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FRAMES FOR HILBERT SPACES WITH RESPECT TO ℓ^1-SUM OF FINITE-DIMENSIONAL SPACES

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Abstract. The classic definition of a frame by Duffin and Schaeffer is well known and very useful in analysis and its applications. There are many examples of constructing frames based on classical systems of functions. However, among some function systems, for example, for sequences of discretized values of the reproducing kernel of a Hilbert space, Duffin and Schaeffer frames do not exist. Therefore, from the point of view of the construction of series expansions of type $h = \sum_{n=1}^{\infty} c_n f_n$, it turned out to be productive to use the more general concept of a frame in the situation of a Hilbert space and, especially, a Banach space. In this regard, we mention Aldroubi concept of the p-frame and its subsequent generalizations. In this paper we consider frames with respect to the ℓ^1-sum of finite-dimensional spaces, give some examples and study some properties of such frames.

1. Introduction

It is well known that the frame approach is useful in solving the so-called synthesis problem. By the synthesis problem we mean the problem of representing an arbitrary function f in some function space F by a series $f = \sum_{n=1}^{\infty} c_n f_n$ in terms of elements of a function sequence $\{f_n\}_{n=1}^{\infty}$, where the coefficients $\{c_n\}_{n=1}^{\infty}$ belong to some space of number sequence X. We mention that the solution to the synthesis problem uses various modern methods of function theory and functional analysis, such as greedy algorithms (see [13]), orthorecursive expansions (see [10]), and Banach frames (see [5]). We note that the synthesis problem is useful in various practical problems ranging from numerical methods for solving differential equations to signal theory.

The concept of a frame was introduced by Duffin and Schaeffer [7] in 1952.

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Let H be a separable complex Hilbert space. A sequence $\{f_n\}_{n=1}^{\infty}$ of non-zero elements of the space H is called *Duffin–Schaeffer frame* if there exist constants $0 < A \le B < \infty$ such that the following inequalities are satisfied for all $h \in H$:

$$A\|h\|^2 \le \sum_{n=1}^{\infty} |\langle h, f_n\rangle|^2 \le B\|h\|^2.$$

It is known, that for any Duffin–Schaeffer frame $\{f_n\}_{n=1}^{\infty} \subset H$ there is a sequence $\{g_n\}_{n=1}^{\infty} \subset H$ such that for each $h \in H$ we have that $\{\langle h, g_n\rangle\}_{n=1}^{\infty} \in \ell^2$ and $h = \sum_{n=1}^{\infty}\langle h, g_n\rangle f_n$. In particular, any Duffin–Schaeffer frame is representing system for H with respect to ℓ^2, i.e., for each $h \in H$ there exist $\{c_n\}_{n=1}^{\infty} \in \ell^2$ such that $h = \sum_{n=1}^{\infty} c_n f_n$. These and many other results are presented in Christensen's book [6].

The following definition is given by Aldroubi, Sun, Tang [1].

Let $1 \le p \le \infty$ and let F be a Banach space and $G = F^*$ be its dual. We say that $\{g_n\}_{n=1}^{\infty} \subset G \setminus \{0\}$ is *a p-frame* for F if there exist constants $0 < A \le B < \infty$ such that for all $f \in F$

$$A\|f\|_F \le \left(\sum_{n=1}^{\infty} |\langle f, g_n\rangle|^p\right)^{\frac{1}{p}} \le B\|f\|_F.$$

Note that when trying to extend the notion of a frame to sequences of elements of Banach spaces, one encounters with ambiguity in the understanding of the duality $\langle h, f_n\rangle$: we either speak of the values of the functional f_n on an element h, or consider the Fourier coefficients of a functional h on the elements f_n. These approaches to the definition of a frame are inequivalent even for reflexive Banach spaces. The first ot these approaches was used by Gröchenig [8] to define the notions of a Banach frame and an atomic decomposition.

As before, let F be a Banach space and $G = F^*$ be its dual.

Let X be a Banach space of sequences $x = \{x_n\}_{n=1}^{\infty}$ such that the standard unit vectors δ_n, $n = 1, 2, \dots$, form a basis in X. Then the dual space $Y = X^*$, clearly, can be identified with the Banach space of sequences $x = \{x_n\}_{n=1}^{\infty}$ such that

$$\|y\|_Y = \sup_{\|x\|_X \le 1} |\langle x, y\rangle|, \qquad \langle x, y\rangle = \sum_{n=1}^{\infty} x_n y_n.$$

Definition 1.1. The pair of sequences $\{f_n\}_{n=1}^{\infty} \subset F \setminus \{0\}$ and $\{g_n\}_{n=1}^{\infty} \subset G \setminus \{0\}$ is called a Banach frame if there are constants $0 < A \le B < \infty$ such that for every vector $f \in F$ the set of values $\{\langle f, g_n\rangle\}_{n=1}^{\infty}$ of the functionals $\{g_n\}_{n=1}^{\infty}$ on f satisfies the inequalities

$$A\|f\|_F \le \|\{\langle f, g_n\rangle\}_{n=1}^{\infty}\|_X \le B\|f\|_F$$

and we have the representation

$$f = \sum_{n=1}^{\infty}\langle f, g_n\rangle f_n.$$

Another definitions of a frame was suggested in [9] and [14]. It is based on inequalities dual to Gröchenig frame inequalities from Definition 1.1.

Definition 1.2. We say that a sequence $\{f_n\}_{n=1}^{\infty} \subset F \setminus \{0\}$ is a frame for a Banach space F with respect to a sequence space X if there are constants $0 < A \le B < \infty$ such that, for every bounded linear functional $g \in G$, the sequence $\{\langle f_n, g\rangle\}_{n=1}^{\infty}$ of its Fourier coefficients satisfies the inequalities

$$A\|g\|_G \le \|\{\langle f_n, g\rangle\}_{n=1}^{\infty}\|_Y \le B\|g\|_G.$$

In particular, if $F = G = H$ is a Hilbert space and $X = Y = \ell^2$, we get the definition of the Duffin–Schaeffer frame.

We emphasize that whenever we talk in this paper about the frame, it will be understood in the sense of Definition 1.2.

Also, in this paper we will consider a special case of the sequence space X, namely, we will assume that the sequence space X is a ℓ^1-sum of finite-dimensional spaces.

2. Frames and absolutely representing systems of subspaces

Let H be a Hilbert space and $\{f_n\}_{n=1}^{\infty} \subset H \setminus \{0\}$ be a sequence in H which consists of blocks (of linearly independent vectors)

$$f_n = f_{k,j}, \qquad j = 0, \dots, n_k - 1, \qquad k \in \{0\} \cup \mathbb{N}.$$

We consider the Banach sequence space

$$X = \left(\bigoplus_{k=0}^{\infty} X_k\right)_{\ell^1},$$

where X_k, $k \in \{0\} \cup \mathbb{N}$, are coordinate spaces of dimension n_k, whose elements are sequences $x_k = \{x_{k,j}\}_{j=0}^{n_k-1}$ of real or complex numbers. When X_k is equipped with the norm

$$\|x_k\|_{X_k} = \left\|\sum_{j=0}^{n_k-1} x_{k,j} f_{k,j}\right\|_H,$$

it is cleary isometric to the subspace

$$H_k = [f_{k,j} : j = 0, \dots, n_k - 1], \qquad k \in \{0\} \cup \mathbb{N}.$$

If $Y_k = X_k^*$ and $Y = X^*$, then the Banach sequence space Y has the form

$$Y = \left(\bigoplus_{k=0}^{\infty} Y_k\right)_{\ell^\infty}.$$

Definition 2.1. We say that a sequence $\{f_n\}_{n=1}^{\infty}$ is a frame for a Hilbert space H with respect to a ℓ^1-sum X of finite-dimensional space if there are constants $0 < A \le B < \infty$ such that for all $h \in H$, the sequence $\{\langle f_n, h\rangle\}_{n=1}^{\infty}$ satisfies the inequalities

$$A\|h\|_H \le \sup_{k \in \{0\} \cup \mathbb{N}} \|\{\langle f_{k,j}, h\rangle\}_{j=0}^{n_k-1}\|_{Y_k} \le B\|h\|_H. \tag{2.1}$$

We define *the synthesis operator* $S : X \to H$ by the formulae

$$Sx = \sum_{k=0}^{\infty} \sum_{j=0}^{n_k-1} x_{k,j} f_{k,j}.$$

Clearly, $\|Sx\|_H \le \sum_{k=0}^{\infty} \left\|\sum_{j=0}^{n_k-1} x_{k,j} f_{k,j}\right\|_H = \|x\|_X$, so that S is well defined and bounded linear operator acting from X to H and $\|S\|_{X \to H} \le 1$.

We define *the analysis operator* $R : H \to Y$ by the equation $Rh = \{\langle f_n, h\rangle\}_{n=1}^{\infty}$. We have

$$\langle Sx, h\rangle = \sum_{k=0}^{\infty}\sum_{j=0}^{n_k-1} x_{k,j}\langle f_{k,j}, h\rangle = \langle x, Rh\rangle,$$

whence the analysis operator R is adjoint to the synthesis operator S: $R = S^*$.

Theorem 2.2. *The following assertions are equivalent:*

(i) $\{f_n\}_{n=1}^{\infty}$ *is a frame for a Hilbert space* H *with respect to a* ℓ^1*-sum of finite-dimensional spaces* X*;*

(ii) $\{H_k\}_{k=0}^{\infty}$ *is a absolutely representing system of subspace, i.e. for all* $h \in H$ *there exist* $h_k \in H_k$*,* $k \in \{0\} \cup \mathbb{N}$*, such that* $h = \sum_{k=0}^{\infty} h_k$ *and* $\sum_{k=0}^{\infty} \|h_k\|_H < \infty$*;*

(iii) there is $0 < \sigma < 1$ *such that for all* $h \in H$

$$\inf_{k\in\{0\}\cup\mathbb{N}} dist(h, H_k) \le \sigma\|h\|_H.$$

Proof. $(i) \Leftrightarrow (ii)$. The frame inequalities (2.1) may be written as

$$A\|h\|_H \le \|Rh\|_Y.$$

Thus they show that the analysis operator R is injective. By the duality between injective and surjective operator (see, for example, [11, Theorem 4.13]), the synthesis operator S must be surjective. Hence, for every vector $h \in H$ there is a sequence $x = \{x_{k,j}\} \in X$ such that $h = Sx$. By the definition we have

$$h = \sum_{k=0}^{\infty}\sum_{j=0}^{n_k-1} x_{k,j} f_{k,j} = \sum_{k=0}^{\infty} h_k, \qquad \sum_{k=0}^{\infty}\|h_k\|_H = \sum_{k=0}^{\infty}\left\|\sum_{j=0}^{n_k-1} x_{k,j} f_{k,j}\right\|_H = \|x\|_X < \infty.$$

Conversely, from the surjectivity of S follows the injectivity of R.

$(i) \Leftrightarrow (iii)$. Note that

$$\|\{\langle f_{k,j}, h\rangle\}_{j=0}^{n_k-1}\|_{Y_k} = \sup_{\|x_k\|_{X_k}\le 1}\left|\sum_{j=0}^{n_k-1} x_{k,j}\langle f_{k,j}, h\rangle\right| = \sup_{h_k\in H_k, \|h_k\|_H\le 1} |\langle h_k, h\rangle| = \|P_k h\|_H,$$

where P_k is the orthogonal projection of H onto H_k. Accordingly, the frame inequalities (2.1) may be written as

$$A\|h\|_H \le \sup_{k\in\{0\}\cup\mathbb{N}} \|P_k h\|_H.$$

By the Pythagorean theorem this is equivalent to the inequality

$$\inf_{k\in\{0\}\cup\mathbb{N}} \mathrm{dist}\,(h, H_k) \le \sqrt{1 - A^2}\,\|h\|_H.$$

□

Remark 2.3. Theorem 1 is shows that frames for a Hilbert space with respect to ℓ^1-sum of finite-dimensional spaces are closely related with absolutely representing systems of subspace and can be characterized in terms of best approximations.

3. Frames and projections of bases

Definition 3.1. A frame $\{f_n\}_{n=1}^{\infty}$ for a Banach space F with respect to a sequence space X is said to be projective if there exist a Banach space F' and a basis $\{e_n\}_{n=1}^{\infty}$ in the direct sum $F \times F'$, which is equivalent to the standard basis $\{\delta_n\}_{n=1}^{\infty}$ in X, such that $f_n = Pe_n$, $n = 1, 2, \dots$, where $P : F \times F' \to F$ is the canonical projection of $F \times F'$ onto F.

Recall that each Duffin–Schaeffer frame is projective [5]. In the case of Banach spaces we can state the following criterion, which is a consequence of some general geometric principles (cf. [15]).

Proposition 3.2. Suppose $\{f_n\}_{n=1}^{\infty}$ is a frame for a Banach space F with respect to a space X. The following conditions are equivalent:
(i) $\{f_n\}_{n=1}^{\infty}$ is a projective frame;
(ii) there is a sequence $\{g_n\}_{n=1}^{\infty} \subset G$ such that for each $f \in F$ we have that $\{\langle f, g_n\rangle\}_{n=1}^{\infty} \in X$ and $f = \sum_{n=1}^{\infty} \langle f, g_n\rangle f_n$;
(iii) the subspace $N = \{x \in X : \sum_{n=1}^{\infty} x_n f_n = 0\}$ is complemented in X.

Theorem 3.3. *Every frame for a Hilbert space H with respect to a ℓ^1-sum of finite-dimensional spaces fails to be projective.*

Proof. On the contrary, assume that there exists a projective frame for a Hilbert space H with respect to a ℓ^1-sum X of finite-dimensional spaces. Then, according to Proposition 1, the subspace N is complemented in X and hence $X = N \oplus M$. It is clear that the restriction of the surjective synthesis operator $S : X \to H$ to the complementary subspace M is an isomorphism from M onto H, whence M is isomorphic to H. On the other hand, in [4, p. 19] Bourgain proved that an arbitrary ℓ^1-sum of finite-dimensional Banach spaces possesses the Schur property (recall that a Banach space E has the Schur property if weak convergence of a sequence in E implies its E-norm convergence). Therefore, the space X as well its subspace M has this property. At the same time, every Hilbert space H fails to have the Schur property. Thus, since the latter property is preserved under isomorphisms, we obtain a contradiction with the fact that M is isomorphic to H. □

Corollary 3.4. There is no frame $\{f_n\}_{n=1}^{\infty}$ for a Hilbert space H with respect to ℓ^1-sum X of finite-dimensional spaces such that for some $\{g_n\}_{n=1}^{\infty} \subset H$ and all $h \in H$ we have

$$h = \sum_{k=0}^{\infty} \sum_{j=0}^{n_k-1} \langle h, g_{k,j}\rangle f_{k,j}, \qquad \sum_{k=0}^{\infty} \left\| \sum_{j=0}^{n_k-1} \langle h, g_{k,j}\rangle f_{k,j} \right\|_H < \infty. \tag{3.1}$$

Proof. Assuming the contrary, suppose that for a Hilbert space H there are a frame $\{f_n\}_{n=1}^{\infty}$ and a sequence $\{g_n\}_{n=1}^{\infty} \subset H$ such that for each $h \in H$ a representation of the form (3.1) exists. Then, by Proposition 1, $\{f_n\}_{n=1}^{\infty}$ is a projective frames for H with respect to ℓ^1-sum X. Since this contradicts Theorem 2, desire result follows. □

4. Examples

4.1. Frames for the Lebesgue space generated by dilations and translations. The sequence $\{f_{k,j}\}_{k=0,j=0}^{\infty,\ 2^k-1}$ of dyadic dilations and translations of a function f defined by

$$f_{k,j}(t) = \begin{cases} f(2^k t - j), & t \in [\frac{j}{2^k}, \frac{j+1}{2^k}], \\ 0, & elsewhere, \end{cases} \quad j = 0, \dots, 2^k - 1, \quad k \in \{0\} \cup \mathbb{N}.$$

Let us denote by ℓ_n^2 a n-dimensional space equipped with the norm

$$\|c\|_2 = \left(\frac{1}{n} \sum_{j=0}^{n} |c_j|^2 \right)^{\frac{1}{2}}.$$

The next theorem follows directly from [2, Theorem 1].

Theorem 4.1. *Let a function $f \in L^2[0,1]$ satisfy condition $\int_0^1 f(t)\,dt \neq 0$.*

Then, the sequence of dilations and translations of f is a frame for $L^2[0,1]$ with respect to the Banach sequence space $X = \big(\bigoplus_{k=0}^{\infty} \ell_{2^k}^2\big)_{\ell^1}$.

4.2. Frames for the Hardy space generated by the Szegö kernel. The Hardy space $H^2(\mathbb{D})$ consists of the holomorphic functions $f(z)$ on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ satisfying

$$\|f\|_{H^2} = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2\,dt \right)^{\frac{1}{2}} < \infty.$$

Let

$$K_{\lambda_n}(z) = \frac{1}{1 - \overline{\lambda_n} z}, \quad z, \lambda_n \in \mathbb{D}, \quad n \in \mathbb{N},$$

denote values of the reproducing Szegö kernel for the Hardy space $H^2(\mathbb{D})$.

The next theorem proved in [12] (see also [3], where obtained more general result).

Theorem 4.2. *Let*

$$\lambda_n = \lambda_{k,j} = r_k e^{2\pi i j / n_k}, \quad j = 0, \dots, n_k - 1, \quad k \in \{0\} \cup \mathbb{N},$$

where natural $n_k \to \infty$ and radia $r_k \to 1$ satisfy the following matching condition:

$$n_k(1 - r_k) \geq a > 0, \qquad k \in \{0\} \cup \mathbb{N}.$$

Then the sequence $\{K_{\lambda_n}\}_{n=1}^{\infty}$ is a frame for the Hardy space $H^2(\mathbb{D})$ with respect to the Banach sequence space $X = \big(\bigoplus_{k=0}^{\infty} \ell_{n_k}^2\big)_{\ell^1}$.

Remark 4.3. In both examples, there are no Duffin–Schaeffer frames for the Lebesgue space and the Hardy space of the form $\{f_{k,j}\}_{k=0,j=0}^{\infty,\,n_k-1}$ and $\{K_{\lambda_n}\}_{n=1}^{\infty}$, respectively.

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