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DYNAMICAL DUAL FRAMES, ANNIHILATING POLYNOMIALS, AND SPECTRAL RADIUS

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Abstract. We use annihilating polynomials to give new proofs and refinements of characterizations of dynamical frames and dynamical dual frames in finite dimensional vector spaces. We use this approach to prove that every redundant finite frame has infinitely many distinct dynamical dual frames that are generated by an operator T with spectral radius $\rho(T) < \Delta$, where $0 < \Delta < 1$ is arbitrary.

1. Introduction

Dynamical sampling [4, 16] is a recent generalization of classical sampling theory [11] that addresses the problem of recovering a signal from spatio-temporal samples of an evolution process. An important motivating difficulty in dynamical sampling is that one might only have access to sub-Nyquist rate spatial samples which must be superresolved using multiple temporal snapshots of a signal under an evolution operator. The seminal work in [3, 4] revealed that many problems on dynamical sampling in both finite and infinite dimensional settings can be reformulated and addressed in terms of frame theory, and this stimulated a burgeoning body of work connecting frames and dynamical sampling, e.g., [1, 2, 5, 6, 8, 10, 13, 14]. This note focuses on dynamical frames and dynamical dual frames in the finite dimensional space $\mathbb{F}^d$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

Given a finite index set $\mathcal{I} \subset \{0, 1, 2, \ldots\}$, a collection of vectors $\{f_n\}_{n\in\mathcal{I}} \subset \mathbb{F}^d$ is a finite frame for $\mathbb{F}^d$ with frame bounds $0 < A \leq B < \infty$ if

$$\forall f \in \mathbb{F}^d, \qquad A\|f\|^2 \leq \sum_{n\in\mathcal{I}} |\langle f, f_n\rangle|^2 \leq B\|f\|^2,$$

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where $\|\cdot\|$ is the Euclidean norm on $\mathbb{F}^d$. The associated synthesis operator $F : \mathbb{F}^{|\mathcal{I}|} \to \mathbb{F}^d$ is defined by $F(a) = \sum_{n\in\mathcal{I}} a_n f_n$. The adjoint $F^* : \mathbb{F}^d \to \mathbb{F}^{|\mathcal{I}|}$ of the synthesis operator is called the analysis operator and is defined by $F^*(x) = \{\langle x, f_n\rangle\}_{n\in\mathcal{I}}$.

Given frames $\{f_n\}_{n\in\mathcal{I}}$ and $\{g_n\}_{n\in\mathcal{I}}$ for $\mathbb{F}^d$, we say that $\{g_n\}_{n\in\mathcal{I}}$ is a dual frame to $\{f_n\}_{n\in\mathcal{I}}$ if

$$\forall f \in \mathbb{F}^d, \qquad f = \sum_{n\in\mathcal{I}} \langle f, f_n\rangle g_n = \sum_{n\in\mathcal{I}} \langle f, g_n\rangle f_n.$$

If F, G are the synthesis operators of $\{f_n\}_{n\in\mathcal{I}}$ and $\{g_n\}_{n\in\mathcal{I}}$, then the dual frame condition is equivalent to requiring that $FG^* = GF^* = I$, where I is the identity operator on $\mathbb{F}^d$.

Definition 1.1. A frame $\{f_n\}_{n\in\mathcal{I}} \subset \mathbb{F}^d$ is *dynamical* if there exists a linear operator $T : \mathbb{F}^d \to \mathbb{F}^d$ and a vector $\phi \in \mathbb{F}^d$ so that $f_n = T^n\phi$ holds for all $n \in \mathcal{I}$, and we say that the frame is *generated by* T.

Definition 1.2. Given a frame $\{f_n\}_{n\in\mathcal{I}}$ for $\mathbb{F}^d$ we say that $\{g_n\}_{n\in\mathcal{I}} \subset \mathbb{F}^d$ is a *dynamical dual* to $\{f_n\}_{n\in\mathcal{I}}$ if $\{g_n\}_{n\in\mathcal{I}}$ is a dual frame to $\{f_n\}_{n\in\mathcal{I}}$, and $\{g_n\}_{n\in\mathcal{I}}$ is dynamical.

We consider the index set $\mathcal{I}_{1,N} = \{1, 2, 3, \cdots, N\}$ as in [9], but also consider $\mathcal{I}_{0,N} = \{0, 1, 2, \cdots, N-1\}$. When necessary, we emphasize the role of the index set $\mathcal{I}$ by saying that a frame or dual frame is $\mathcal{I}$-dynamical. We shall see that the evolution operator T need not be invertible for $\mathcal{I}_{0,N}$-dynamical frames, but as noted in [9], T must be invertible for $\mathcal{I}_{1,N}$-dynamical frames.

Theorem 3.7 in [9] shows that every redundant finite frame for $\mathbb{F}^d$ admits infinitely many distinct dynamical dual frames. Dynamical duals are used to introduce a dynamical frame quantization algorithm for digitally encoding frame coefficients, see Algorithm 1 in [9]. Theorem 3.7 in [9] shows that dynamical duals exist in abundance, but error bounds for the dynamical quantization algorithm crucially require dynamical duals that are generated by an evolution operator T with operator norm $\|T\| < 1$. Unfortunately, Section 6 in [9] illustrates that there exist finite frames that do not have any dynamical duals generated by an evolution operator with norm strictly less than one.

Overview and main results. Our main contributions are as follows. We introduce a novel approach based on annihilating polynomials to study dynamical frames and dynamical duals. A key consequence of our results is that one can construct dynamical duals generated by T with spectral radius $\rho(T) < 1$, even in the case that no evolution operator satisfying $\|T\| < 1$ exists.

We address the following main objectives:

- We use annihilating polynomials to characterize dynamical frames and dynamical duals in $\mathbb{F}^d$, see Theorem 2.5 and Proposition 2.8. This approach gives new proofs and refinements of results in [9], and elucidates the role played by the characteristic polynomial and invertibility of the evolution operator T.
- We prove that every redundant finite frame has infinitely many distinct dynamical duals that are generated by an operator T with spectral radius less than Δ, where $0 < \Delta < 1$ is arbitrary, see Theorem 3.4.
- We extend error bounds for the dynamical frame quantization algorithm from [9] to the setting where $\|T\| < 1$ does not hold, see Section 5.

The remainder of the paper is organized as follows. Section 2 uses annihilating polynomials to study dynamical frames and dynamical duals in $\mathbb{F}^d$. Section 3 proves that every redundant finite frame has infinitely many dynamical dual frames generated by an operator with small spectral radius. Section 4 gives examples to illustrate the theorems in Section 3. Section 5 discusses error bounds for the dynamical quantization algorithm in [9].

2. Characterizations of dynamical frame and dynamical duals in $\mathbb{F}^d$

In this section we use annihilating polynomials to characterize dynamical frames and dynamical duals in finite dimensional spaces. This shows that annihilating polynomials are a natural tool for studying dynamical frames in finite dimensions, but this can be contrasted with the infinite dimensional setting, e.g., [13, 10]. See [7] for previous work that connects annihilating polynomials and dynamical sampling. Annihilating polynomials are widely studied because they provide useful decompositions of finite-dimensional vector spaces into direct sums of operator-invariant subspaces, e.g., Jordan and primary decompositions [15]. We begin by recalling necessary background definitions on annihilating polynomials.

Let $\mathbb{F}[x]$ denote the set of polynomials with coefficients in $\mathbb{F}$. Given a linear operator $T : \mathbb{F}^d \to \mathbb{F}^d$ we abuse notation and let T also refer to a matrix representation of the operator, typically with respect to the canonical basis for $\mathbb{F}^d$ unless otherwise stated. The *characteristic polynomial* $p_T \in \mathbb{F}[x]$ of T is defined by $p_T(x) = \det(xI - T)$. The characteristic polynomial p_T is monic, has degree d, and annihilates T by the Cayley-Hamilton theorem, that is, $p_T(T) = 0$. Given a vector $f \in \mathbb{F}^d$, *the minimal T-annihilating polynomial of f* is defined to be the unique monic polynomial $p_{T,f} \in \mathbb{F}[x]$ of minimal degree such that $p_{T,f}(T)f = 0$, e.g., see Chapter 4 and Sections 6.3, 6.4, 7.1 in [15]. Note that $p_{T,f}$ has degree at most d, since $p_T(T)f = 0$.

The uniqueness of minimal T-annihilating polynomials is well known, but, since this is explicitly used in several of our results, we briefly recall some details for the convenience of the reader. Suppose p_1, p_2 are two distinct monic polynomials of minimal equal degree that satisfy $p_1(T)f = p_2(T)f = 0$. If p_1, p_2 have degree k, then $p = p_1 - p_2$ is a polynomial of degree at most $k-1$ that satisfies $p(T)f = 0$. Since p_1, p_2 are distinct, p is not the zero polynomial. Therefore p can be rescaled to a monic polynomial q with degree at most $k-1$ that still satisfies $q(T)f = 0$. Since this contradicts the minimality of degree for p_1, p_2, it follows that the minimal T-annihilating polynomial of f is unique.

2.1. Dynamical frames. We now revisit the characterizations of dynamical frames from Section 2 of [9] and give new proofs based on annihilating polynomials.

Lemma 2.1. *Let $T : \mathbb{F}^d \to \mathbb{F}^d$ be a linear operator and $\phi \in \mathbb{F}^d$. If $\{T^n\phi\}_{n=0}^{N-1}$ is a frame for $\mathbb{F}^d$, then $\{T^n\phi\}_{n=0}^{d-1}$ is a basis for $\mathbb{F}^d$, and* $\mathrm{Rank}(T) \geq d-1$. *Moreover, the characteristic polynomial of T is the minimal T-annihilating polynomial of ϕ.*

Proof. Let

$$p_{T,\phi}(x) = c_0 + c_1 x + \cdots + c_{k-1}x^{k-1} + x^k$$

be the minimal T-annihilating polynomial of ϕ, that is, $p_{T,\phi}$ is the monic polynomial of least degree such that $p_{T,\phi}(T)\phi = 0$. So

$$T^k\phi = -c_0\phi - c_1 T\phi - \cdots - c_{k-1}T^{k-1}\phi. \tag{2.1}$$

By the Cayley-Hamilton theorem, the characteristic polynomial p_T of T satisfies $p_T(T) = 0$, and hence $p_T(T)\phi = 0$. Since p_T is monic and has degree d, the minimality of $p_{T,\phi}$ gives $k \leq d$. By (2.1), one has $T^k\phi \in \operatorname{span}\{\phi, T\phi, \cdots, T^{k-1}\phi\}$. So,

$$T^{k+1}\phi \in \operatorname{span}\{T\phi, T^2\phi, \ldots, T^k\phi\} \subset \operatorname{span}\{\phi, T\phi, \ldots, T^{k-1}\phi\}.$$

Repeating this argument gives

$$\forall k \leq j \leq N-1, \qquad T^j\phi \in \operatorname{span}\{\phi, T\phi, \ldots, T^{k-1}\phi\}.$$

So, $\mathbb{F}^d = \operatorname{span}\{T^n\phi\}_{n=0}^{N-1} = \operatorname{span}\{T^n\phi\}_{n=0}^{k-1}$, which gives $k \geq d$. However, since $k \leq d$, one has $k = d$. So, $\mathbb{F}^d = \operatorname{span}\{T^n\phi\}_{n=0}^{d-1}$ and $\{T^n\phi\}_{n=0}^{d-1}$ is a basis for $\mathbb{F}^d$. Since $\{T^n\phi\}_{n=0}^{d-1}$ is linearly independent and $\{T^n\phi\}_{n=1}^{d-1} \subset \operatorname{Range}(T)$, one has $\operatorname{Rank}(T) \geq d-1$.

Recall that the characteristic polynomial p_T satisfies $p_T(T)\phi = 0$. Since p_T and $p_{T,\phi}$ are both monic polynomials of degree d and satisfy $p_T(T)\phi = p_{T,\phi}(T)\phi = 0$, the uniqueness of $p_{T,\phi}$ implies that $p_T = p_{T,\phi}$. □

A vector $f \in \mathbb{F}^d$ is *cyclic* for a linear operator $T : \mathbb{F}^d \to \mathbb{F}^d$ if

$$\operatorname{span}\{f, Tf, \ldots, T^{d-1}f\} = \mathbb{F}^d.$$

It can be shown that f is cyclic for T if and only if $p_{T,f} = p_T$, e.g., see Theorem 1 and the subsequent remark in Section 7.1 of [15]. In particular, the vector ϕ in Lemma 2.1 is a cyclic vector for T.

The operator T in Lemma 2.1 need not be invertible when the index set $\mathcal{I}_{0,N}$ is used, but the situation is different for the index set $\mathcal{I}_{1,N}$. In particular, the following result recovers Lemma 2.1 from [9].

Corollary 2.2. Let $T : \mathbb{F}^d \to \mathbb{F}^d$ be a linear operator and $\phi \in \mathbb{F}^d$. If $\{T^n\phi\}_{n=1}^N$ is a frame for $\mathbb{F}^d$, then T is invertible and every d consecutive elements of $\{T^n\phi\}_{n=1}^N$ form a basis for $\mathbb{F}^d$. Moreover, the characteristic polynomial of T is the minimal T-annihilating polynomial of $T\phi$.

Proof. The operator T is surjective because it maps the set $\{T^n\phi\}_{n=0}^{N-1}$ to the set $\{T^n\phi\}_{n=1}^N$ which spans $\mathbb{F}^d$. So, since $\mathbb{F}^d$ is a finite-dimensional space, $T : \mathbb{F}^d \to \mathbb{F}^d$ is invertible. Applying Lemma 2.1 to the frame $\{T^n(T\phi)\}_{n=0}^{N-1} = \{T^n\phi\}_{n=1}^N$ shows that $\{T^n\phi\}_{n=1}^d$ is a basis for $\mathbb{F}^d$, and that the characteristic polynomial of T is the minimal T annihilating polynomial of $T\phi$. Since T is invertible, this implies that every d consecutive elements of $\{T^n\phi\}_{n=1}^N$ form a basis for $\mathbb{F}^d$. □

Lemma 2.1 shows that the operator T generating a redundant dynamical frame is completely determined by its action on the basis given by first d elements of the frame. Since our later characterizations of dynamical frames and dynamical duals are formulated using the characteristic polynomical p_T, see Theorem 2.5 and Proposition 2.8, it will be useful to observe that two redundant dynamical frames are distinct when they are generated by operators with distinct characteristic polynomials.

Corollary 2.3. Let $\{f_n\}_{n=0}^{N-1}$ and $\{g_n\}_{n=0}^{N-1}$ be dynamical frames for $\mathbb{F}^d$, with $N > d$, where $f_n = (T_1)^n\phi_1$ and $g_n = (T_2)^n\phi_2$ for some $\phi_1, \phi_2 \in \mathbb{F}^d$ and linear operators $T_1, T_2 : \mathbb{F}^d \to \mathbb{F}^d$. If T_1, T_2 have distinct characteristic polynomials, i.e., $p_{T_1} \neq p_{T_2}$, then the dynamical frames $\{f_n\}_{n=0}^{N-1}$, $\{g_n\}_{n=0}^{N-1}$ are distinct in the sense that $f_j \neq g_j$ for some $0 \leq j \leq d$.

Proof. By Lemma 2.1, the characteristic polynomial p_{T_1} is the minimal T_1-annihilating polynomial of ϕ_1. So $p_{T_1}(T_1)\phi_1 = 0$. Since the minimal T_1-annihilating polynomial of ϕ_1 is unique, and since p_{T_2} is monic, has degree d, and satisfies $p_{T_1} \neq p_{T_2}$, it follows that $p_{T_2}(T_1)\phi_1 \neq 0$. However, by Lemma 2.1, p_{T_2} is the minimal T_2-annihilating polynomial of ϕ_2 and satisfies $p_{T_2}(T_2)\phi_2 = 0$. So

$$p_{T_2}(T_2)\phi_2 \neq p_{T_2}(T_1)\phi_1. \tag{2.2}$$

If one denotes $p_{T_2}(x) = x^d + \sum_{n=0}^{d-1} b_n x^n$, then (2.2) gives

$$(T_2)^d\phi_2 + \sum_{n=0}^{d-1} b_n (T_2)^n \phi_2 \neq (T_1)^d\phi_1 + \sum_{n=0}^{d-1} b_n (T_1)^n \phi_1.$$

Hence

$$g_d + \sum_{n=0}^{d-1} b_n g_n \neq f_d + \sum_{n=0}^{d-1} b_n f_n.$$

It follows that there exists $0 \leq j \leq d$ such that $f_j \neq g_j$. □

Corollary 2.3 need not be true when $N = d$. For example, consider the canonical basis $\{e_1, e_2\}$ for $\mathbb{R}^2$ with $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then $e_2 = T_1 e_1 = T_2 e_1$ with $T_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $T_2 = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$, but T_1 and T_2 do not have the same characteristic polynomial.

Corollary 2.4. Let $\{f_n\}_{n=1}^N$ and $\{g_n\}_{n=1}^N$ be dynamical frames for $\mathbb{F}^d$, with $N > d$, where $f_n = (T_1)^n \phi_1$ and $g_n = (T_2)^n \phi_2$ for some $\phi_1, \phi_2 \in \mathbb{F}^d$ and linear operators $T_1, T_2 : \mathbb{F}^d \to \mathbb{F}^d$. If T_1, T_2 have distinct characteristic polynomials, i.e., $p_{T_1} \neq p_{T_2}$, then the dynamical frames $\{f_n\}_{n=1}^N$, $\{g_n\}_{n=1}^N$ are distinct in the sense that $f_j \neq g_j$ for some $1 \leq j \leq d+1$.

Proof. Apply Corollary 2.3 to the dynamical frames $\{(T_1)^n(T_1\phi_1)\}_{n=0}^{N-1} = \{(T_1)^n\phi_1\}_{n=1}^N$ and $\{(T_2)^n(T_2\phi_2)\}_{n=0}^{N-1} = \{(T_2)^n\phi_2\}_{n=1}^N$. □

The next result characterizes dynamical frames in finite-dimensional spaces. The right shift operator $R : \mathbb{F}^N \to \mathbb{F}^N$ is defined by $R : (a_0, a_1, a_2, \ldots, a_{N-1}) \mapsto (0, a_0, a_1, \ldots, a_{N-2})$.

Theorem 2.5. *Let $\{f_n\}_{n=0}^{N-1}$ be a frame for $\mathbb{F}^d$ with $N > d$ and synthesis operator F. The following are equivalent:*

(1) $\{f_n\}_{n=0}^{N-1}$ is a dynamical frame with $f_n = T^n f_0$, where $T : \mathbb{F}^d \to \mathbb{F}^d$ is a linear operator with characteristic polynomial $p_T(x) = x^d + \sum_{n=0}^{d-1} b_n x^n$.

(2) $\mathrm{Ker}(F) = \mathrm{span}\{R^{i-1}b : 1 \leq i \leq N-d\}$ with $b = (b_0, b_1, \ldots, b_{d-1}, 1, 0, \ldots, 0) \in \mathbb{F}^N$.

Proof. We first prove (1) $\implies$ (2). Suppose $\{T^n f_0\}_{n=0}^{N-1}$ is a frame for $\mathbb{F}^d$ and let $p_T(x) = x^d + \sum_{n=0}^{d-1} b_n x^n$ be the characteristic polynomial of T. By the Cayley-Hamilton theorem, $p_T(T) = 0$ and one has

$$b_0 f_0 + b_1 T f_0 + \cdots + b_{d-1} T^{d-1} f_0 + T^d f_0 = 0. \tag{2.3}$$

Let $b = (b_0, b_1, \ldots, b_{d-1}, 1, 0, \ldots, 0) \in \mathbb{F}^N$.

Recall that the synthesis operator $F : \mathbb{F}^N \to \mathbb{F}^d$ is defined by

$$(a_0, a_1, \dots, a_{N-1}) \longmapsto \sum_{n=0}^{N-1} a_n T^n f_0.$$

Note that F has full rank, i.e., $\mathrm{Rank}(F) = d$, and $b \in \mathrm{Ker}(F)$. Moreover, by (2.3), the right shift $Rb = (0, b_0, b_1, \dots, b_{d-1}, 1, 0, \dots, 0) \in \mathbb{F}^N$ satisfies

$$F(Rb) = T\left(T^d f_0 + \sum_{n=0}^{d-1} b_n T^n f_0\right) = T(0) = 0.$$

Repeating this argument gives that $R^j b \in \mathrm{Ker}(F)$ for all $0 \leq j \leq N - d - 1$. It remains to show that $\{b, Rb, \dots, R^{N-d-1}b\}$ is a basis for $\mathrm{Ker}(F)$. To see this, note that $\mathrm{Ker}(F)$ has dimension $N - d$ and the following $N \times (N - d)$ matrix, which has the $N - d$ vectors $b, Rb, \dots, R^{N-d-1}b$ as its columns, has rank $N - d$

$$\begin{bmatrix} b_0 & 0 & \cdots & 0 \\ \vdots & b_0 & \ddots & \vdots \\ b_{d-1} & \vdots & \ddots & 0 \\ 1 & b_{d-1} & & b_0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & b_{d-1} \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

We next prove (2) $\implies$ (1). Let $\{f_n\}_{n=0}^{N-1}$ be a frame for $\mathbb{F}^d$ with synthesis operator F such that $\mathrm{Ker}(F) = \mathrm{span}\{R^{i-1}b : 1 \leq i \leq N - d\}$ for some $b = (b_0, \dots, b_{d-1}, 1, 0, \dots, 0)$.

Since $b \in \mathrm{Ker}(F)$,

$$b_0 f_0 + b_1 f_1 + \cdots + b_{d-1} f_{d-1} + f_d = 0, \tag{2.4}$$

f_d can be written as a linear combination of $f_0, \dots, f_{d-1}$. Using this and $Rb \in \mathrm{Ker}(F)$, we obtain that f_{d+1} is a linear combination of $f_0, \dots, f_{d-1}$. Repeating this argument gives that for each $d \leq k \leq N - 1$, the vector f_k is a linear combination of $f_0, \dots, f_{d-1}$. Therefore, $\{f_n\}_{n=0}^{d-1}$ is a frame for $\mathbb{F}^d$ with d elements, and so is a basis for $\mathbb{F}^d$.

Since $\{f_n\}_{n=0}^{d-1}$ is a basis for $\mathbb{F}^d$ define the operator $T : \mathbb{F}^d \to \mathbb{F}^d$ by $Tf_n = f_{n+1}$ for $0 \leq n \leq d - 1$, and extend T to act on all of $\mathbb{F}^d$ by linearity. We next show that $Tf_n = f_{n+1}$ also holds for $d \leq n \leq N - 2$.

Since $Rb \in \mathrm{Ker}(F)$, one has $b_0 f_1 + b_1 f_2 + \cdots + b_{d-1} f_d + f_{d+1} = 0$. By (2.4), one has

$$Tf_d = T(-b_0 f_0 - b_1 f_1 - \cdots - b_{d-1} f_{d-1}) = -b_0 f_1 - b_1 f_2 - \cdots - b_{d-1} f_d = f_{d+1}.$$

Repeating this argument gives that $Tf_n = f_{n+1}$ for all $d \leq n \leq N - 2$, and hence for all $0 \leq n \leq N - 2$. In particular, $T^n f_0 = f_n$ holds for all $0 \leq n \leq N - 1$, and the frame $\{f_n\}_{n=0}^{N-1}$ is dynamical.

By (2.4) and $T^n f_0 = f_n$, $0 \leq n \leq N - 1$, one has

$$b_0 f_0 + b_1 T f_0 + \cdots + b_{d-1} T^{d-1} f_0 + T^d f_0 = 0.$$

So, $q(x) = x^d + \sum_{n=0}^{d-1} b_n x^n$ is a monic polynomial of degree d that satisfies $q(T)f_0 = 0$. Since $\{T^n f_0\}_{n=0}^{N-1}$ is a frame for $\mathbb{F}^d$, Theorem 2.1 shows that the characteristic polynomial

p_T of T is the minimal T-annihilating polynomial of f_0, i.e., $p_T = p_{T,f_0}$. Since q and p_T are both monic and of degree d, and satisfy $q(T)f_0 = p_T(T)f_0 = p_{T,f_0}(T)f_0 = 0$, and since the minimal T-annihilating polynomial of f_0 is unique, it follows that $q = p_T = p_{T,f_0}$. □

In the proof of Theorem 2.5, note that the operator $T : \mathbb{F}^d \to \mathbb{F}^d$ has matrix representation with respect to the basis $\{T^n f_0\}_{n=0}^{d-1}$ given by

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & -b_0 \\ 1 & 0 & \cdots & 0 & -b_1 \\ 0 & 1 & \cdots & 0 & -b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -b_{d-1} \end{bmatrix} \tag{2.5}$$

The matrix C defined by (2.5) is called the *companion matrix* of the operator T, or the *rational form* of T. Equation (2.5) illustrates that $\mathrm{Rank}(T) = \mathrm{Rank}(C) \geq d-1$, cf. Lemma 2.1.

It is possible that $b_0 = 0$ in Theorem 2.5 for the index set $\mathcal{I}_{0,N}$, but the situation changes when the index set $\mathcal{I}_{1,N}$ is used. In particular, the next result recovers Theorem 2.4 from [9], where the vector b is called a *d-suitable vector*.

Corollary 2.6. Let $\{f_n\}_{n=1}^N$ be a frame for $\mathbb{F}^d$ with $N > d$ and synthesis operator F. The following are equivalent:

(1) $\{f_n\}_{n=1}^N$ is a dynamical frame with $f_n = T^n\phi$, where $\phi \in \mathbb{F}^d$ and $T : \mathbb{F}^d \to \mathbb{F}^d$ is a linear operator with characteristic polynomial $p_T(x) = x^d + \sum_{n=0}^{d-1} b_n x^n$.

(2) $\mathrm{Ker}(F) = \mathrm{span}\{R^{i-1}b : 1 \leq i \leq N-d\}$ where $b = (b_0, b_1, \dots, b_{d-1}, 1, 0, \dots, 0) \in \mathbb{F}^N$ and $b_0 \neq 0$.

Proof. The implication (1) $\implies$ (2) follows by applying Theorem 2.5 to the dynamical frame $\{T^n(T\phi)\}_{n=0}^{N-1} = \{T^n\phi\}_{n=1}^N$. Since T is invertible by Corollary 2.2, the characteristic polynomial $p_T(x) = \det(xI - T)$ satisfies $0 \neq \det(-T) = p_T(0) = b_0$.

The implication (2) $\implies$ (1) follows by considering the frame $\{f'_n\}_{n=0}^{N-1}$ with $f'_n = f_{n+1}$, which has the same synthesis operator F as $\{f_n\}_{n=1}^N$. Theorem 2.5 shows that $\{f'_n\}_{n=0}^{N-1}$ is dynamical with $f'_n = T^n f'_0$, where T has characteristic polynomial $p_T(x) = x^d + \sum_{n=0}^{d-1} b_n x^n$. Since $0 \neq b_0 = p_T(0) = \det(-T)$, it follows that T is invertible. Thus, letting $\phi = T^{-1}f_1$ gives that $f_n = f'_{n-1} = T^{n-1}f'_0 = T^{n-1}f_1 = T^n(T^{-1}f_1) = T^n\phi$. □

Theorem 2.5 and Corollary 2.6 characterize redundant dynamical frames with $N > d$ elements in $\mathbb{F}^d$. The non-redundant case $N = d$ is much simpler since every basis for $\mathbb{F}^d$ is automatically dynamical, see Proposition 2.5 in [9].

2.2. Dynamical dual frames. Let $\{f_n\}_{n=0}^{N-1}$ be a dynamical frame for $\mathbb{F}^d$ with synthesis operator F. Theorem 2.5 shows that $\mathrm{Ker}(F) = \{R^{i-1}b : 1 \leq i \leq N-d\}$ for an appropriate vector $b \in \mathbb{F}^d$ and hence

$$\mathbb{F}^N = \mathrm{Range}(F^*) \oplus \mathrm{span}\{R^{i-1}b : 1 \leq i \leq N-d\}. \tag{2.6}$$

Since $\mathrm{Range}(F^*)$ and $\mathrm{Ker}(F)$ are orthogonal subspaces of $\mathbb{F}^N$, the direct sum in (2.6) is an orthogonal decomposition. Recall that the direct sum $\mathbb{F}^N = S_1 \oplus S_2$ of subspaces S_1, S_2 holds if and only if $\mathbb{F}^N = S_1 + S_2$ and $S_1 \cap S_2 = \{0\}$.

If $\{f_n\}_{n=0}^{N-1}$ is not dynamical, it is still possible for the direct sum (2.6) to hold with nonorthogonal subspaces. In this case, $\{f_n\}_{n=0}^{N-1}$ admits a dynamical dual frame and (2.6) is, in fact, a necessary and sufficient condition, cf. Corollary 3.2 in [9], which we revisit below.

It will be useful to note the following lemma.

Lemma 2.7. *If F, G are $d \times N$ matrices with $N \geq d$ and $GF^* = I$, then*

$$\mathbb{F}^N = \text{Range}(F^*) \oplus \text{Ker}(G). \tag{2.7}$$

Proof. Since $GF^* = I$, the matrices F, G, F^*, G^* have full rank with $\dim[\text{Range}(F^*)] = d$ and $\dim[\text{Range}(G)] = d$. The rank-nullity theorem gives that $\dim[\text{Ker}(G)] = N - d$. Moreover, since $GF^* = I$, one has $\text{Range}(F^*) \cap \text{Ker}(G) = \{0\}$. Thus,

$$\begin{aligned} &\dim[\text{Range}(F^*) + \text{Ker}(G)] \\ &= \dim[\text{Range}(F^*)] + \dim[\text{Ker}(G)] - \dim[\text{Range}(F^*) \cap \text{Ker}(G)] = N. \end{aligned}$$

It follows that $\mathbb{F}^N = \text{Range}(F^*) + \text{Ker}(G)$. So (2.7) holds. □

Given a frame $\{f_n\}_{n=0}^{N-1}$ for $\mathbb{F}^d$ with synthesis operator F, and a dual frame $\{g_n\}_{n=0}^{N-1}$ with synthesis operator G, the direct sum decomposition (2.7) holds by Lemma 2.7. Conversely, if a subspace W is a direct complement for $\text{Range}(F^*)$, that is,

$$\mathbb{F}^N = \text{Range}(F^*) \oplus W,$$

then we can define a dual frame synthesis operator

$$G = (FP_{W^\perp}F^*)^{-1}FP_{W^\perp} \tag{2.8}$$

as the Moore-Penrose pseudoinverse of $P_{W^\perp}F^*$, where $W^\perp$ is the orthogonal complement of W, and $P_{W^\perp} : \mathbb{F}^N \to \mathbb{F}^N$ the orthogonal projection onto $W^\perp$. The operator $P_{W^\perp}F^* : \mathbb{F}^d \to \mathbb{F}^N$ is injective and $\text{Ker}(G) = W$, see the proofs of Lemma 3.1 and Proposition 3.3 in [9]. This ensures that the columns of G form a dual frame for $\{f_n\}_{n=0}^{N-1}$, and allows one to characterize dynamical dual frames, cf. Corollary 3.2 and Proposition 3.3 in [9].

Proposition 2.8. Let $\{f_n\}_{n=0}^{N-1}$ be a frame for $\mathbb{F}^d$ with $N > d$ and synthesis operator F. The following are equivalent:

(1) $\{f_n\}_{n=0}^{N-1}$ has a dynamical dual frame $\{g_n\}_{n=0}^{N-1}$ with $g_n = T^n g_0$, where $T : \mathbb{F}^d \to \mathbb{F}^d$ is a linear operator with characteristic polynomial $p_T(x) = x^d + \sum_{n=0}^{d-1} b_n x^n$,
(2) The direct sum $\mathbb{F}^N = \text{Range}(F^*) \oplus W$ holds with $W = \text{span}\{R^{i-1}b : 1 \leq i \leq N - d\}$ and $b = (b_0, b_1, \dots, b_{d-1}, 1, 0, \dots, 0) \in \mathbb{F}^N$.

Proof. We first prove (1) $\implies$ (2). Suppose $\{g_n\}_{n=0}^{N-1}$ has synthesis operator G and is a dynamical dual frame of $\{f_n\}_{n=0}^{N-1}$. Since $\{g_n\}_{n=0}^{N-1}$ is a dual frame of $\{f_n\}_{n=0}^{N-1}$, (2.7) holds. Since $\{g_n\}_{n=0}^{N-1}$ is dynamical, Theorem 2.5 gives that $\text{Ker}(G) = \text{span}\{R^{i-1}b : 1 \leq i \leq N - d\}$ with b of the required form. Thus (2) holds by letting $W = \text{Ker}(G)$.

We next prove (2) $\implies$ (1). If one defines G by (2.8) then the column vectors $\{g_n\}_{n=0}^{N-1}$ in G form a dual frame of the frame $\{f_n\}_{n=0}^{N-1}$. Moreover, note that $\text{Ker}(G) = W = \text{span}\{R^{i-1}b : 1 \leq i \leq N - d\}$. By Theorem 2.5, $\{g_n\}_{n=0}^{N-1}$ is a dynamical dual frame, and the characteristic polynomial p_T has the desired form. □

We note that a similar version of Proposition 2.8 holds for the index set $\mathcal{I}_{1,N}$, with the only change being that the vector b must additionally have its first component be nonzero.

Proposition 2.9. Let $\{f_n\}_{n=1}^N$ be a frame for $\mathbb{F}^d$ with $N > d$ and synthesis operator F. The following are equivalent:

(1) $\{f_n\}_{n=1}^N$ has a dynamical dual frame $\{g_n\}_{n=1}^N$, with $\phi \in \mathbb{F}^d$ and $g_n = T^n\phi$, where $T : \mathbb{F}^d \to \mathbb{F}^d$ is a linear operator with characteristic polynomial $p_T(x) = x^d + \sum_{n=0}^{d-1} b_n x^n$,

(2) The direct sum $\mathbb{F}^N = \mathrm{Range}(F^*) \oplus W$ holds with $W = \mathrm{span}\{R^{i-1}b : 1 \le i \le N-d\}$ and $b = (b_0, b_1, \ldots, b_{d-1}, 1, 0, \ldots, 0) \in \mathbb{F}^N$ with $b_0 \neq 0$.

We conclude this section with a note about the structure of dynamical dual frames under the action of an invertible transformation. Let $\{f_n\}_{n=0}^{N-1}$ be a frame for $\mathbb{F}^d$ with dynamical dual frame $\{T^n\phi\}_{n=0}^{N-1}$ and let $S : \mathbb{F}^d \to \mathbb{F}^d$ be an invertible linear operator. Observe that, for every $x \in \mathbb{F}^d$,

$$\begin{aligned} x &= \sum_{n=0}^{N-1} \langle x, f_n\rangle T^n\phi = S^{-1}\left(\sum_{n=0}^{N-1} \langle Sx, f_n\rangle T^n\phi\right) \\ &= \sum_{n=0}^{N-1} \langle x, S^* f_n\rangle\, S^{-1}T^n S S^{-1}\phi = \sum_{n=0}^{N-1} \langle x, S^* f_n\rangle (S^{-1}TS)^n (S^{-1}\phi), \end{aligned}$$

which shows that $\{(S^{-1}TS)^n(S^{-1}\phi)\}_{n=0}^{N-1}$ is a dynamical dual frame for the frame $\{S^* f_n\}_{n=0}^{N-1}$.

3. Dynamical dual frames with small spectral radius

Section 7 of [9] posed the following question:

Question 3.1. Given a finite frame $\{f_n\}_{n=1}^N$ for $\mathbb{F}^d$, is there a simple algorithm for finding a dynamical dual $\{T^n\phi\}_{n=1}^N$ with minimal $\|T\|$, especially in the case when $\|T\| < 1$ is possible?

It was of particular interest in [9] to determine when a frame admits a dynamical dual with $\|T\| < 1$, but it was shown that this is not always possible. This section proves a positive counterpart if one relaxes the operator norm condition $\|T\| < 1$ to $\rho(T) < 1$, where $\rho(T)$ is the spectral radius of T. Given any $0 < \Delta < 1$, we show that every redundant finite frame $\{f_n\}_{n=1}^N$ for $\mathbb{F}^d$ has infinitely many distinct dynamical dual frames $\{T^n\phi\}_{n=1}^N$ that are generated by T with $\rho(T) < \Delta$.

Given a linear operator $T : \mathbb{C}^d \to \mathbb{C}^d$, its *spectral radius* $\rho(T)$ is defined as

$$\rho(T) = \max\{|\lambda_j| : 1 \le j \le d\},$$

where $\{\lambda_j\}_{j=1}^d \subset \mathbb{C}$ are the eigenvalues of T. By Gelfand's formula $\rho(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$. The operator norm bounds the spectral radius by $\rho(T) \le \|T\|$, but in general one can have $\rho(T) < \|T\|$.

To directly bound the spectral radius of an operator $T : \mathbb{F}^d \to \mathbb{F}^d$ it will be useful to recall the following result about the variation of roots for perturbations of polynomials, see Theorem 1 in [12]. We state the result with a different indexing of the polynomial coefficients than [12].

Theorem 3.2. *Let $p(z) = z^d + \sum_{j=0}^{d-1} a_j z^j$ and $q(z) = z^d + \sum_{j=0}^{d-1} b_j z^j$ be two monic polynomials of degree d with complex coefficients. The roots of p and q can be enumerated as $\alpha_1, \dots, \alpha_d$, $\beta_1, \dots, \beta_d$, respectively, such that*

$$\max_{1\leq i\leq d} |\alpha_i - \beta_i| \leq \frac{4}{2^{1/d}} \left(\sum_{k=0}^{d-1} |a_k - b_k| \gamma^k \right)^{1/d} \tag{3.1}$$

where $\gamma = 2\max_{0\leq j\leq d-1}\{|a_j|^{1/(d-j)}, |b_j|^{1/(d-j)}\}$.

We also need the following measure-theoretic lemma. Given a set $U \subset \mathbb{R}^d$, denote its d-dimensional Lebesgue measure by $|U| = |U|_d$. Given a set $V \subset \mathbb{C}^d$, denote $|V| = |V|_d = |\widehat{V}|_{2d}$ where $\widehat{V} \subset \mathbb{R}^{2d}$ is the natural identification of V as a subset of $\mathbb{R}^{2d}$.

Lemma 3.3. *Suppose $E \subset \mathbb{F}^{d+1}$ is a Lebesgue measurable set that satisfies*

$$\forall t \in \mathbb{F}\backslash\{0\}, \quad E = tE, \tag{3.2}$$

and suppose that $A \subset \mathbb{F}^d$ satisfies

$$(x_1, x_2, \dots, x_d) \in A \quad \Longleftrightarrow \quad (x_1, x_2, \dots, x_d, 1) \in E. \tag{3.3}$$

Then A is Lebesgue measurable. Moreover, if $|\mathbb{F}^{d+1}\backslash E|_{d+1} = 0$, then $|\mathbb{F}^d\backslash A|_d = 0$.

Proof. We first show that A is Lebesgue measurable. By the Fubini-Tonelli theorem, e.g., Theorem 6.7 in [20], the section $E_z = \{(x_1, x_2, \dots, x_d) \in \mathbb{F}^d : (x_1, x_2, \dots, x_d, z) \in E\}$ is Lebesgue measurable for almost every $z \in \mathbb{F}$. If $z \in \mathbb{F}\backslash\{0\}$ then (3.2) and (3.3) give

$$(x_1, x_2, \dots, x_d) \in E_z \iff \left(\frac{x_1}{z}, \frac{x_2}{z}, \dots, \frac{x_d}{z}\right) \in A \iff (x_1, x_2, \dots, x_d) \in zA. \tag{3.4}$$

Thus $zA = E_z$ is Lebesgue measurable for almost every $z \in \mathbb{F}\backslash\{0\}$. Since each section E_z is a rescaling of A, the measurabilty of one section implies the measurability of every section, and thus A must be Lebesgue measurable.

We next prove that $\mathbb{F}^d\backslash A$ has measure zero. If $|\mathbb{F}^{d+1} \setminus E|_{d+1} = 0$, then, by the Fubini-Tonelli theorem, the sections

$$(\mathbb{F}^{d+1}\backslash E)_z = \{(x_1, x_2, \dots, x_d) \in \mathbb{F}^d : (x_1, x_2, \dots, x_d, z) \in \mathbb{F}^{d+1}\backslash E\}$$

satisfy $|(\mathbb{F}^{d+1} \setminus E)_z|_d = 0$ for almost every $z \in \mathbb{F}$. Notice that, for $z \in \mathbb{F} \setminus \{0\}$,

$$(\mathbb{F}^{d+1} \setminus E)_z = \mathbb{F}^d \setminus E_z = \mathbb{F}^d \setminus zA = z(\mathbb{F}^d \setminus A).$$

If $\mathbb{F} = \mathbb{R}$ then $|z(\mathbb{F}^d\backslash A)|_d = |z|^d|\mathbb{F}^d\backslash A|_d$, whereas if $\mathbb{F} = \mathbb{C}$ then $|z(\mathbb{F}^d\backslash A)|_d = |z|^{2d}|\mathbb{F}^d\backslash A|_d$, e.g., see Theorem 3.35 in [20]. Thus, it follows that $|\mathbb{F}^d\backslash A|_d = 0$. ☐

We are now ready to state the main result of this section.

Theorem 3.4. *Fix $0 < \Delta < 1$. Given a frame $\{f_n\}_{n=0}^{N-1}$ for $\mathbb{F}^d$, with $N > d$, there exist infinitely many distinct dynamical dual frames $\{T^n\phi\}_{n=0}^{N-1}$ for which $\rho(T) < \Delta$ and T has non-repeated non-zero eigenvalues.*

Proof. The outline of the proof has two components. First, the results in [9] imply that almost every choice of characteristic polynomial p_T in Proposition 2.8 yields a dynamical dual. Second, one can guarantee small spectral radius for the evolution operator T, in addition to non-repeated non-zero eigenvalues, by taking p_T to be a small perturbation of $x^d - \epsilon$, which has small, distinct, non-zero roots when $\epsilon > 0$ is close to zero.

Let $\mathcal{C}_0$ be the set of all points $(c_0, c_1, \ldots, c_d) \in \mathbb{F}^{d+1}$ such that $c_d \neq 0$ and

$$\mathbb{F}^N = \mathrm{Range}(F^*) \oplus \mathrm{span}\{R^{j-1}b : \ 1 \leq j \leq N-d\},$$

where $b = (c_0, c_1, \ldots, c_{d-1}, c_d, 0, \ldots, 0) \in \mathbb{F}^N$. The proof of Theorem 3.7 in [9] shows that $\mathbb{F}^{d+1}\backslash\mathcal{U}$ has measure zero, $|\mathbb{F}^{d+1}\backslash\mathcal{U}|_{d+1} = 0$, where $\mathcal{U}$ is the set of all points $(c_0, c_1, \ldots, c_d) \in \mathcal{C}_0$ such that $c_0 \neq 0$. Since $\mathbb{F}^{d+1}\backslash\mathcal{C}_0 \subset \mathbb{F}^{d+1}\backslash\mathcal{U}$, it follows that $|\mathbb{F}^{d+1}\backslash\mathcal{C}_0|_{d+1} \leq |\mathbb{F}^{d+1}\backslash\mathcal{U}|_{d+1} = 0$.

Let $\mathcal{C}_1$ be the set of all points $(c_0, c_1, \ldots, c_{d-1}) \in \mathbb{F}^d$ such that

$$\mathbb{F}^N = \mathrm{Range}(F^*) \oplus \mathrm{span}\{R^{j-1}b : \ 1 \leq j \leq N-d\},$$

where $b = (c_0, c_1, \ldots, c_{d-1}, 1, 0, \ldots, 0) \in \mathbb{F}^N$. Note that $\mathcal{C}_0 = t\mathcal{C}_0$ holds for all $t \in \mathbb{F}\backslash\{0\}$, and

$$(c_0, c_1, \ldots, c_{d-1}) \in \mathcal{C}_1 \iff (c_0, c_1, \ldots, c_{d-1}, 1) \in \mathcal{C}_0.$$

By Lemma 3.3, it follows that $|\mathbb{F}^d\backslash\mathcal{C}_1|_d = 0$.

Let $\mathcal{C}$ be the set of all points $(c_0, c_1, \ldots, c_{d-1}) \in \mathcal{C}_1$ such that all roots of $x^d + \sum_{n=0}^{d-1} c_n x^n$ are non-repeated, non-zero, and have complex modulus strictly less than Δ. Given $\epsilon, \delta > 0$ define

$$E_{\epsilon,\delta} = \{(x_0, x_1, \ldots, x_{d-1}) \in \mathbb{F}^d : |x_0 + \epsilon| < \delta \text{ and } |x_j| < \delta \text{ for } 1 \leq j \leq d-1\}.$$

We claim that if $\epsilon, \delta > 0$ are suitably small then $\mathcal{C}_1 \cap E_{\epsilon,\delta} \subset \mathcal{C}$. To see this, first assume that

$$0 < \delta < \epsilon < \left(\frac{1}{2}\right)\left(\frac{\Delta}{5}\right)^d. \tag{3.5}$$

Let $c = (c_0, c_1, \ldots, c_{d-1}) \in \mathcal{C}_1 \cap E_{\epsilon,\delta}$ and apply Theorem 3.2 with $p(x) = x^d + \sum_{n=0}^{d-1} c_n x^n$ and $q(x) = x^d - \epsilon$. Note that q has roots $\beta_n = \epsilon^{1/d} e^{2\pi i n/d}$ for $1 \leq n \leq d$. By Theorem 3.2, the polynomial p has roots $\{\alpha_n\}_{n=1}^d$ which can be enumerated so that

$$\max_{1 \leq n \leq d} |\alpha_n - \epsilon^{1/d} e^{2\pi i n/d}| \leq \frac{4}{2^{1/d}} \left(|c_0 + \epsilon| + \sum_{n=1}^{d-1} |c_n| \gamma^n \right)^{1/d}, \tag{3.6}$$

where

$$\gamma = 2 \max \Big[\{|c_0|^{1/d}, \epsilon^{1/d}\} \cup \{|c_n|^{1/(d-n)} : 1 \leq n \leq d-1\} \Big].$$

Using $c \in E_{\epsilon,\delta}$ and (3.5) yields $|c_0| \leq |c_0 + \epsilon| + \epsilon < \delta + \epsilon$ and

$$\begin{aligned}
\gamma &\leq 2 \max \Big[\{(\delta + \epsilon)^{1/d}, \epsilon^{1/d}\} \cup \{\delta^{1/(d-n)} : 1 \leq n \leq d-1\} \Big] \\
&\leq 2 \max \Big\{ (\delta + \epsilon)^{1/d}, \delta^{1/(d-1)} \Big\} \\
&\leq 2(\epsilon + \delta)^{1/d} < 2(2\epsilon)^{1/d} < \frac{2\Delta}{5} < \frac{\Delta}{2}.
\end{aligned} \tag{3.7}$$

So, (3.5), (3.6), (3.7), $0 < \Delta < 1$, and $c \in E_{\epsilon,\delta}$ imply

$$\begin{aligned}\max_{1\leq n\leq d} |\alpha_n - \epsilon^{1/d} e^{2\pi i n/d}| &\leq \frac{4}{2^{1/d}} \left(\sum_{n=0}^{d-1} \delta \gamma^n \right)^{1/d} \\ &\leq \frac{4\delta^{1/d}}{2^{1/d}} \left(\frac{1}{1-\gamma} \right)^{1/d} \\ &< 4\delta^{1/d}. \end{aligned} \tag{3.8}$$

It follows from (3.5) and (3.8) that

$$|\alpha_n| \leq |\alpha_n - \epsilon^{1/d} e^{2\pi i n/d}| + |\epsilon^{1/d} e^{2\pi i n/d}| < 4\delta^{1/d} + \epsilon^{1/d} < 5\epsilon^{1/d} < \Delta.$$

This shows that all roots of p have complex modulus strictly less than Δ.

It remains to show that all roots of p are non-zero and non-repeated when δ is sufficiently small. For this, we assume the following condition in addition to (3.5)

$$0 < \delta < \left(\frac{\epsilon}{8^d} \right) [2 - 2\cos(2\pi/d)]^{d/2}. \tag{3.9}$$

Since (3.9) implies $0 < \delta < \epsilon/4^d$, it follows from (3.8) that all roots of p are non-zero. Using (3.8) and recalling that $\{\beta_j\}_{j=1}^d$ are the dth roots of ϵ gives that for $m \neq n$

$$\begin{aligned}\epsilon^{1/d}[2 - 2\cos(2\pi/d)]^{1/2} &= |\beta_1 - \beta_2| \\ &\leq |\beta_m - \beta_n| \\ &\leq |\alpha_n - \beta_n| + |\alpha_n - \alpha_m| + |\alpha_m - \beta_m| \\ &< 8\delta^{1/d} + |\alpha_n - \alpha_m|. \end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10) gives that for $m \neq n$

$$|\alpha_n - \alpha_m| > \epsilon^{1/d}[2 - 2\cos(2\pi/d)]^{1/2} - 8\delta^{1/d} > 0.$$

In particular, the roots $\{\alpha_n\}_{n=1}^d$ are distinct.

We have shown that if ϵ, δ satisfy (3.5) and (3.9), then $\mathcal{C}_1 \cap E_{\epsilon,\delta} \subset \mathcal{C}$. By Proposition 2.8, for each $c = (c_0, c_1, \ldots, c_{d-1}) \in \mathcal{C}_1 \cap E_{\epsilon,\delta}$ the frame $\{f_n\}_{n=0}^{N-1}$ has a dynamical dual frame $\{T^n \phi\}_{n=0}^{N-1}$ for some $\phi \in \mathbb{F}^d$ and an operator $T : \mathbb{F}^d \to \mathbb{F}^d$ with characteristic polynomial $p_T(x) = x^d + \sum_{n=0}^{d-1} c_n x^n$. Since $c \in \mathcal{C}_1 \cap E_{\epsilon,\delta} \subset \mathcal{C}$, all roots of p_T are distinct, non-zero, and have modulus strictly less than Δ. This means that all eigenvalues of T are non-repeated, non-zero, and have modulus strictly less than Δ. Hence T has non-repeated non-zero eigenvalues and $\rho(T) < \Delta$. Moreover, by Corollary 2.3, these dual frames are distinct for distinct choices of c. Finally, since $E_{\epsilon,\delta}$ has positive Lebesgue measure, and since $\mathbb{F}^d \backslash \mathcal{C}_1$ has zero Lebesgue measure, one has that $\mathcal{C}_1 \cap E_{\epsilon,\delta}$ has positive Lebesgue measure. This ensures that $\{f_n\}_{n=0}^{N-1}$ has infinitely many distinct dynamical dual frames generated by an operator T with non-repeated eigenvalues and $\rho(T) < \Delta$. □

For perspective, Example 6.2 in [9] shows that there exist redundant frames for which no reordering of the frame admits a dynamical dual that is generated by an operator with $\|T\| < 1$. However, Theorem 3.4 shows that every redundant frame has a dynamical dual for which $\rho(T) < 1$.

Corollary 3.5. Fix $0 < \Delta < 1$. Given a frame $\{f_n\}_{n=1}^N$ for $\mathbb{F}^d$, with $N > d$, there exist infinitely many distinct dynamical dual frames $\{T^n\phi\}_{n=1}^N$ for which $\rho(T) < \Delta$ and T has non-repeated non-zero eigenvalues.

Proof. Let $f'_n = f_{n+1}$. By Theorem 3.4, the frame $\{f'_n\}_{n=0}^{N-1}$ has infinitely many distinct dynamical dual frames $\{T^n\phi\}_{n=0}^{N-1}$ where T has non-repeated non-zero eigenvalues and $\rho(T) < \Delta$. Note that $\{f_n\}_{n=1}^N$ has the dynamical dual $\{T^n(T^{-1}\phi)\}_{n=1}^N$. $\square$

Given a frame $\{f_n\}_{n=0}^{N-1}$, the proof of Theorem 3.4 indicates a probabilistic method for constructing dynamical dual frames which, with probability one, have non-repeated roots and satisfy $\rho(T) < \Delta$. Let $E_{\epsilon,\delta}$ be the set from the proof of Theorem 3.4 with $0 < \delta < \epsilon$ satisfying (3.5) and (3.9). If $c = (c_0, c_1, \ldots, c_{d-1}) \in \mathbb{F}^d$ is chosen at uniformly at random from $E_{\epsilon,\delta}$, then, with probability one, the vector $b = (c_0, c_1, \ldots, c_{d-1}, 1, 0, \ldots, 0) \in \mathbb{F}^d$ satisfies

$$\mathbb{F}^d = \operatorname{span}\{f_n\}_{n=0}^{d-1} \ \oplus \ W$$

with $W = \operatorname{span}\{R^{i-1}b : 1 \leq i \leq N-d\}$. Then G defined by (2.8) is the synthesis operator of a dynamical dual frame $\{g_n\}_{n=0}^{N-1}$ which, as in the proof of Theorem 3.4, is generated by an operator T with non-repeated roots and $\rho(T) < \Delta$.

Theorem 3.4 and Corollary 3.5 give insight into why the dynamical dual frames from Example 3.8 and Figure 1 in [9] can satisfy $\|T^n\phi\| \to 0$ despite the fact that this example satisfies $\|T\| \geq 1$ by Theorem 6.3 in [9]. See Examples 4.5 and 4.6 for further illustrations of this. In particular, one has $\rho(T) < 1$ if and only if $\lim_{n\to\infty} T^n = 0$, see Section 7.10 in [17].

The final theorem in this section provides a deterministic proof, in the case when the first d frame vectors form a basis, of the existence of dynamical dual frames associated with an evolution operator T having a prescribed characteristic polynomial.

Theorem 3.6. *Let $\{f_n\}_{n=0}^{N-1}$ be a frame of $N > d$ vectors for $\mathbb{F}^d$ such that $\{f_0, f_1, \ldots, f_{d-1}\}$ is a basis for $\mathbb{F}^d$. For sufficiently small $\varepsilon > 0$, $\{f_n\}_{n=0}^{N-1}$ has a dynamical dual frame of the form $\{T^n\phi\}_{n=0}^{N-1}$ with $p_T(x) = x^d + \sum_{j=0}^{d-1} c_j x^j$ whenever $|c_j| < \varepsilon$ for $0 \leq j \leq d-1$.*

Proof. By Proposition 2.8, $\{f_n\}_{n=0}^{N-1}$ has a dynamical dual frame if and only if

$$\mathbb{F}^N = \operatorname{Range}(F^*) \oplus \operatorname{span}\{R^{i-1}b : 1 \leq i \leq N-d\}, \tag{3.11}$$

where F is the synthesis operator of $\{f_n\}_{n=0}^{N-1}$, and $b = (c_0, c_1, \ldots, c_{d-1}, 1, 0, \ldots, 0) \in \mathbb{F}^N$. Moreover, if (3.11) holds, then the dynamical dual frame is generated by a linear operator $T : \mathbb{F}^d \to \mathbb{F}^d$ with characteristic polynomial $p_T(x) = x^d + \sum_{n=0}^{d-1} c_n x^n$.

Recall that Range (F^*) is the row space of the matrix F. Let the coordinates of f_n with respect to the standard basis be $f_n(j)$, $1 \leq j \leq d$, and consider the $N \times N$ matrix

$$B = \begin{bmatrix} f_0(1) & \cdots & f_0(d) & c_0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & c_0 & \ddots & \vdots \\ f_{d-1}(1) & \cdots & f_{d-1}(d) & c_{d-1} & \vdots & \ddots & 0 \\ & & & 1 & c_{d-1} & & c_0 \\ \vdots & \ddots & \vdots & 0 & 1 & \ddots & \vdots \\ & & & \vdots & \ddots & \ddots & c_{d-1} \\ f_{N-1}(1) & \cdots & f_{N-1}(d) & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{3.12}$$

Notice that (3.11) is equivalent to $\det B \neq 0$.

Let Σ_N represent the permutation group of $\{0, 1, 2, \ldots, N-1\}$ and let Σ_d represent the subgroup of Σ_N consisting of permutations σ such that $\sigma(k) = k$ for $d \leq k \leq N-1$. Adopting this notation and employing the Leibniz formula for the determinant of B, it follows that

$$\det B = \det F_0 + \sum_{\sigma \in \Sigma_N \setminus \Sigma_d} (-1)^\sigma B_{\sigma(1)1} \cdot B_{\sigma(2)2} \cdots B_{\sigma(N-1)(N-1)}, \tag{3.13}$$

where F_0 is the top-left $d \times d$ block of B. In (3.13) we used that $B_{\sigma(k)k} = 1$ for $d \leq k \leq N-1$ when $\sigma \in \Sigma_d$. Observe that each product in the sum (3.13) over $\Sigma_N \setminus \Sigma_d$ must contain at least one factor c_j, $0 \leq j \leq d-1$. Under the assumption that $|c_j| < \epsilon$, $0 \leq j \leq d-1$, it follows that $\det B = \det F_0 + O(\epsilon)$ and the fact that $\{f_0, f_1, \ldots, f_{d-1}\}$ forms a basis for $\mathbb{F}^d$ guarantees that $\det F_0 \neq 0$. Consequently, $\det B \neq 0$ for sufficiently small $\epsilon > 0$, completing the proof. $\square$

Theorem 3.6 motivates several remarks. The matrix B in (3.12) is $M_{F,b}$ in the article [9]. Theorem 3.6 provides a self-contained proof of Theorem 3.7 in [9], without requiring the Gröbner basis techniques from [18], for the special case when the first d elements of the frame are a basis.

If $\{f_n\}_{n=0}^{N-1}$ is a frame for $\mathbb{F}^d$ with $N > d$, then it is possible to reorder the frame vectors so that $\{f_0, f_1, \ldots, f_{d-1}\}$ forms a basis for $\mathbb{F}^d$. In this sense, Theorem 3.6 can be applied to any frame of $N > d$ vectors for $\mathbb{F}^d$ after reordering the frame vectors.

Let $\{f_n\}_{n=0}^{N-1}$ be a frame of $N > d$ vectors for $\mathbb{F}^d$ such that $\{f_0, f_1, \ldots, f_{d-1}\}$ is a basis for $\mathbb{F}^d$. Fix $0 < \Delta < 1$. Consider the set $E_{\epsilon,\delta}$ introduced in the proof of Theorem 3.4 and assume that $\epsilon, \delta > 0$ satisfy both (3.5) and (3.9). The perturbation argument given in the proof of Theorem 3.4 shows that the roots of $p(x) = x^d + \sum_{j=0}^{d-1} c_j x^j$ are non-repeated, non-zero, and of modulus strictly less than Δ whenever $c \in E_{\epsilon,\delta}$. Notice that $c \in E_{\epsilon,\delta}$ implies that $|c_j| < 2\epsilon$ for $0 \leq j \leq d-1$ and thus it is possible to combine these observations with Theorem 3.6 to conclude that $E_{\epsilon,\delta} \subseteq \mathcal{C}$ for sufficiently small ϵ, provided that both (3.5) and (3.9) are satisfied. Hence, Theorem 3.6 makes it possible to recover Theorem 3.4 using a deterministic argument under the assumption that $\{f_0, f_1, \ldots, f_{d-1}\}$ forms a basis for $\mathbb{F}^d$.

Notice that Theorem 3.4 demonstrates the existence of infinitely many dynamical dual frames for which the evolution operator T has small spectral radius as well as nonzero and non-repeated eigenvalues. As a byproduct of the fact that the eigenvalues are nonzero, the evolution operator T must be invertible. It is natural to ask whether any frame of $N > d$ vectors for $\mathbb{F}^d$ has a dynamical dual frame for which the evolution operator T has small spectral radius, but is not invertible. In the event that the frame vectors $\{f_0, f_1, \ldots, f_{d-1}\}$ form a basis for $\mathbb{F}^d$, Theorem 3.6 can be used to show that there are infinitely many distinct dynamical dual frames for which the evolution operator T has non-repeated eigenvalues, non-trivial kernel, and $\rho(T) < \Delta$. The main difference in the proof stems from the fact that c_0 must be zero, which means the perturbation argument should be applied to the polynomials $p_T(x) = x^d + \sum_{j=1}^{d-1} c_j x^j$ and $q(x) = x^d - \epsilon x$, with appropriate modifications.

4. Examples

This section gives examples which illustrate Theorem 3.4 and Corollary 3.5.

Example 4.1. Consider the frame $\{e_n\}_{n=1}^4$ for $\mathbb{R}^2$ where $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $e_3 = -e_1$, $e_4 = -e_2$. This frame has synthesis operator

$$F = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}. \tag{4.1}$$

Note that $\{e_n\}_{n=1}^4$ is a dynamical frame with $e_n = U^n e$, where $U = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $e = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

It follows from Theorem 6.3 in [9] that if $\{T^n\phi\}_{n=1}^4$ is a dynamical dual frame of $\{e_n\}_{n=1}^4$, then $\|T\| \geq 1$. This can also be seen directly by using the dynamical dual frame expansion

$$\forall v \in \mathbb{R}^4, \qquad v = \sum_{j=1}^{4} \langle v, e_j \rangle T^j \phi = \langle v, e_1 \rangle (T\phi - T^3\phi) + \langle v, e_2 \rangle (T^2\phi - T^4\phi). \tag{4.2}$$

Applying (4.2) to $v = e_1$ and $v = e_2$ yields

$$e_1 = T\phi - T^3\phi \qquad \text{and} \qquad e_2 = T^2\phi - T^4\phi,$$

which gives $Te_1 = e_2$. This implies $\|T\| \geq 1$, and also shows that T is of the form

$$T = \begin{bmatrix} 0 & -c_0 \\ 1 & -c_1 \end{bmatrix}, \tag{4.3}$$

for some $c_0, c_1 \in \mathbb{R}$. The characteristic polynomial of T is $p_T(x) = x^2 + c_1 x + c_0$.

By Proposition 2.8, $\{f_n\}_{n=1}^4$ has a dynamical dual frame $\{T^n\phi\}_{n=1}^4$ if and only if

$$\mathbb{R}^4 = \underbrace{\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\}}_{\operatorname{Range}(F^*)} \oplus \operatorname{span}\left\{ b = \begin{bmatrix} c_0 \\ c_1 \\ 1 \\ 0 \end{bmatrix}, Rb = \begin{bmatrix} 0 \\ c_0 \\ c_1 \\ 1 \end{bmatrix} \right\}, \tag{4.4}$$

with $c_0 \neq 0$. Equation (4.4) holds if and only if

$$0 \neq \det\left(\begin{bmatrix} 1 & 0 & c_0 & 0 \\ 0 & 1 & c_1 & c_0 \\ -1 & 0 & 1 & c_1 \\ 0 & -1 & 0 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 0 & c_0 & 0 \\ 0 & 0 & c_1 & 1+c_0 \\ 0 & 0 & 1+c_0 & c_1 \\ 0 & -1 & 0 & 1 \end{bmatrix}\right) = (1+c_0)^2 - c_1^2.$$

Hence, $\{e_n\}_{n=1}^4$ has a dynamical dual of the form $\{T^n\phi\}_{n=1}^4$ if and only if T satisfies (4.3) with

$$c_0 \neq 0 \qquad \text{and} \qquad |c_1| \neq |1+c_0|. \tag{4.5}$$

The condition $c_0 \neq 0$ guarantees that T is invertible. Moreover, T has non-repeated eigenvalues if and only if $c_1^2 \neq 4c_0$. If

$$\mathcal{U} = \{(c_0, c_1) \in \mathbb{R}^2 : c_0 \neq 0 \text{ and } |c_1| \neq |1+c_0| \text{ and } c_1^2 \neq 4c_0\}$$

then $\mathbb{R}^2 \backslash \mathcal{U}$ has Lebesgue measure zero.

It remains to control the spectral radius of T. If λ is an eigenvalue of T then

$$|\lambda| \leq \frac{|c_1| + |(c_1^2 - 4c_0)^{1/2}|}{2} = \frac{|c_1| + |c_1^2 - 4c_0|^{1/2}}{2} \leq \frac{|c_1| + |c_1| + 2|c_0|^{1/2}}{2} = |c_1| + |c_0|^{1/2}.$$

In particular, if $0 < \Delta < 1$ and $|c_1|, |c_0| < \frac{\Delta^2}{4}$ then $|\lambda| < \Delta$. So, given $0 < \Delta < 1$, $(c_0, c_1) \in \mathcal{U} \cap [-\frac{\Delta^2}{2}, \frac{\Delta^2}{2}]^2$ implies that $\rho(T) < \Delta$.

In summary, $\mathcal{U} \cap [-\frac{\Delta^2}{2}, \frac{\Delta^2}{2}]^2$ has positive Lebesgue measure, and for each $(c_0, c_1) \in \mathcal{U} \cap [-\frac{\Delta^2}{2}, \frac{\Delta^2}{2}]^2$ the operator T defined by (4.3) generates a dynamical dual frame where T has non-repeated, non-zero roots and $\rho(T) < \Delta < 1$. By Corollary 2.4, these dual frames are distinct since they are generated by operators with distinct characteristic polynomials.

Finally, $\{e_n\}_{n=1}^4$ has a dynamical dual of the form $\{T^n\phi\}_{n=0}^3$ if and only if T satisfies (4.3) with $|c_1| \neq |1+c_0|$. This relaxes (4.5) by not requiring $c_0 \neq 0$.

The next example provides intuition for what the dynamical dual frames in Example 4.1 look like when c_0, c_1 are sufficiently small.

Example 4.2. Let $\{e_n\}_{n=1}^4 \subset \mathbb{R}^2$ be the frame from Example 4.1 with synthesis operator F from (4.1), and dynamical dual frame $\{T^n\phi\}_{n=1}^4$ with T as in (4.3). By (2.8), the synthesis operator of $\{T^n\phi\}_{n=1}^4$ is $G = (FP_{W^\perp}F^*)^{-1}FP_{W^\perp}$, where W is the column span of

$$N := \begin{bmatrix} c_0 & 0 \\ c_1 & c_0 \\ 1 & c_1 \\ 0 & 1 \end{bmatrix}. \tag{4.6}$$

The projection $P_{W^\perp}$ can be written as $P_{W^\perp} = I_{4\times 4} - N(N^*N)^{-1}N^*$.

If $c_0 = \frac{1}{17}$ and $c_1 = \frac{1}{5\pi}$ then computing G in MATLAB gives

$$G \approx \begin{bmatrix} 0.9442 & 0.0034 & -0.0558 & 0.0034 \\ -0.0570 & 0.9479 & -0.0570 & -0.0521 \end{bmatrix}.$$

More generally, we informally claim that if $c_0, c_1 \approx 0$ then

$$G \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{4.7}$$

To see this, note that the parameter $(c_0, c_1) = (0, 0)$ is forbidden in Example 4.1 to guarantee distinct eigenvalues of the operator T, but it gives insight into what G looks like when $c_0, c_1 \approx 0$. If $c_0, c_1 \approx 0$, then

$$N \approx N_0 := \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad P_{W^\perp} \approx I_{4\times 4} - N_0(N_0^* N_0)^{-1} N_0^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} =: P_0.$$

This gives $G = (FP_{W^\perp}F^*)^{-1}FP_{W^\perp} \approx (FP_0F^*)^{-1}FP_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

For perspective, note that the matrix in (4.7) is not the synthesis operator of an $\mathcal{I}_{1,4}$-dynamical frame, but it is the synthesis operator of an $\mathcal{I}_{0,4}$-dynamical frame.

The following two examples show how reordering a frame affects the manner in which the norms of a dynamical dual frame evolve.

Example 4.3. Given the set $\{e_n\}_{n=1}^4 \subset \mathbb{R}^2$ from Example 4.1, there are various ways of ordering the vectors. In Example 4.2, we considered the frame ordered as e_1, e_2, e_3, e_4, having synthesis matrix F given by (4.1). We mentioned how the vectors $\{T^n\phi\}$ of a dynamical dual frame would look in the cases where the coefficients c_0, c_1 of the characteristic polynomial of T are small. It is worth noting that, with this ordering of frame vectors, the first two vectors $(1, 0)$ and $(0, 1)$ already form a basis for $\mathbb{R}^2$, and we claimed that the last two vectors of dynamical dual frames tend to be close to $(0, 0)$ as c_1 and c_2 approach zero.

In the subsequent discussion, we aim to visualize this behavior a bit more and compare it to the outcomes obtained by considering a different order for the original frame vectors. Specifically, we will explore two options for the synthesis operator of the original frame:

$$\text{Case 1: } F = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \qquad \text{Case 2: } \widetilde{F} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

For our MATLAB simulation, we generated two decreasing random sequences of coefficients, one for c_0 and another for c_1. For each choice, we constructed the synthesis operators $G = (FP_{W^\perp}F^*)^{-1}FP_{W^\perp}$ and $(\widetilde{F}P_{W^\perp}\widetilde{F}^*)^{-1}\widetilde{F}P_{W^\perp}$ of a dynamical dual. Here, W represents the column span of the matrix N defined by (4.6). Additionally, for each choice of c_0, c_1 we computed the spectral radius of the generator operator T (the companion matrix (4.3) is the same for both cases, but the vector ϕ that is cyclical for T differs). The following are some of the outputs that we obtain in our simulations, and we copy them for the purpose of illustrating a pattern:

- Coefficients of characteristic polynomial: $c_0 = 0.80623, c_1 = 0.86705$
  Spectral radius: 0.8979

Synthesis operator of Dynamical Dual

Case 1: **Case 2:**

$$\begin{bmatrix} 0.4200 & 0.2784 & -0.5800 & 0.2784 \\ -0.3453 & 0.7194 & -0.3453 & -0.2806 \end{bmatrix} \quad \begin{bmatrix} 0.3506 & -0.6494 & 0.2804 & 0.2804 \\ -0.4314 & -0.4314 & 0.7219 & -0.2781 \end{bmatrix}$$

- Coefficients of characteristic polynomial: $c_0 = 0.62985, c_1 = 0.56805$
  Spectral radius: 0.79363

Synthesis operator of Dynamical Dual

Case 1: **Case 2:**

$$\begin{bmatrix} 0.5601 & 0.1533 & -0.4399 & 0.1533 \\ -0.2434 & 0.6984 & -0.2434 & -0.3016 \end{bmatrix} \quad \begin{bmatrix} 0.2090 & -0.7910 & 0.3177 & 0.3177 \\ -0.8010 & -0.8010 & 0.9595 & -0.0405 \end{bmatrix}$$

- Coefficients of characteristic polynomial: $c_0 = 0.38494, c_1 = 0.41617$
  Spectral radius: 0.62044

Synthesis operator of Dynamical Dual

Case 1: **Case 2:**

$$\begin{bmatrix} 0.6945 & 0.0918 & -0.3055 & 0.0918 \\ -0.2385 & 0.7937 & -0.2385 & -0.2063 \end{bmatrix} \quad \begin{bmatrix} 0.2727 & -0.7273 & 0.1977 & 0.1977 \\ -1.3341 & -1.3341 & 1.0688 & 0.0688 \end{bmatrix}$$

- Coefficients of characteristic polynomial: $c_0 = 0.16091, c_1 = 0.35971$
  Spectral radius: 0.40113

Synthesis operator of Dynamical Dual

Case 1: **Case 2:**

$$\begin{bmatrix} 0.8467 & 0.0475 & -0.1533 & 0.0475 \\ -0.2952 & 0.9529 & -0.2952 & -0.0471 \end{bmatrix} \quad \begin{bmatrix} 0.6000 & -0.4000 & 0.0473 & 0.0473 \\ -1.8282 & -1.8282 & 0.9518 & -0.0482 \end{bmatrix}$$

- Coefficients of characteristic polynomial: $c_0 = 0.0046115, c_1 = 0.093381$
  Spectral radius: 0.067908

Synthesis operator of Dynamical Dual

Case 1: **Case 2:**

$$\begin{bmatrix} 0.9954 & 0.0004 & -0.0046 & 0.0004 \\ -0.0933 & 1.0041 & -0.0933 & 0.0041 \end{bmatrix} \quad \begin{bmatrix} 0.9508 & -0.0492 & 0.0002 & 0.0002 \\ -9.7531 & -9.7531 & 0.9557 & -0.0443 \end{bmatrix}$$

It can be seen that for Case 1, as c_0, c_1 decrease, G tends to the matrix (4.7), as we suggested in Example 4.2. Notice that the first two columns of F (i.e., e_1, e_2) span $\mathbb{R}^2$, and this is related to the fact that, in Case 1, the first two columns of G approach the dual basis of $\{e_1, e_2\}$ (in this case, $\{e_1, e_2\}$), while the last two columns approach the zero vector. In Case 2, the fourth vector of the dynamical dual approaches the zero vector, and this is related to the fact that for $\widetilde{F}$ the first three columns contain a basis for $\mathbb{R}^2$.

Example 4.4. This example performs an analogous numerical experiment to that of Example 4.3 but uses a random frame $\{f_n\}_{n=1}^N$ for $\mathbb{R}^d$, with $d = 15$ and $N = 30$. We choose coefficients $c = (c_0, \dots, c_{d-1})$ randomly from the uniform distribution on $[0, 1/2]^d$, and generate the sequence $(c_0^k, \dots, c_{d-1}^k) = (1-(0.1)(k-1))^2 c$, for $k = 1, \dots, 10$. For each $1 \le k \le 10$ we compute a dynamical dual frame $\{(T_k)^n \phi_k\}_{n=1}^N$, where the characteristic polynomial of T_k is $p_{T_k}(x) = x^d + \sum_{j=0}^{d-1} c_j^k x^j$. In the experiments, we observe a cut-off phenomenon on the norms of the dual frame vectors. That is, for k fixed there is an abrupt decrease in the norm in the tail of the set $\{(T_k)^n \phi_k\}_{n=1}^N$. This cut-off phenomenon is accentuated when the coefficients of p_{T_k} tend to zero.

We run this experiment in three different setups:

Setup 1: We choose $\{f_n\}_{n=1}^N$ so that the first d vectors form a basis for $\mathbb{R}^d$. We observe the cut-off at $n = d$. See Figure 1a.

Setup 2: We choose the first two vectors of $\{f_n\}_{n=1}^N$ as zero and the subsequent d vectors linearly independent. We observe the cut-off occurs at $n = d+2$. See Figure 1b.

Setup 3: We choose $\{f_n\}_{n=1}^N$ so that $f_1 = f_5$ and $f_2 = f_8$. We observe the cut-off occurs at $n = d + 8$. See Figure 1c.

The remaining examples provide visualizations of the evolution of the elements in dynamical dual frames in dimensions two and three.

Example 4.5. We now revisit the Nth roots of unity frame with $N = 32$ investigated in Example 3.8 of [9]. Recall that this frame consists of the vectors $\{f_n\}_{n=1}^N$ where

$$f_n = \begin{pmatrix} \cos(2\pi n/N) \\ \sin(2\pi n/N) \end{pmatrix}, \quad 1 \le n \le N. \tag{4.8}$$

Figure 2 depicts four distinct dynamical dual frames for $\{f_n\}_{n=1}^{32}$. The roots of the characteristic polynomial for the corresponding evolution operator are provided in the caption below each graph. Figure 3 depicts the progression of $\|T^n\phi\|$ for the dynamical dual frames shown in Figure 2.

Example 4.6. Consider the frame $\{f_n\}_{n=1}^{32}$ for $\mathbb{R}^3$ given by

$$f_n = \begin{pmatrix} \sin((2j_n+1)\pi/16)\cos((2k_n+1)\pi/8) \\ \sin((2j_n+1)\pi/16)\sin((2k_n+1)\pi/8) \\ \cos((2j_n+1)\pi/16) \end{pmatrix},$$

where $j_n = \lceil \frac{n}{8} \rceil$ and $k_n = 1 + (n \bmod 8)$. This frame is depicted in Figure 4. Figure 5 depicts four distinct dynamical dual frames for $\{f_n\}_{n=1}^{32}$. The roots of the characteristic polynomial for the corresponding evolution operator are provided in the caption below each graph. Figure 6 depicts the progression of $\|T^n\phi\|$ for the dynamical dual frames shown in Figure 5.

5. Error bounds for the dynamical quantization algorithm

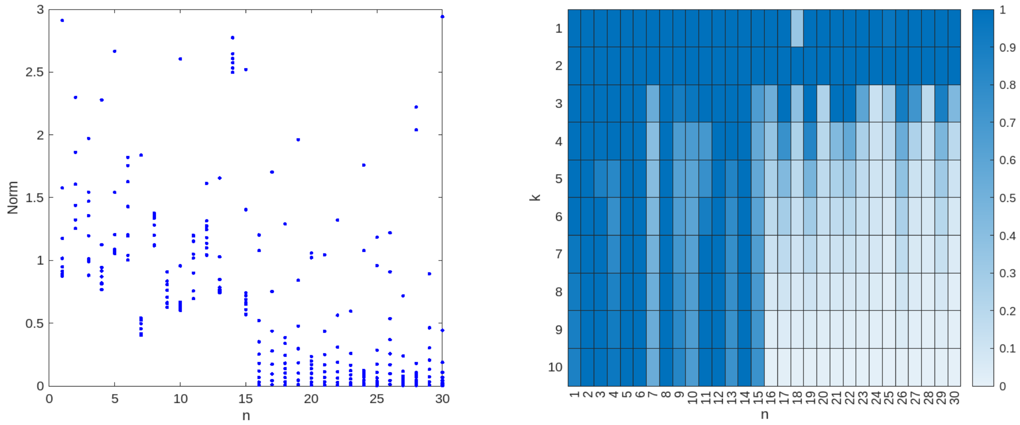
In this section, we briefly sketch how error bounds for the dynamical quantization algorithm from [9] can be adapted to the setting of dynamical duals with $\rho(T) < 1$.

Let $\{f_n\}_{n=1}^N$ be a frame for $\mathbb{F}^d$ with dynamical dual frame $\{T^n\phi\}_{n=1}^N$. The dynamical quantization algorithm from Section 4 in [9] is used to approximate the full-precision frame expansion $x = \sum_{n=1}^N \langle x, f_n\rangle T^n\phi$ by a quantized expansion $\widetilde{x} = \sum_{n=1}^N q_n T^n\phi$, with $q_n \in \mathcal{A}_{\delta,L}$, where the quantization alphabet $\mathcal{A}_{\delta,L}$ is a finite set of L uniformly spaced real numbers with gap δ. See Algorithm 1 in [9] for a description of the method.

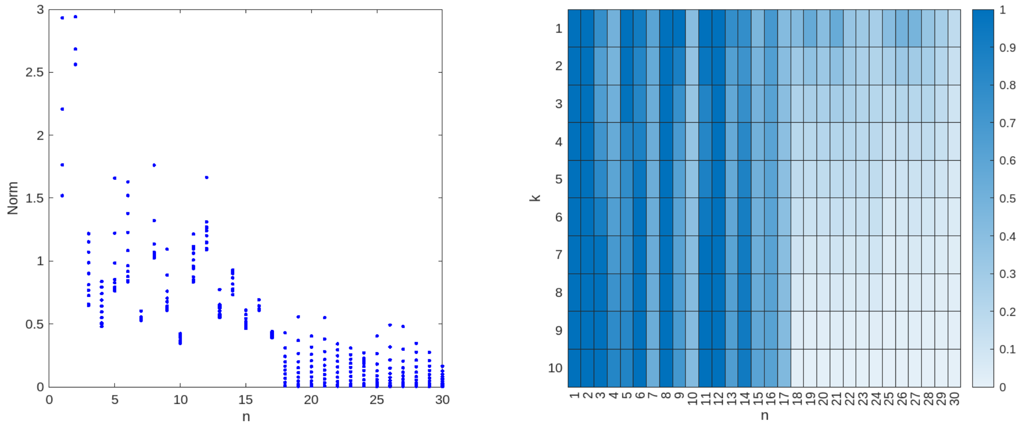
Corollary 4.4 in [9] shows that if the dynamical quantization algorithm is implemented with order $r = d$ then, under suitable conditions, the following error bound holds

$$\|x - \widetilde{x}\| \le \left(\frac{\delta}{2} \sum_{l=1}^d |\alpha_l| \right) \|T\|^N \|\phi\| \sum_{j=1}^d \|T\|^j, \tag{5.1}$$

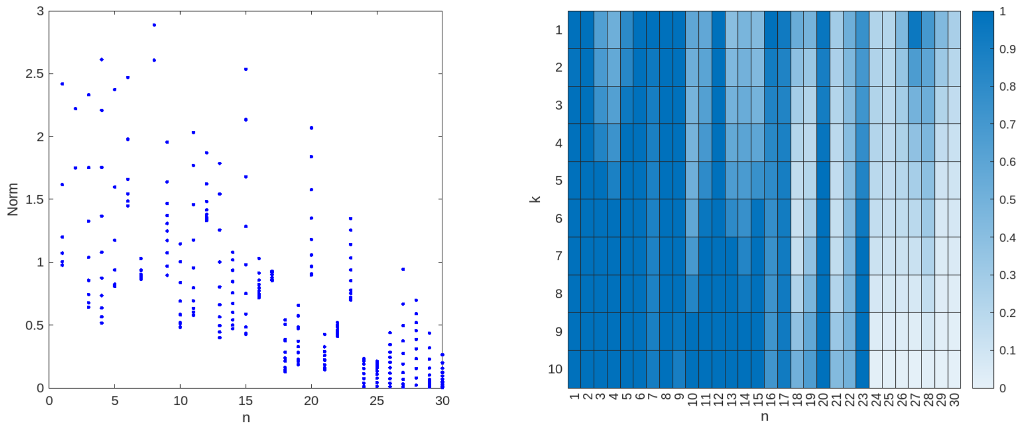
where $\{\alpha_l\}_{l=1}^d$ are constants in the algorithm that depend on T and ϕ, and which indicate how $T^{d+1}\phi$ is represented in terms of the basis $\{T^n\phi\}_{n=1}^d$, see Section 4 of [9] for details.



(A) Setup 1. Abrupt decrease at $n = d = 15$.

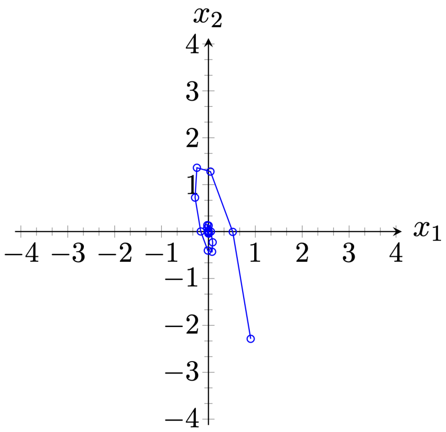


(B) Setup 2. Abrupt decrease in norm at $n = d + 2 = 17$.

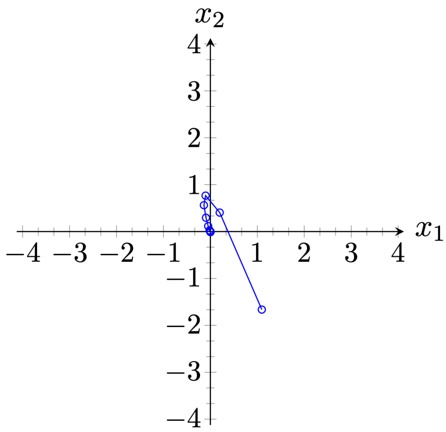


(C) Setup 3. Abrupt decrease at $n = d + 8 = 23$.

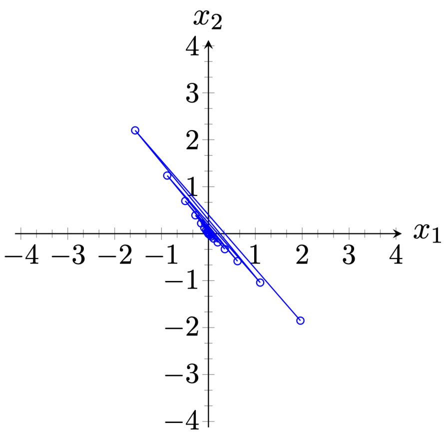
FIGURE 1. Left: For each $1 \leq n \leq N = 30$, we have 10 points representing the 2-norms of the dual vectors $\{(T_k)^n \phi_k\}_{k=1}^{10}$ (for visualization we plotted only norms less or equal than 3). Right: Heat map to represent the norms of the vectors $\{(T_k)^n \phi_k : 1 \leq n \leq 30, 1 \leq k \leq 10\}$. Color darkness is chosen so that it saturates when the 2-norm is bigger or equal than 1.



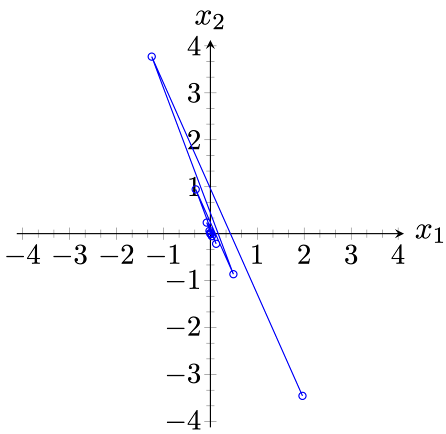
(A) Roots: $0.75e^{\pi i/4}$, $0.75e^{-\pi i/4}$.



(B) Roots: $0.5e^{\pi i/6}$, $0.5e^{-\pi i/6}$.



(C) Roots: -0.75, 0.75.



(D) Roots: -0.5, 0.5.

FIGURE 2. Examples of dynamical dual frames for the roots of unity frame for $\mathbb{R}^2$ with $N = 32$.

In view of (5.1) it is desirable to have $\|T\| < 1$ for the error $\|x - \widetilde{x}\|$ to be small, and Section 5 in [9] includes numerical examples where the quantization error becomes exponentially small as a function of the frame size N. The practical applicability of this approach is limited by the fact that not all frames admit a dynamical dual frame with $\|T\| < 1$. However, Theorem 3.4 and Corollary 3.5 show that every redundant finite frame has a dynamical dual frame that is generated by T with non-repeated roots and $\rho(T) < 1$; we adapt the error bound (5.1) to this setting. Specifically, we show that the following version of (5.1) holds with the role of $\|T\|$ replaced by $\rho(T)$ and an eigenvector

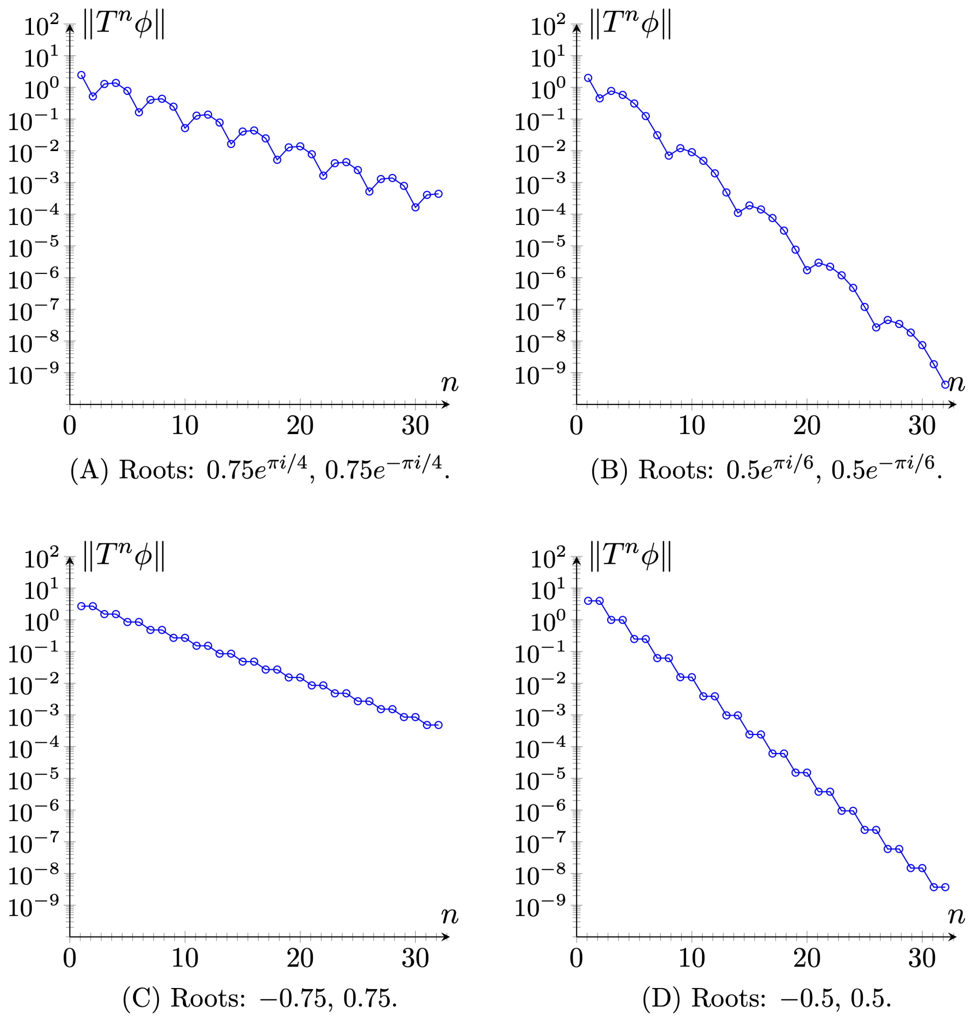


FIGURE 3. Progression of $\|T^n\phi\|$ for the Dynamical Dual Frames of Figure 2.

condition number $\kappa(T)$

$$\|x - \widetilde{x}\| \leq \left(\frac{\delta}{2}\sum_{l=1}^{d}|\alpha_l|\right)\kappa(T)[\rho(T)]^N\|\phi\|\sum_{j=1}^{d}[\rho(T)]^j. \tag{5.2}$$

Given a diagonalizable matrix T, its *eigenvector condition number*, e.g., [19], is defined by

$$\kappa(T) = \inf\ \|P\|\|P^{-1}\|$$

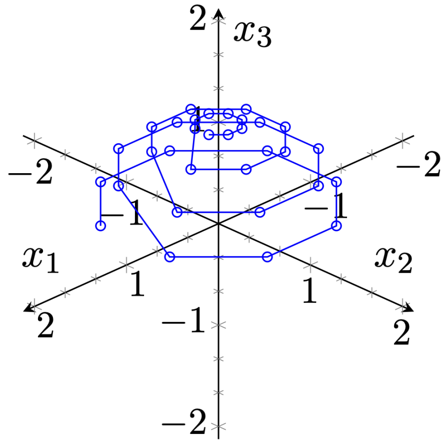
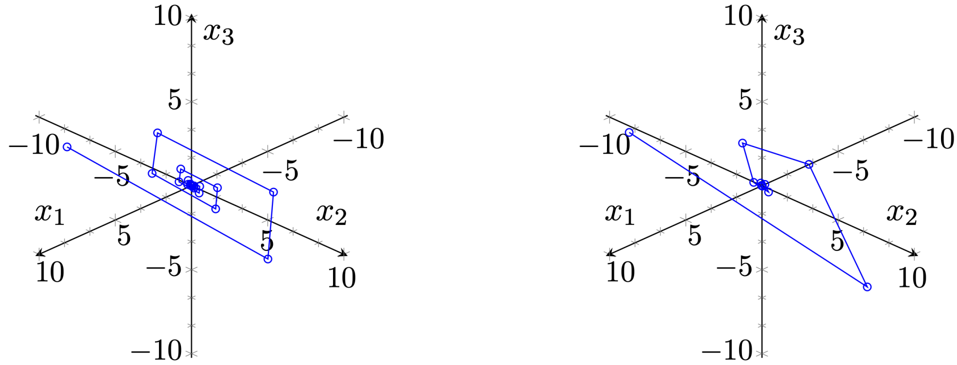


FIGURE 4. Frame of 32 unit vectors in $\mathbb{R}^3$ described in Example 4.6.



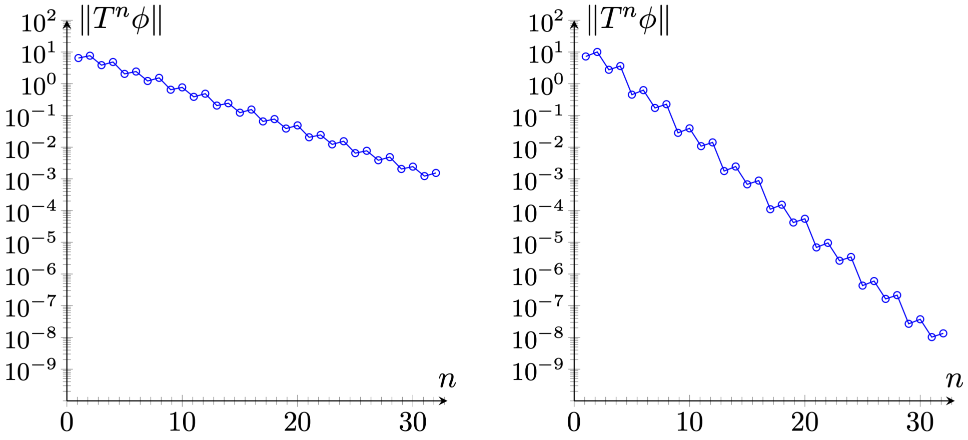
(A) Roots: 0.75, $0.75i$, $-0.75i$.

(B) Roots: 0.5, $0.5i$, $-0.5i$.

FIGURE 5. Examples of dynamical dual frames for the frame of Example 4.6.

where the infimum is taken over all invertible matrices P such that $T = PDP^{-1}$ for some diagonal matrix D. Recall that D contains the eigenvalues of T on its diagonal and P contains eigenvectors of T as its columns. The operator norm of T^n can be bounded as

$$\|T^n\| = \|PD^nP^{-1}\| \leq \|D^n\|\|P\|\|P^{-1}\| \leq \|D\|^n\|P\|\|P^{-1}\| = [\rho(T)]^n\|P\|\|P^{-1}\|. \quad (5.3)$$



(A) Roots: 0.75, $0.75i$, $-0.75i$.

(B) Roots: 0.5, $0.5i$, $-0.5i$.

FIGURE 6. Progression of $\|T^n\phi\|$ for the dynamical dual frames of Figure 5.

Taking an infimum over P in (5.3) gives

$$\|T^n\| \leq [\rho(T)]^n \kappa(T). \tag{5.4}$$

In particular, since every matrix with non-repeated eigenvalues is diagonalizable, (5.4) applies whenever T has non-repeated eigenvalues.

To sketch the derivation of (5.2), the same reasoning as for Corollary 4.4 and equation (4.15) in [9] gives that $x - \widetilde{x} = \sum_{j=1}^d w_j^N T^{j+N}\phi$, for scalars w_j^N that satisfy $|w_j^N| \leq \frac{\delta}{2}\sum_{l=1}^d |\alpha_l|$ by Lemma 4.2 in [9]. Combining this with (5.4) yields the bound (5.2) as follows

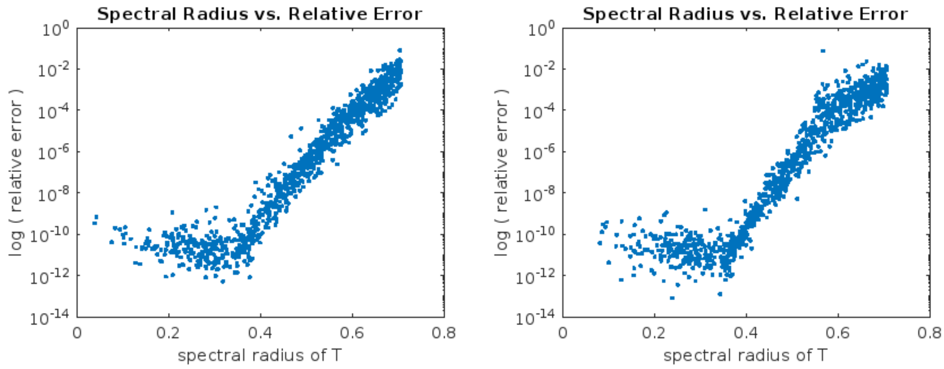
$$\begin{aligned}\|x - \widetilde{x}\| = \Big\|\sum_{j=1}^d w_j^N T^{j+N}\phi\Big\| &\leq \left(\frac{\delta}{2}\sum_{l=1}^d |\alpha_l|\right)\sum_{j=1}^d \|T^{j+N}\|\|\phi\| \\ &\leq \left(\frac{\delta}{2}\sum_{l=1}^d |\alpha_l|\right)\kappa(T)[\rho(T)]^N\|\phi\|\sum_{j=1}^d [\rho(T)]^j.\end{aligned}$$

Since Section 3 shows that every redundant finite frame admits dynamical dual frames generated by an operator T with non-repeated roots and $\rho(T) < 1$, the error bound (5.2) suggests that the dynamical quantization algorithm can perform well even if $\|T\| < 1$ is not possible.

The following numerical experiments illustrate the performance of the dynamical quantization algorithm when implemented with dual frames satisfying $\rho(T) < 1$.

Example 5.1 (Relative Error versus Spectral Radius)**.** This example shows that the error of the dynamical quantization algorithm decreases until saturation when implemented using dynamical duals with decreasing spectral radius $\rho(T)$, see Figure 7.

Let $\{f_n\}_{n=1}^{32}$ be a frame for $\mathbb{R}^2$; we consider when $\{f_n\}_{n=1}^{32}$ is either the 32nd roots of unity frame (4.8) or a random frame. We repeat the following experiment 1000 times. Let $x \in \mathbb{R}^2$ be a random vector with independent standard normal entries. Generate a dynamical dual frame $\{T^n\phi\}_{n=1}^{32}$ with $p_T(z) = z^2 + c_1 z + c_0$, where c_0, c_1 are drawn uniformly at random from $[0, 1/2]$. The dynamical quantization algorithm from [9] is applied to x with parameter $r = 2$ and a quantization alphabet with stepsize $\delta = 1$, and the output is denoted $\widetilde{x} \in \mathbb{R}^2$. Figure 7 plots the relative error $\frac{\|x-\widetilde{x}\|}{\|\widetilde{x}\|}$ against the computed spectral radius $\rho(T)$ for two different choices of frame: the 32nd roots of unity frame is used for Figure 7a and a random frame is used in Figure 7b.

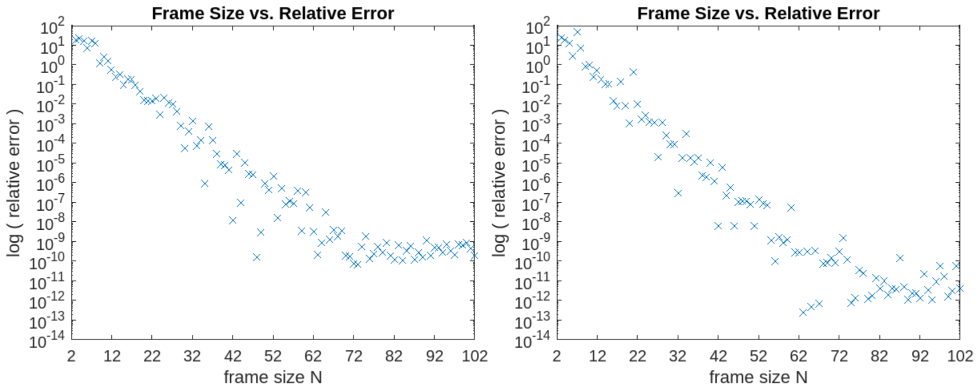


(A) Nth roots of unity frame. (B) Random frame.

FIGURE 7. Figures 7a and 7b depict the relationship between spectral radius and relative error of the dynamical quantization algorithm from [9] for two different frames.

Example 5.2 (Relative Error versus Frame Size)**.** This example shows that the error of the dynamical quantization algorithm decreases when considering frames of increasing size N while enforcing the dynamical dual to be generated by an operator T with $\rho(T) < 1$, see Figure 8.

For each $2 \leq N \leq 102$, let $\{f_n\}_{n=1}^{N}$ be a frame for $\mathbb{R}^2$; we consider when $\{f_n\}_{n=1}^{N}$ is either the Nth roots of unity frame (4.8) or a random frame. After having fixed the frame, we repeat the following experiment 1000 times. Let $x \in \mathbb{R}^2$ be a random vector with independent standard normal entries. Generate a dynamical dual frame $\{T^n\phi\}_{n=1}^{N}$ with $p_T(z) = z^2 + c_1 z + c_0$, where c_0, c_1 are drawn uniformly at random from $[0, 1/2]$. The dynamical quantization algorithm from [9] is applied to x with parameter $r = 2$ and a quantization alphabet with stepsize $\delta = 1$, and the output is denoted $\widetilde{x} \in \mathbb{R}^2$. Figure 8 plots the relative error $\frac{\|x-\widetilde{x}\|}{\|\widetilde{x}\|}$, averaged over the 1000 instances of x, against the frame size N for two different choices of frames: the Nth roots of unity frames are used in Figure 8a and random frames are used in Figure 8b.



(A) Nth roots of unity frame. (B) Random frame

FIGURE 8. Figures 8a and 8b depict the relationship between the frame size N and the relative error of the dynamical quantization algorithm from [9] for two different frames.

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