1}} + \frac{\theta v}{2^{k-1}}\right) - f^{(M)}\left(\frac{n-1}{2^{k-1}}\right)\right| \end{aligned}$$

$$\begin{aligned} \text{Then,}\ \ (E_{2^{k-1},M}(f))^2 &= \min||f - (S_{2^{k-1},M}f)||_2^2 \\ &\leq \left\|f - \sum_{n=1}^{2^{k-1}} \sum_{m=1}^{M} c_{n,m}\psi_{n,m}(t)\right\|_2^2 \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{n=1}^{2^{k-1}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \frac{\left(\frac{v}{2^{k-1}}\right)^{2M}}{(M!)^2} \left| f^{(M)}\left(\frac{n-1}{2^{k-1}} + \frac{\theta v}{2^{k-1}}\right)\right. \\
&- \left. f^{(M)}\left(\frac{n-1}{2^{k-1}}\right)\right|^2 dv \\
&\leq \sum_{n=1}^{2^{k-1}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \frac{\left(\frac{v}{2^{k-1}}\right)^{2M}}{(M!)^2} \left(\frac{\theta_1 v}{2^{k-1}}\right)^{2\alpha} dv, \\
&\qquad\qquad f \in C^{M,\alpha}[0,1),\ 0 < \theta_1 \leq 1 \\
&\leq \int_0^1 \frac{1}{2^{2M(k-1)+2\alpha}(M!)^2} v^{2M+2\alpha} dv \\
&= \frac{1}{2^{2M(k-1)+2\alpha}(M!)^2(2M+2\alpha+1)}
\end{aligned}
$$

$$
\begin{aligned}
\text{Therefore,}\quad E_{2^{k-1},M}(f) &\leq \frac{1}{2^{M(k-1)+\alpha}(M!)\sqrt{2M+2\alpha+1}} \\
&= O\left(\frac{1}{2^{M(k-1)+\alpha}(M!)\sqrt{2M+2\alpha+1}}\right),\ M \geq 1. \quad \square
\end{aligned}
$$

5. Algorithm of Chandrasekhar's white dwarfs and Pantograph differential equation

In this section, the algorithm for the solution of Chandrasekhar's white dwarfs and the Pantograph differential equation will be discussed. For this purpose, let us derive the four basis functions of the Genocchi wavelet for $k = 1$, $M = 4$ as follows:

$$
\left.
\begin{aligned}
\psi_{1,1}(t) &= 1 \\
\psi_{1,2}(t) &= \sqrt{3}(2t-1) \\
\psi_{1,3}(t) &= \sqrt{\frac{10}{3}}(3t^2-3t) \\
\psi_{1,4}(t) &= \sqrt{\frac{35}{17}}(4t^3-6t^2+1)
\end{aligned}
\right\},\ t \in [0,1). \tag{5.1}
$$

Let $y(t)$ be the solution of Chandrasekhar's white dwarf and Pantograph differential equation. Then

$$
y(t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{n,m}\psi_{n,m}(t) \tag{5.2}
$$

and $(2^{k-1}, M)^{th}$ partial sum of series (5.2) as:

$$
y(t) = (S_{2^{k-1},M}f)(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=1}^{M} c_{n,m}\psi_{n,m}(t) = C^T\psi(t) \tag{5.3}
$$

For initial conditions of Chandrasekhar's white dwarf and Pantograph differential equation, the Eqn. (5.3) reduces to

$$y(0)=\sum_{n=1}^{2^{k-1}}\sum_{m=1}^{M}c_{n,m}\psi_{n,m}(0)=a, \tag{5.4}$$

$$y'(0)=\frac{d}{dt}\left(\sum_{n=1}^{2^{k-1}}\sum_{m=1}^{M}c_{n,m}\psi_{n,m}(t)\right)_{t=0}=b, \tag{5.5}$$

$$y''(0)=\frac{d^2}{dt^2}\left(\sum_{n=1}^{2^{k-1}}\sum_{m=1}^{M}c_{n,m}\psi_{n,m}(t)\right)_{t=0}=c. \tag{5.6}$$

In Eqn. (5.3), C^T contains $2^{k-1}M$ unknown coefficients. Consequently, if initial conditions are excluded, an additional $(2^{k-1}M-\beta)$ conditions are required to solve the differential equation, where β denotes the order of the differential equation. For the values of the $2^{k-1}M$ unknown coefficients $c_{n,m}$, collocation points $t_j=\frac{j-1}{2^{k-1}M}, j=1,2,...,2^{k-1}M$, are substituted into Eqn. (5.3) to obtain $(2^{k-1}M-\beta)$ equations. Hence, the values of the unknown coefficients $c_{n,m}$ are determined by these $2^{k-1}M$ systems of equations. This algorithm is also applicable to the solution of higher-order linear and nonlinear differential equations.

6. Results and Effectiveness of the Algorithm

In this section, the effectiveness of the proposed algorithm for the wavelet solution of Chandrasekhar's white dwarf and the Pantograph differential equation has been discussed. A comparison has been made between the solutions of Chandrasekhar's white dwarf and the Pantograph differential equations obtained using the proposed method and those obtained through the Chebyshev wavelet method, Legendre wavelet method, and ODE-45 method.

Example 6.1. Consider the Pantograph differential equation

$$y'''(t)-ty''(2t)+y'(t)+y\left(\frac{t}{2}\right)=t\cos(2t)+\cos\left(\frac{t}{2}\right),\ t\in(0,1], \tag{6.1}$$

$$y(0)=1,\ y'(0)=0,\ y''(0)=-1. \tag{6.2}$$

The exact solution of the Eqn. (6.1) is $y(t)=\cos(t)$.

$$\begin{aligned} y^{(M)}(x+t)-y^{(M)}(x) &= \begin{cases} (-1)^{[\frac{M}{2}]}(\cos(x+t)-\cos(x)), & \text{if M is even;}\\ (-1)^{[\frac{M+1}{2}]}(\sin(x+t)-\sin(x)), & \text{if M is odd,}\end{cases}\\ &\le \begin{cases} |(-1)^{[\frac{M}{2}]}((x+t)-x)|, & \text{if M is even;}\\ |(-1)^{[\frac{M+1}{2}]}((x+t)-x)|, & \text{if M is odd,}\end{cases}\\ &= |t| = O(|t|^{\alpha}),\ \alpha=1.\end{aligned}$$

$$\text{Therefore, } y(t)\in C^{M,\alpha}[0,1), \text{ since } C^{M,1}[0,1)\subsetneq C^{M,\alpha}[0,1), \forall\alpha\in(0,1]. \tag{6.3}$$

Hence, the solution of the Pantograph differential equation $y(t)\in C^{M,\alpha}[0,1)$ class.

By using the algorithm of the Genocchi wavelet approach described in Section 5, the wavelet solution of the Pantograph differential equation has been obtained. For the approximate solution of Eqn. (6.1), take $k = 1$, and $M = 4$. Then the approximate solution $y(t)$ of the Pantograph differential equation will be:

$$\begin{aligned} y(t) &= \sum_{m=1}^{4} c_{1,m}\psi_{1,m}(t) = c_{1,1}\psi_{1,1}(t) + c_{1,2}\psi_{1,2}(t) + c_{1,3}\psi_{1,3}(t) + c_{1,4}\psi_{1,4}(t) \\ &= c_{1,1} + \sqrt{3}(2t-1)c_{1,2} + \sqrt{\frac{10}{3}}(3t^2-3t)c_{1,3} + \sqrt{\frac{35}{17}}(4t^3-6t^2+1)c_{1,4}. \quad (6.4) \end{aligned}$$

For the values of the unknowns $c_{1,1}$, $c_{1,2}$, $c_{1,3}$, and $c_{1,4}$, we collocate the equations obtained from Eqns. (6.1) and (6.4) at $t = 0.5$. Using the initial conditions in Eqn. (6.4), four systems of linear equations are obtained. Solving these systems of equations yields the values of the unknowns as follows:

$$\begin{aligned} &c_{1,1} = 0.785892258873987,\ c_{1,2} = -0.113253959313326, \\ &c_{1,3} = -0.071628093092430,\ c_{1,4} = 0.012507232821827. \end{aligned} \quad (6.5)$$

Substituting the values of $c_{1,1}$, $c_{1,2}$, $c_{1,3}$, and $c_{1,4}$ from Eqn. (6.5) into Eqn. (6.4),

$$\begin{aligned} y(t) &= 0.785892258873987 - 0.113253959313326\sqrt{3}(2t-1) - 0.07162809309243 \\ &\quad \sqrt{\frac{10}{3}}(3t^2-3t) + 0.012507232821827\sqrt{\frac{35}{17}}(4t^3-6t^2+1),\ t \in [0,1)\,. \quad (6.6) \end{aligned}$$

The exact solution (E.S.) and wavelet solutions of the Pantograph differential equation by the Genocchi wavelet method (G.W.M.) for different values of t in [0,1) have been obtained. Also, a comparison of this solution with the Legendre wavelet method (L.W.M.) [31], the first kind Chebyshev wavelet method (F.K.C.W.M.) [22], and the second kind Chebyshev wavelet method (S.K.C.W.M.) [13] is given in Table 1.

t	E.S.	F.K.C.W.M.	S.K.C.W.M.	L.W.M.	G.W.M.
0.1	0.9950041653	0.9950717845	0.9950717845	0.9944433024	0.995071784517748
0.2	0.9800665778	0.9805742760	0.9805742759	0.9755464196	0.980574276141984
0.3	0.9553364891	0.9569381820	0.9569381818	0.9399691660	0.956938181979195
0.4	0.9210609940	0.9245942091	0.9245942090	0.8843713562	0.924594209135870
0.5	0.8775825619	0.8839730646	0.8839730646	0.8054128049	0.883973064718497
0.6	0.8253356149	0.8355054558	0.8355054556	0.6997533267	0.835505455833562
0.7	0.7648421873	0.7796220895	0.7796220894	0.5640527366	0.779622089587555
0.8	0.6967067093	0.7167536730	0.7167536731	0.3949708486	0.716753673086963
0.9	0.6216099683	0.6473309133	0.6473309133	0.1891674780	0.647330913438273

Table 1

The graphs of the exact solution and approximate solutions of the Pantograph differential equation by the Genocchi wavelet method, Legendre wavelet method, and Chebyshev wavelet method are shown in Figure 1.

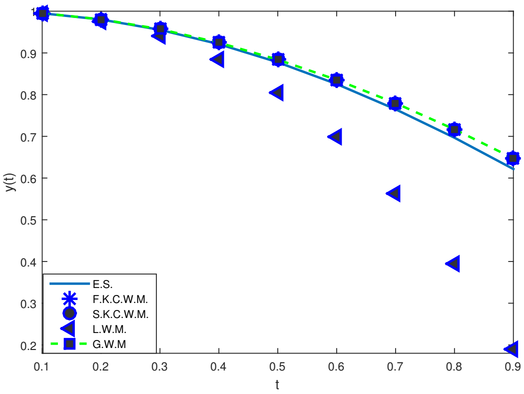

Figure 1

Based on Table 1 and Figure 1, it is evident that the exact and Genocchi wavelet solutions of the Pantograph differential equation coincide almost everywhere.
The absolute error of solution of Pantograph equation with different methods is given in Table 2.

t	F.K.C.W.M.	S.K.C.W.M.	L.W.M.	G.W.M.
0.1	0.0000676192	0.0000676192	0.0005608629	0.000067619217748
0.2	0.0005076982	0.0005076981	0.0045201582	0.000507698341984
0.3	0.0016016929	0.0016016927	0.0153673231	0.001601692879195
0.4	0.0035332151	0.0035332150	0.0366896378	0.003533215135870
0.5	0.0063905027	0.0063905027	0.0721697570	0.006390502818497
0.6	0.0101698409	0.0101698407	0.1255822882	0.010169840933562
0.7	0.0147799022	0.0147799021	0.2007894507	0.014779902287555
0.8	0.0200469637	0.0200469638	0.3017358607	0.020046963786963
0.9	0.025720945	0.0257209450	0.4324424903	0.025720945138273

Table 2

Example 6.2. Consider the Chandrasekhar's white dwarfs equation

$$y'' + \frac{2}{t}y' + (y^2 - c^2)^{\frac{3}{2}} = 0,\ t \in (0,1],\ y(0) = 1,\ y'(0) = 0. \tag{6.7}$$

Chandrasekhar's white dwarfs equation has only an existing analytic solution. By employing the algorithm of the Genocchi wavelet approach described in Section 5, and following the process of example (6.1), the wavelet solution of the differential equation (6.7) has been obtained.

$$y(t) = c_{1,1} + \sqrt{3}(2t-1)c_{1,2} + \sqrt{\frac{10}{3}}(3t^2 - 3t)c_{1,3} + \sqrt{\frac{35}{17}}(4t^3 - 6t^2 + 1)c_{1,4}. \tag{6.8}$$

For $c = 0$, the values of unknowns,

$$\begin{aligned} c_{1,1} &= 0.92783149959866283,\ c_{1,2} = -0.03673738735101032, \\ c_{1,3} &= -0.02323476386261047,\ c_{1,4} = 0.00595004274262933. \end{aligned} \tag{6.9}$$

Substituting the values of $c_{1,1}$, $c_{1,2}$, $c_{1,3}$, and $c_{1,4}$ from Eqn. (6.9) into Eqn. (6.8),

$$\begin{aligned} y(t) &= 0.92783149959866283 - 0.03673738735101032\sqrt{3}(2t-1) - 0.02323476386261047 \\ &\quad \sqrt{\frac{10}{3}}(3t^2-3t) + 0.00595004274262933\sqrt{\frac{35}{17}}(4t^3-6t^2+1),\ t \in [0,1)\,. \end{aligned} \tag{6.10}$$

For $c = 0.1$, the values of unknowns,

$$\begin{aligned} c_{1,1} &= 0.92886797607973291,\ c_{1,2} = -0.03623037555718233, \\ c_{1,3} &= -0.02291410144879764,\ c_{1,4} = 0.00583971401215223. \end{aligned} \tag{6.11}$$

$$\begin{aligned} y(t) &= 0.92886797607973291 - 0.03623037555718233\sqrt{3}(2t-1) - 0.02291410144879764 \\ &\quad \sqrt{\frac{10}{3}}(3t^2-3t) + 0.00583971401215223\sqrt{\frac{35}{17}}(4t^3-6t^2+1),\ t \in [0,1)\,. \end{aligned} \tag{6.12}$$

For $c = 0.2$, the values of unknowns,

$$\begin{aligned} c_{1,1} &= 0.93195029445680128,\ c_{1,2} = -0.0347205939689622, \\ c_{1,3} &= -0.02195923173116526,\ c_{1,4} = 0.0055140376933618. \end{aligned} \tag{6.13}$$

$$\begin{aligned} y(t) &= 0.93195029445680128 - 0.0347205939689622\sqrt{3}(2t-1) - 0.02195923173116526 \\ &\quad \sqrt{\frac{10}{3}}(3t^2-3t) + 0.0055140376933618\sqrt{\frac{35}{17}}(4t^3-6t^2+1),\ t \in [0,1)\,. \end{aligned} \tag{6.14}$$

For $c = 0.3$, the values of unknowns,

$$\begin{aligned} c_{1,1} &= 0.93699552602828853,\ c_{1,2} = -0.03224251334462935, \\ c_{1,3} &= -0.02039195593148045,\ c_{1,4} = 0.00498919905528592. \end{aligned} \tag{6.15}$$

$$\begin{aligned} y(t) &= 0.93699552602828853 - 0.03224251334462935\sqrt{3}(2t-1) - 0.02039195593148045 \\ &\quad \sqrt{\frac{10}{3}}(3t^2-3t) + 0.00498919905528592\sqrt{\frac{35}{17}}(4t^3-6t^2+1),\ t \in [0,1)\,. \end{aligned} \tag{6.16}$$

For $c = 0$, $c = 0.1$, $c = 0.2$, and $c = 0.3$, the wavelet solutions of Chandrasekhar's white dwarfs equation by the Genocchi wavelet method for different values of t in [0,1) are as follows:

t	G.W.M. for $c = 0$	G.W.M. for $c = 0.1$	G.W.M. for $c = 0.2$	G.W.M. for $c = 0.3$
0.1	0.998249280748980	0.998275709305189	0.998354178471217	0.998482193623542
0.2	0.993133722659471	0.993236903983604	0.993543303848386	0.994043315337291
0.3	0.984858225226803	0.985084684179515	0.985757261076788	0.986855176405928
0.4	0.973627687946305	0.974020150037191	0.975185935101702	0.977089588094135
0.5	0.959647010313307	0.960244401700904	0.962019210868407	0.964918361666596
0.6	0.943121091823137	0.943958539314922	0.946446973322185	0.950513308387993
0.7	0.924254831971125	0.925363663023518	0.928659107408313	0.934046239523006
0.8	0.903253130252599	0.904660872970960	0.908845498072073	0.915688966336320
0.9	0.880320886162890	0.882051269301519	0.887196030258743	0.895613300092616

Table 3

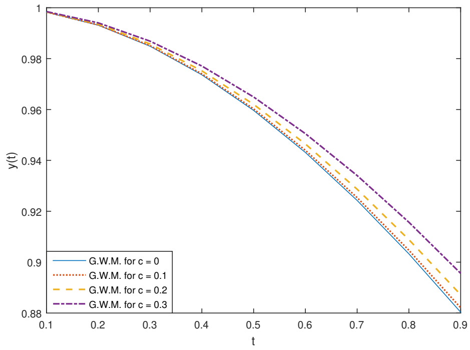

Figure 2

Absolute errors of numerical solution of Chandrasekhar's white dwarfs eqution by the method in [20] is given in Table 4 for different values of c=0, c=0.1, c=0.2, c=0.3 respectively. The graphs of the wavelet solutions of Chandrasekhar's white dwarf equation by the Genocchi wavelet method for $c = 0$, $c = 0.1$, $c = 0.2$, and $c = 0.3$ are shown in Figure 2.

t	For $c = 0$	For $c = 0.1$	For $c = 0.2$	For $c = 0.3$
0.1	0.000083919251020	0.000327619251020	0.001738719251020	0.001740919251020
0.3	0.000142974773197	0.002335474773197	0.004409274773197	0.006357274773197
0.5	0.001251910313307	0.004824189686693	0.010573089686693	0.015974689686693
0.7	0.005400331971125	0.006413368028875	0.017601668028875	0.028123568028875
0.9	0.012959686162890	0.006219013837110	0.024418713837110	0.041568313837110

Table 4

For $c = 0$, $c = 0.1$, $c = 0.2$, and $c = 0.3$, the wavelet solutions of Chandrasekhar's white dwarfs equation by the Genocchi wavelet method for different values of t in [0,1) have been obtained. A comparison of this solution with the first kind Chebyshev wavelet method [22], second kind Chebyshev wavelet method [13], Legendre wavelet method [31], and ODE-45 method is provided in Table 5, Table 6, Table 7, and Table 8, respectively.

t	F.K.C.W.M.	S.K.C.W.M.	L.W.M.	ODE-45	G.W.M.
0.1	0.997382219289834	0.998249280748979	1.000094478583123	0.998335829791912	0.998249280748980
0.2	0.991742700897040	0.993133722659471	1.000034216145318	0.993373096941448	0.993133722659471
0.3	0.983249583333342	0.984858225226803	0.999303665405824	0.985199789749438	0.984858225226803
0.4	0.972071005110464	0.973627687946305	0.997387279083880	0.973958256297232	0.973627687946305
0.5	0.958375104740129	0.959647010313307	0.993769509898726	0.959839069744109	0.959647010313307
0.6	0.942330020734064	0.943121091823137	0.987934810569599	0.943073172180013	0.943121091823137
0.7	0.924103891603992	0.924254831971125	0.979367633815740	0.923922837519243	0.924254831971125
0.8	0.903864855861636	0.903253130252600	0.967552432356387	0.902672088871812	0.903253130252599
0.9	0.881781052018723	0.880320886162890	0.951973658910779	0.879617167054250	0.880320886162890

Table 5 (for $c = 0$)

t	F.K.C.W.M.	S.K.C.W.M.	L.W.M.	ODE-45	G.W.M.
0.1	0.997424299732278	0.998275709305189	1.000094092979523	0.998360716830938	0.998275709305189
0.2	0.991870838389425	0.993236903983604	1.000037382499589	0.993472057338686	0.993236903983604
0.3	0.983504702512733	0.985084684179515	0.999321384432440	0.985420270606134	0.985084684179515
0.4	0.972490978643491	0.974020150037191	0.997437614650323	0.974344928803696	0.974020150037191
0.5	0.958994753322988	0.960244401700904	0.993877589025480	0.960432949337120	0.960244401700904
0.6	0.943181113092514	0.943958539314922	0.988132823430156	0.943910944423214	0.943958539314922
0.7	0.925215144493359	0.925363663023518	0.979694833736594	0.925036410500163	0.925363663023518
0.8	0.905261934066812	0.904660872970960	0.968055135817040	0.904088375362684	0.904660872970960
0.9	0.883486568354164	0.882051269301519	0.952705245543736	0.881358082261526	0.882051269301519

Table 6 (for $c = 0.1$)

t	F.K.C.W.M.	S.K.C.W.M.	L.W.M.	ODE-45	G.W.M.
0.1	0.997549017676401	0.998354178471216	1.000092824193383	0.998434625620456	0.998354178471217
0.2	0.992250998754142	0.993543303848386	1.000046343317419	0.993765959515237	0.993543303848386
0.3	0.984262013717921	0.985757261076788	0.999373127187940	0.986075140568471	0.985757261076788
0.4	0.973738133052438	0.975185935101701	0.997585745620778	0.975493568121377	0.975185935101702
0.5	0.960835427242390	0.962019210868407	0.994196768431766	0.962197389251163	0.962019210868407
0.6	0.945709966772478	0.946446973322185	0.988718765436734	0.946400441882476	0.946446973322185
0.7	0.928517822127400	0.928659107408313	0.980664306451514	0.928346103406621	0.928659107408313
0.8	0.909415063791856	0.908845498072072	0.969545961291939	0.908298608184600	0.908845498072073
0.9	0.888557762250543	0.887196030258743	0.954876299773841	0.886534363112111	0.887196030258743

Table 7 (for $c = 0.2$)

t	F.K.C.W.M.	S.K.C.W.M.	L.W.M.	ODE-45	G.W.M.
0.1	0.997751721249008	0.998482193623542	1.000090342590997	0.998555257353742	0.998482193623542
0.2	0.992870191676970	0.994043315337291	1.000059499071984	0.994245710666062	0.994043315337291
0.3	0.985496928246993	0.986855176405928	0.999454662504954	0.987144327610629	0.986855176405928
0.4	0.975773447922188	0.977089588094135	0.997823025951899	0.977369411485696	0.977089588094135
0.5	0.963841267665664	0.964918361666596	0.994711782474813	0.965079819226243	0.964918361666596
0.6	0.949841904440528	0.950513308387992	0.989668125135689	0.950468825192674	0.950513308387993
0.7	0.933916875209892	0.934046239523006	0.982239246996519	0.933757000904524	0.934046239523006
0.8	0.916207696936863	0.915688966336320	0.971972341119297	0.915184587175592	0.915688966336320
0.9	0.896855886584551	0.895613300092616	0.958414600566015	0.895003811020111	0.895613300092616

Table 8 (for $c = 0.3$)

The graphs of the wavelet solutions of Chandrasekhar's white dwarfs equation by the Genocchi wavelet method, Legendre wavelet method, Chebyshev wavelet method, and ODE-45 method for $c = 0$, $c = 0.1$, $c = 0.2$, and $c = 0.3$ are shown in Figures 3, 4, 5, and 6, respectively.

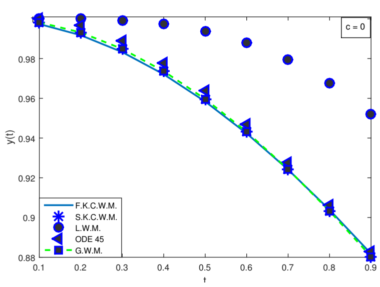

Figure 3

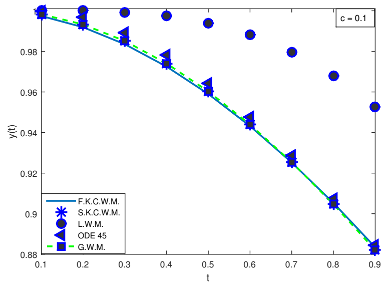

Figure 4

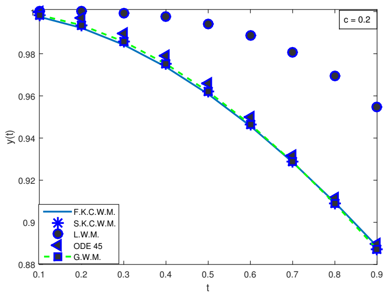

Figure 5

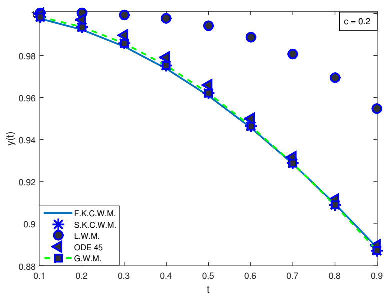

Figure 6

7. Conclusions

1. The Genocchi wavelet approximation of Theorems 4.1 is given by:
$$E_{2^{k-1},M}(f) = O\left(\frac{1}{2^{M(k-1)+\alpha}(M!)\sqrt{2M+2\alpha+1}}\right) \to 0 \text{ as } k \to \infty, M \to \infty.$$
Therefore, the approximation $E_{2^{k-1},M}(f)$ of solution function f belonging to $C^{M,\alpha}[0,1)$ is the best possible in wavelet analysis.
2. The exact solutions of the Pantograph differential equations and Genocchi wavelet solutions almost overlap in the range [0,1), as compared to the Chebyshev and Legendre wavelet solutions shown in Table 1 and Figure 1.
3. Based on Table 3 and Figure 2, for $c = 0$, $c = 0.1$, $c = 0.2$, and $c = 0.3$, the wavelet solutions of Chandrasekhar's white dwarfs equation by the Genocchi wavelet method for different values of t in [0,1) exhibit similar properties.
4. Based on Tables 5, 6, 7, 8 and Figures 3, 4, 5, 6, for $c = 0$, $c = 0.1$, $c = 0.2$, and $c = 0.3$, the wavelet solutions of Chandrasekhar's white dwarfs equation by the Genocchi wavelet method are better compared to ODE-45, Chebyshev, and Legendre wavelet methods.
5. The illustrated example demonstrates the validity and accuracy of the proposed method in Section 5 for solving linear and nonlinear differential equations of any order. Various differential equations related to computing the flow or movement of electricity, growth and decay problems, and equations describing the thermodynamic properties of a system can be solved using the proposed method. This method can also be employed to solve differential equations formulated for addressing biological processes.

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