Poincare Journal of Analysis & Applications
Vol. 10, No. 3 (2023), Special issue, 55-67

DOI: 10.46753/pjaa.2023.v010i03.004

FIBONACCI WAVELETS APPROACH FOR SOLVING NON LINEAR FREDHOLM INTEGRAL EQUATIONS

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Date of Receiving	:	04. 04. 2023
Date of Revision	:	06. 08. 2023
Date of Acceptance	:	24. 08. 2023

Abstract. In this research article, an effective computational technique is introduced for the numerical solutions of nonlinear Fredholm integral equations (NLFIE) by utilizing the Fibonacci wavelets collocation method. The primary aim of this study is to use Fibonacci wavelets to approximate the nonlinear part of the integral equations. The NLFIE is then reduced to a set of nonlinear algebraic equations. Further, these equations are computed by appropriate techniques, such as the Newton-Raphson technique. Also, the error and convergence analysis of the suggested technique is presented. Several examples are presented to show the effectiveness of the suggested technique. This allows us to evaluate the precision of the number of results. It is worth mentioning that the findings are accurate even when there are few grid points.

1. Introduction

Integral equations play a significant role in various scientific and technological challenges. One of the fundamental methods employed across various domains of applied mathematics involves the utilization of integral equations. They are used in numerous mathematical models and engineering disciplines, including cosmic radiation, image processing, spectroscopy, radiography [8, 29]. Numerous numerical techniques have been created to find the answers to integral equations, such as the triangular function approach [16], the linear multistep method [6], Bernstein's approximation [30], radial basis functions (RBFs) [17], wavelet methods [32, 15, 25] and the modified homotopy perturbation method [18]. Due to the prevalence of integral equations and the fact that there are often no precise solutions, researchers have created a variety of numerical methods to solve these equations by approximating them [25, 11, 7, 4, 20].
Wavelets are mathematical operations that separate data into different frequency

2010 *Mathematics Subject Classification.* 11B39, 45B05, 65D30, 65T60.
Key words and phrases. Collocation points, Fibonacci polynomials, Fibonacci wavelets, Non linear integral equations.
Communicated by: Javad Mashreghi
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components. Each frequency component is then individually analyzed at a degree of detail that corresponds directly to its scaler value. In the last two decades, wavelets have become increasingly popular. They have a variety of uses in scientific computation, so it is not surprising that modern relevant work has significantly employed numerical approximation. This is primarily because wavelets offer a built-in method for breaking down a problem into a series of scale- and location-dependent coefficients. Wavelets have been applied to numerical approximations using a variety of techniques, such as the wavelet-meshless method [9], wavelet-based finite element method [33] wavelet Galerkin method [14]. Researchers have employed various types of wavelets for the numerical solutions of integrodifferential equations [5, 24], integral differential equations [32], fractional partial differential equations [23], partial differential equations [27] and ordinary differential equations [19].

Different wavelets have been employed to numerically solve various types of integral equations. Which includes Chebyshev, Legendre, Haar, CAS, Bernstein, and Fibonacci wavelets [32, 25, 24, 3, 10, 26]. The Fibonacci wavelets, which are derived from the Fibonacci polynomials, are a recent addition to the category of wavelet families. Sedigheh Sabermahani et al. [22] introduced Fibonacci wavelets for the first time in 2019. Fibonacci wavelets captured the interest of many researchers due to their unique characteristics and benefits over other wavelets [22]. Due to features like compact support and vanishing moment, the Fibonacci wavelet is a function that can be generated at multiple scales and has a wide range of applications. During our comprehensive literature survey, we discovered several valuable works related to the Fibonacci wavelet approach. These works encompassed diverse topics such as time-varying delay problems [22], Hunter-Saxton equation [28], fractional optimal control problems with bibliometric analysis [21], nonlinear Stratonovich Volterra integral equations [25], time-fractional telegraph equations [23], Fredholm integro-differential-difference equations [15], Pennes bioheat transfer equation [12], fractional Bagley-Torvik equation [31], linear Fredholm integral equations [32], logistic growth model [2], fractional electric circuit equations [1] and a time-fractional bioheat transfer model [13]. Notably, we observed that no existing research had explored the utilization of the Fibonacci wavelet for handling NLFIE. This realization motivated us to introduce and propose the Fibonacci wavelet collocation method as a novel approach for addressing NLFIE. With this proposed algorithm, we aim to open new avenues of research in this field and potentially uncover new insights and solutions for NLFIE. So in this work, an algorithm is presented to approximate the solution of the NLFIE by utilizing the Fibonacci wavelets. The results obtained from the Fibonacci wavelet collocation method demonstrate the effectiveness and proficiency of this method, as clearly illustrated in the tables and figures. The computed solutions obtained through the Fibonacci wavelet collocation method exhibit good approximate outcomes, indicating the reliability of the technique. The proposed wavelet-based numerical approach is straightforward to use, effective, and computationally reliability. In the present work, an algorithm is proposed to numerically solve the nonlinear Fredholm integral equation by using the Fibonacci wavelets collocation technique. Considering the following NLFIE as:

$$\mu(\nu) = f(\nu) + \int_a^b k(\nu, s)[\mu(s)]^q ds, \quad q > 0, \tag{1.1}$$

where ν,s $\in$ [0,1], $k(\nu, s)$ is the function of ν and s, the unknown function, $\mu(\nu)$ is to be determined and take [a,b] = [0,1] without loss of generality. The fundamental idea behind our work is to determine the unknown function μ by using the Fibonacci wavelets collocation technique.

The article is organized as follows: In Section 2, an overview of the Fibonacci polynomials, wavelets, and their properties is provided, along with the Fibonacci wavelets function approximation. The solution to the proposed numerical method is given in Section 3. Section 4 interprets the error estimation and convergence analysis. Section 5 discussed the four problems presented to justify the proposed method's efficiency and accuracy. Lastly, a detailed conclusion is drawn.

2. Fibonacci Wavelets

In this section, the basic and most important characteristics of Fibonacci polynomials, along with the Fibonacci wavelets is given. Further, the function approximation of the Fibonacci wavelets is provided.

Fibonacci Polynomial

In [22], the authors defined that for any $\nu \in \mathbb{R}^+$, the recurrence relation defines the Fibonacci polynomials as:

$$F_{\mathrm{u}+2}(\nu) = \nu F_{\mathrm{u}+1}(\nu) + F_{\mathrm{u}}(\nu), \quad (\mathrm{u} > 0),$$

initial conditions are provided as $F_0(\nu) = 0, F_1(\nu) = 1$.

Following is the generic formula for the Fibonacci polynomials:

$$F_{\mathrm{u}}(\nu) = \begin{cases} 1, & \mathrm{u} = 0, \\ \nu, & \mathrm{u} = 1, \\ \nu F_{\mathrm{u}-1}(\nu) + F_{\mathrm{u}-2}(\nu), & \mathrm{u} > 1, \end{cases}$$

and further the formula in closed form represented as:

$$F_{\mathrm{u}-1}(\nu) = \frac{\alpha^{\mathrm{u}} - \eta^{\mathrm{u}}}{\alpha - \eta} \quad (\mathrm{u} \geq 1),$$

where α , η represent the roots of the recursion's partner polynomial, $x^2 - \nu x - 1$.

In [22], the authors defined the Fibonacci polynomials power-form representation as:

$$F_{\mathrm{u}}(\nu) = \sum_{i=0}^{\lfloor \mathrm{u}/2 \rfloor} \begin{pmatrix} \mathrm{u} - i \\ i \end{pmatrix} \nu^{\mathrm{u}-2i} \quad (\mathrm{u} \geq 0),$$

where $\lfloor \cdot \rfloor$ stands for the well recognised floor function.

The Fibonacci polynomial has the following properties:

$$\int_0^\nu F_{\mathrm{u}}(s) ds = \frac{1}{\mathrm{u}+1} \left[F_{n+1}(\nu) + F_{\mathrm{u}-1}(\nu) - F_{\mathrm{u}+1}(0) + F_{\mathrm{u}-1}(0) \right],$$

$$\int_0^1 F_{\mathrm{u}}(\nu) F_{\mathrm{w}}(\nu) d\nu = \sum_{i=0}^{\lfloor \mathrm{u}/2 \rfloor} \sum_{j=0}^{\lfloor \mathrm{w}/2 \rfloor} \begin{pmatrix} \mathrm{u} - i \\ i \end{pmatrix} \begin{pmatrix} \mathrm{w} - j \\ j \end{pmatrix} \frac{1}{\mathrm{u} + \mathrm{w} - 2i - 2j + 1}. \quad (2.1)$$

Wavelets and their properties
In [22], the authors defined that by stretching and moving the mother wavelet generates a family of functions. They were called wavelets. The family of wavelets is given by:

$$\phi_{a,b}(\nu) = |a|^{-\frac{1}{2}} \phi\left(\frac{\nu - b}{a}\right), \quad a, b \in \mathbb{R}, a \neq 0,$$

such that, b and a are the translation and dilation parameter respectively. Additionally, b varying continuously.
Let us choose $a = a_0^{-k}$, $b = \mathrm{w} a_0^{-k} b_0$, $a_0 > 1$, $b_0 > 1$, and w, $k \in \mathbb{Z}^+$, and discrete wavelets family is given by

$$\phi_{\mathrm{w,u}}(\nu) = |a_0|^{\frac{k}{2}} \phi\left(a_0^k \nu - \mathrm{w} b_0\right),$$

in which wavelet basis $\phi_{\mathrm{w,u}}(\nu)$ is defined over $L^2(R)$.

Fibonacci Wavelet
From [23], we define the Fibonacci wavelets as:

$$\phi_{\mathrm{w,u}}(\nu) = \begin{cases} 2^{\frac{k-1}{2}} \tilde{F}_{\mathrm{u}}\left(2^{k-1}\nu - \mathrm{w} + 1\right), & \frac{\mathrm{w}-1}{2^{k-1}} \leq \nu < \frac{\mathrm{w}}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.2}$$

with

$$\tilde{F}_{\mathrm{u}}(\nu) = \frac{1}{\sqrt{\omega_{\mathrm{u}}}} F_{\mathrm{u}}(\nu),$$

and

$$\omega_{\mathrm{u}} = \int_0^1 F_{\mathrm{u}}^2(\nu) d\nu,$$

where $w_{\mathrm{u}}, \mathrm{u} = 0, 1, \ldots, M-1$, are obtained using equation (2.1), the Fibonacci polynomials is of order w, $\mathrm{w} = 1, 2, \ldots, 2^{k-1}$ where $k \in \mathbb{Z}^+$.

Function Approximation
Let $f(s) \in [0, 1)$ is expressed in form of the Fibonacci wavelet by

$$f(\nu) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} r_{n,m} \phi_{n,m}(\nu) = R^T \phi(\nu). \tag{2.3}$$

By truncating the equation (2.3), we obtain

$$f(\nu) = \sum_{m=0}^{M-1} \sum_{n=1}^{2^{k-1}} r_{n,m} \phi_{n,m}(\nu) = R^T \phi(\nu), \tag{2.4}$$

as R, $\phi(\nu)$ are the $\hat{m} \times 1$ ($\hat{m} = (2^{k-1}M)$) vectors given by

$$R = \left[r_{0,0}, r_{0,1}, .., r_{0,M}, r_{1,0}, .., r_{1,M}, .., r_{2^k-1,0}, .., r_{2^k-1,M}\right]^T, \tag{2.5}$$

and

$$\phi(\nu) = \left[\phi_{0,0}, \phi_{0,1}, .., \phi_{0,M}, \phi_{1,0}, .., \phi_{1,M}, .., \phi_{2^k-1,0}, .., \phi_{2^k-1,M}\right]^T. \tag{2.6}$$

For instance, if k = 2, M = 4, the eight basis functions are given by

$$\left.\begin{aligned} \phi_{1,0}(\nu) &= \sqrt{2}, \\ \phi_{1,1}(\nu) &= 2\sqrt{6}\nu, \\ \phi_{1,2}(\nu) &= \sqrt{\tfrac{15}{14}}\left(1+4\nu^2\right), \\ \phi_{1,3}(\nu) &= \sqrt{\tfrac{960}{38}}\left(2\nu^3+\nu\right) \end{aligned}\right\}, \quad 0 \le \nu < \frac{1}{2},$$

$$\left.\begin{aligned} \phi_{2,0}(\nu) &= \sqrt{2}, \\ \phi_{2,1}(\nu) &= \sqrt{6}(2\nu-1), \\ \phi_{2,2}(\nu) &= \sqrt{\tfrac{30}{7}}\left(2\nu^2-2s+1\right), \\ \phi_{2,3}(\nu) &= \sqrt{\tfrac{480}{304}}\left(8\nu^3-12\nu^2+10\nu-3\right) \end{aligned}\right\}, \quad \frac{1}{2} \le \nu < 1.$$

Using the collocation point $\nu_j = \frac{j-0.5}{\hat{m}}$, $j = 1, 2, \cdots \hat{m}$ equation (2.6) reduces to $\hat{m} \times \hat{m}$ Fibonacci wavelet coefficient matrix. For particular if we take $k = 2, M = 4$, from (2.6), it become

$$\phi(\nu) = \begin{bmatrix} \phi_{10}(\nu_i) \\ \phi_{11}(\nu_i) \\ \phi_{12}(\nu_i) \\ \phi_{13}(\nu_i) \\ \phi_{20}(\nu_i) \\ \phi_{21}(\nu_i) \\ \phi_{22}(\nu_i) \\ \phi_{23}(\nu_i) \end{bmatrix},$$

$$= \begin{pmatrix} \phi_{10}(\nu_1) & \phi_{10}(\nu_2) & \phi_{10}(\nu_3) & \phi_{10}(\nu_4) & \phi_{10}(\nu_5) & \phi_{10}(\nu_6) & \phi_{10}(\nu_7) & \phi_{10}(\nu_8) \\ \phi_{11}(\nu_1) & \phi_{11}(\nu_2) & \phi_{11}(\nu_3) & \phi_{11}(\nu_4) & \phi_{11}(\nu_5) & \phi_{11}(\nu_6) & \phi_{11}(\nu_7) & \phi_{11}(\nu_8) \\ \phi_{12}(\nu_1) & \phi_{12}(\nu_2) & \phi_{12}(\nu_3) & \phi_{12}(\nu_4) & \phi_{12}(\nu_5) & \phi_{12}(\nu_6) & \phi_{12}(\nu_7) & \phi_{12}(\nu_8) \\ \phi_{13}(\nu_1) & \phi_{13}(\nu_2) & \phi_{13}(\nu_3) & \phi_{13}(\nu_4) & \phi_{13}(\nu_5) & \phi_{13}(\nu_6) & \phi_{13}(\nu_7) & \phi_{13}(\nu_8) \\ \phi_{20}(\nu_1) & \phi_{20}(\nu_2) & \phi_{20}(\nu_3) & \phi_{20}(\nu_4) & \phi_{20}(\nu_5) & \phi_{20}(\nu_6) & \phi_{20}(\nu_7) & \phi_{20}(\nu_8) \\ \phi_{21}(\nu_1) & \phi_{21}(\nu_2) & \phi_{21}(\nu_3) & \phi_{21}(\nu_4) & \phi_{21}(\nu_5) & \phi_{21}(\nu_6) & \phi_{21}(\nu_7) & \phi_{21}(\nu_8) \\ \phi_{22}(\nu_1) & \phi_{22}(\nu_2) & \phi_{22}(\nu_3) & \phi_{22}(\nu_4) & \phi_{22}(\nu_5) & \phi_{22}(\nu_6) & \phi_{22}(\nu_7) & \phi_{22}(\nu_8) \\ \phi_{23}(\nu_1) & \phi_{23}(\nu_2) & \phi_{23}(\nu_3) & \phi_{23}(\nu_4) & \phi_{23}(\nu_5) & \phi_{23}(\nu_6) & \phi_{23}(\nu_7) & \phi_{23}(\nu_8) \end{pmatrix}. \tag{2.7}$$

$$F_{8\times 8} = \begin{pmatrix} 1.4142 & 1.4142 & 1.4142 & 1.4142 & 0 & 0 & 0 & 0 \\ 0.3062 & 0.9186 & 1.5309 & 2.1433 & 0 & 0 & 0 & 0 \\ 1.0513 & 1.1807 & 1.4394 & 1.8276 & 0 & 0 & 0 & 0 \\ 0.3166 & 1.0087 & 1.8775 & 3.0408 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.4142 & 1.4142 & 1.4142 & 1.4142 \\ 0 & 0 & 0 & 0 & 0.3062 & 0.9186 & 1.5309 & 2.1433 \\ 0 & 0 & 0 & 0 & 1.0513 & 1.1807 & 1.4394 & 1.8276 \\ 0 & 0 & 0 & 0 & 0.3166 & 1.0087 & 1.8775 & 3.0408 \end{pmatrix}. \tag{2.8}$$

Remark 2.1. For a $\hat{m}$-vector G, we have

$$\phi(\nu)\phi^T(\nu)G = \tilde{G}\phi(\nu),$$

now, the Taylor wavelet coefficient matrix is denoted by $\phi(\nu)$ and $\tilde{G}$ is a matrix of order $\hat{m} \times \hat{m}$ shown below

$$\tilde{G} = \phi(\nu)\bar{G}\phi^{-1}(\nu),$$

here $\bar{G} = \operatorname{diag}\left(\phi^{-1}(\nu)G\right)$. Also, for a $\hat{m} \times \hat{m}$ matrix S, we consider

$$\phi^T(\nu)Y\phi(\nu) = \hat{S}^T\phi(\nu),$$

where, $\hat{S}^T = W\phi^{-1}(\nu)$ and W is a $\hat{m}$ vector as $W = \operatorname{diag}\left(\phi^T(\nu)S\phi(\nu)\right)$.
For, further studies related to Fibonacci wavelet and its various applications, one can see [22, 28, 21, 12, 31].

3. Description of Method

In this section, the Fibonacci wavelet collocation method is proposed to solve the NLFIE. First, consider the NLFIE as:

$$\mu(\nu) = f(\nu) + \int_a^b k(\nu, s)[\mu(s)]^q ds, \quad , q > 0. \tag{3.1}$$

Now, assume that $(a, b) = [0, 1]$. Further, by approximating $\mu(\nu)$, $f(\nu)$ and $k(\nu, s)$, and $[\mu(\nu)]^q$ w.r.t Fibonacci wavelets collocation method is given by:

$$\mu(\nu) \simeq R^T\phi(\nu) = R\phi^T(\nu), \tag{3.2}$$

where R is given by equation (2.5) which is a undefined vector to be calculated.

$$f(\nu) \simeq Z^T\phi(\nu) = Z\phi^T(\nu), \tag{3.3}$$

$$k(\nu, s) \simeq \phi^T(\nu)K\phi(s) = \phi^T(s)K^T\phi(\nu), \tag{3.4}$$

and

$$[\mu(s)]^q \simeq (R^*)^T\phi(\nu) = (R^*)\phi^T(\nu), \tag{3.5}$$

where R and Z are Fibonacci wavelet collocation technique coefficient vectors, K is the Fibonacci wavelets matrices and R^* is a column vector function of the elements of vector R. Substituting equations (3.2), (3.3), (3.4), and (3.5) in equation (3.1), we get

$$R^T\phi(\nu) = Z^T\phi(\nu) + (R^*)^T\left(\int_0^1 \phi(s)\phi^T(s)ds\right)K\phi(\nu).$$

Using the relation $\int_0^1 \phi(s)\phi^T(s)ds = 1$, we compute

$$R^T\phi(\nu) = Z^T\phi(\nu) + (R^*)^TK\phi(\nu),$$

hence

$$R^T - (R^*)^TK = Z^T. \tag{3.6}$$

By solving this nonlinear system of equations, the undefined vector R is calculated. Substituting this unknown vector in equation (3.2), the solution of NLFIE equation (3.1) is obtained.

4. Error Estimation and Convergence Analysis

Theorem 4.1. [28] *Suppose* $\Theta \in C^M[0,1)$ *and* $Y_M = span\{\psi_0(\nu), \psi_1(\nu), ..., \psi_{M-1}(\nu)\}$. *If* $\Theta_M(\nu) = A^T \hat{F}(\nu)$ *is the best approximation of* $\Theta(\nu)$ *out of* Y_M *on the interval* $[\frac{\mathrm{u}-1}{2^{k-1}}, \frac{\mathrm{u}}{2^{k-1}}]$ *the error bound of the approximate solution* $\Theta^*(\nu)$ *by using Fibonacci wavelet on interval* $[0,1)$ *would be obtained in the following form:*

$$\|e(\nu)\|_2 = \|\Theta - \Theta^*\|_2 \leq \frac{H}{M!\sqrt{2M+1}}.$$

Theorem 4.2. [28] *If a continuous function* $\mu(\nu) \in L^2[0,1)$ *which is bounded by some constant* $\tilde{M}$ *i.e,* $|\mu(\nu)| \leq \tilde{M}$, *then the function* $\mu(\nu)$ *can be extended as the sum of Fibonacci wavelet and the series converges to* $\mu(\nu)$ *uniformly, i.e,*

$$\mu(\nu) = \sum_{\mathrm{m}=1}^{2^{k-1}} \sum_{\mathrm{n}=0}^{M-1} r_{\mathrm{m,n}} \phi_{\mathrm{m,n}}(\nu),$$

where

$$r_{\mathrm{m,n}} = \langle \mu(\nu), \phi_{\mathrm{m,n}}(\nu) \rangle .$$

Error Estimation

The absolute error of the proposed technique is calculated as:

$$E_{abs} = |\mu(\nu) - \mu^*(\nu)|, \quad \nu \in [0,1),$$

where, $\mu^*(\nu)$ and $\mu(\nu)$ is the approximate and exact solution respectively.

5. Numerical Problems

In this section, numerical problems are provided to demonstrate the viability and usefulness of the proposed technique. The results obtained from the proposed methodology are shown with the use of tables and graphs. MATLAB (R2020a) software is used throughout this paper for the computation work.

Example 5.1. Let us consider the NLFIE (1.1), by taking $f(\nu) = \nu e^{\nu} - \frac{1}{288}(3+e^2)\nu$, $k(\nu,s) = \frac{1}{36}\nu s$ and $q = 2$, where $(a,b) =$ [0,1], $\nu \in$ [0,1). The problem has exact solution $\mu(\nu) = \nu e^{\nu}$.

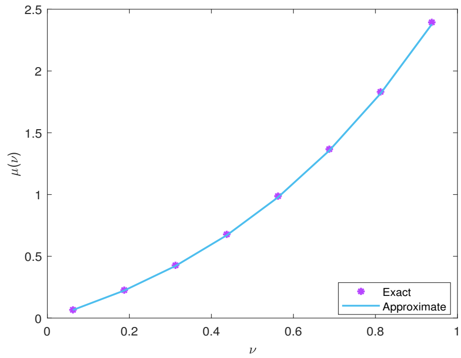

Figure 1. Comparison plot of exact and approximate solutions for Example 5.1.

ν	Exact	Approximate	Absolute error
0.0	0.046782	0.046777	4.6157e − 06
0.1	0.114422	0.112729	1.6922e − 03
0.2	0.246265	0.242882	3.3829e − 03
0.3	0.407039	0.401966	5.0731e − 03
0.4	0.602470	0.595705	6.7648e − 03
0.5	0.832415	0.823962	8.4532e − 03
0.6	0.832415	0.823962	8.4532e − 03
0.7	1.413631	1.401793	1.1838e − 02
0.8	1.784623	1.771087	1.3536e − 02
0.9	2.225092	2.209863	1.5229e − 02

Table 1. Numerical results of exact, approximate solutions and absolute error for k=2 M=4 of Example 5.1

We obtain an approximate solution to example 5.1 by applying the description of method presented in section 3. The comparison of exact and approximate solutions are plotted in fig 1. The absolute error of the problem for $\hat{m} = 8$ is given in table 1.

Example 5.2. Let us consider the NLFIE (1.1), by taking $f(\nu) = \frac{5}{6}\nu^2 - \frac{8}{105}\nu - 1$, $k(\nu, s) = \nu^2 s + \nu s^2$ and $q = 2$, where $(a, b) = [0,1]$, $\nu \in [0,1)$. The problem has exact solution $\mu(\nu) = \nu^2 - 1$.

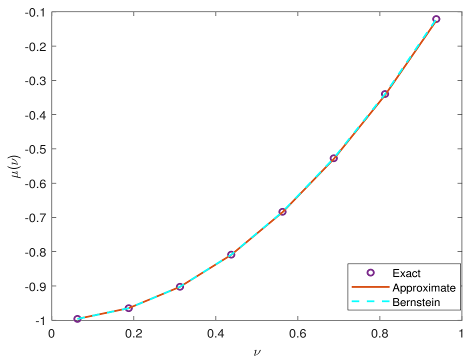

FIGURE 2. Comparison plot of exact, approximate and Bernstein wavelet solutions for Example 5.2.

ν	$\hat{m}$=4	$\hat{m}$=6	$\hat{m}$=8
0.1	6.4387e − 03	7.3665e − 04	8.7150e − 06
0.3	4.4295e − 03	6.7649e − 04	9.6168e − 05
0.5	9.4863e − 02	2.5664e − 03	1.2404e − 05
0.7	7.5475e − 02	2.6385e − 03	4.4143e − 04
0.9	8.7514e − 01	2.6597e − 03	9.0957e − 04

TABLE 2. Comparison of absolute error at resolution $\hat{m} = 4,6,8$ of Example 5.2

We approximate the solution of example 5.2 by applying the methodology presented in section 3. The comparison plot of exact, approximate, and Bernstein polynomial multiwavelet [26] solutions are geometrically represented by fig 2. The absolute error of the problem for different resolutions $\hat{m} = 4$, 6 and 8 is shown in table 2.

Example 5.3. Let us consider the NLFIE (1.1), by taking $f(\nu) = sin(\pi\nu)$, $k(\nu, s) = \frac{1}{5}cos(\pi\nu)sin(\pi s)$ and $q = 3$, where $(a, b) = [0,1]$, $\nu \in [0,1)$. The problem has exact solution $\mu(\nu) = sin(\pi\nu) + \frac{20-\sqrt{391}}{3}cos(\pi\nu)$.

ν	Example 5.3		Example 5.4	
	Exact	Approximate	Exact	Approximate
0.0	0.032343	0.048819	2.000000	2.000000
0.1	0.373832	0.386401	1.986718	1.986722
0.2	0.643794	0.654589	1.958593	1.958601
0.3	0.847865	0.855694	1.908593	1.908605
0.4	0.958862	0.962936	1.836718	1.836734
0.5	0.980785	0.980786	1.746093	1.746112
0.6	0.980785	0.980786	1.746093	1.746112
0.7	0.759893	0.752064	1.508593	1.508620
0.8	0.522526	0.511728	1.358593	1.358624
0.9	0.232635	0.220062	1.186718	1.186753

TABLE 3. Numerical results of exact and approximate solutions for k=2 M=4

We obtain an approximate solution to example 5.3 by applying the methodology presented in section 3. The exact and approximate solutions are graphically represented by fig 3. Table 3 is presented to show the values of approximate and exact solutions at different grid points.

Example 5.4. Let us consider the NLFIE (1.1), by taking $f(\nu) = 2 - \frac{2\sqrt{2}-1}{3}\nu - \nu^2$, $k(\nu, s) = \nu s$ and $q = \frac{1}{2}$, where $(a, b) =$ [0,1], $\nu \in$ [0,1). The problem has exact solution $\mu(\nu) = 2 - \nu^2$.
We acquire an approximate solution to example 5.4 by applying the methodology presented in section 3. The exact and approximate solutions are geometrically represented by fig 4. Table 3 is presented to show the values of approximate and exact solutions at different grid points.

6. Conclusion

The relevance of numerical methods has expanded due to the numerous applications of integral equations and the shortcomings of analytical methods for solving them. In the current study, we proposed a numerical approach to solving NLFIE using

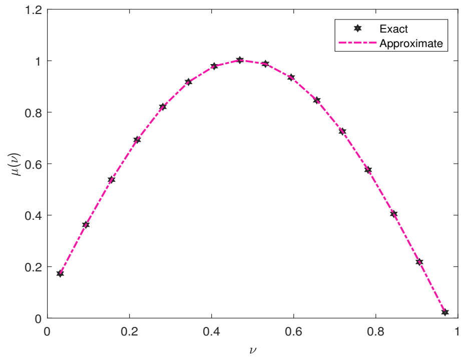

FIGURE 3. Comparison plot of exact and approximate solutions for Example 5.3.

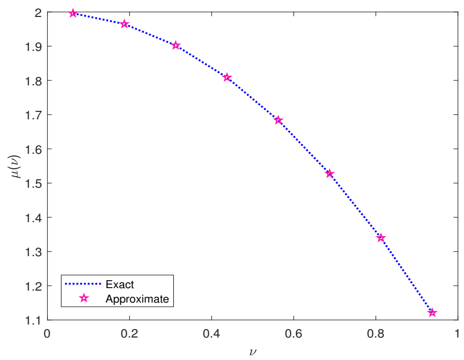

FIGURE 4. Comparison plot of exact and approximate solutions for Example 5.4.

Fibonacci wavelets. The Fibonacci wavelets matrices are constructed for different levels of resolution. The proposed technique describes the approximate solution of a given NLFIE by converting it into a nonlinear algebraic system and then solving it by Newton's method. An error and convergence analysis of the proposed technique is given. Test problems are proposed for different values of $q = \frac{1}{2}, 2, 3$. The geometrical representations are shown by fig 1, fig 2, fig 3, fig 4. The numerical trials show that the outcomes and the exact answer are in good agreement. The findings reveal a clear trend wherein higher levels of resolution correspond to reduced errors. As the resolution increases, the accuracy of the results improves, leading to a decrease in the overall error. Thus, the suggested approach is effective in resolving NLFIE.
In future, the suggested Fibonacci wavelet collocation technique can be extended to approximate the solutions of Abel's and Stochastic Abel's integral equations.

References

[1] S. Ahmed, K. Shah, S. Jahan, and T. Abdeljawad, An efficient method for the fractional electric circuits based on Fibonacci wavelet, Results Phys., 52 (2023), Art. ID 106753, 10 pp.

[2] S. Ahmed, S. Jahan and K. S. Nisar, Hybrid Fibonacci wavelet method to solve fractional-order logistic growth model, Math. Methods Appl. Sci., 46 (2023), no. 15, 16218–16231.

[3] P. Assari and M. Dehghan, Application of dual-Chebyshev wavelets for the numerical solution of boundary integral equations with logarithmic singular kernels, Eng. Comput., 35 (2019), 175–190.

[4] K. Atkinson, Iterative variants of the Nyström method for the numerical solution of integral equations, Numer. Math., 22 (1974), 17–31.

[5] S. Behera and S. S. Ray, An efficient numerical method based on Euler wavelets for solving fractional order pantograph Volterra delay-integro-differential equations, J. Comput. Appl. Math., 406 (2022), Paper No. 113825, 23 pp.

[6] H. Chen, L. Liu and J. Ma, Analysis of generalized multistep collocation solutions for oscillatory Volterra integral equations, Mathematics, 8 (2020), no. 11, Paper No. 2004, 16 pp.

[7] A. Deep, A. Kumar, S. Abbas and M. Rabbani, Solvability and numerical method for non-linear Volterra integral equations by using Petryshyn's fixed point theorem, Int. J. Nonlinear Anal. Appl., 13 (2022), no. 1, 1–28.

[8] O. Diekmann, Thresholds and travelling waves for the geographical spread of infection, J. Math. Biol., 6 (1978), no. 2, 109–130.

[9] M. Hosseininia, M. H. Heydari and Z. Avazzadeh, A hybrid wavelet-meshless method for variable-order fractional regularized long-wave equation, Eng. Anal. Bound. Elem., 142 (2022), 61–70.

[10] B. Hussain, A. Afroz and S. Jahan, Approximate solution for proportional delay Reccati differential equations by Haar wavelet method, Poincare J. Anal. Appl., 8 (2021), no. 2, 155–168.

[11] F. Fryklund, M. C. A. Kropinski and A.K. Tornberg, An integral equation–based numerical method for the forced heat equation on complex domains, Adv. Comput. Math., 46 (2020), no. 5, Paper No. 69, 36 pp.

[12] M. Irfan, F. A. Shah and K. S. Nisar, Fibonacci wavelet method for solving Pennes bioheat transfer equation, Int. J. Wavelets Multiresolution Inf. Process., 19 (2021), no. 6, Paper No. 2150023, 19 pp.

[13] M. Irfan and F. A. Shah, Fibonacci wavelet method for solving the time-fractional bioheat transfer model, Optik, 241 (2021), Art. ID 167084, 13 pp.

[14] M. Kumar, S. Upadhyay and K. N. Rai, A study of cryosurgery of lung cancer using Modified Legendre wavelet Galerkin method, J. Therm. Biol., 78 (2018), 356–366.

[15] A. Kurt, S. Yalçınbaş and M. Sezer, Fibonacci collocation method for solving high-order linear Fredholm integro-differential-difference equations, Int. J. Math. Math. Sci., (2013), Art. ID 846013, 9 pp.

[16] K. Maleknejad, H. Almasieh and M. Roodaki, Triangular functions (TF) method for the solution of nonlinear Volterra-Fredholm integral equations, Commun. Nonlinear Sci. Numer. Simul., 15 (2010), no. 11, 3293–3298.

[17] F. Mirzaee and N. Samadyar, On the numerical solution of fractional stochastic integro-differential equations via meshless discrete collocation method based on radial basis functions, Eng. Anal. Bound. Elem., 100 (2019), 246–255.

[18] S. Noeiaghdam, M. A. F. Araghi and D. Sidorov, Dynamical strategy on homotopy perturbation method for solving second kind integral equations using the CESTAC method, J. Comput. Appl. Math., 411 (2022), Paper No. 114226, 13 pp.

[19] A. Rayal and S.R. Verma, Two-dimensional Gegenbauer wavelets for the numerical solution of tempered fractional model of the nonlinear Klein-Gordon equation, Appl. Numer. Math., 174 (2022), 191–220.

[20] G. R. Richter, Numerical solution of integral equations of the first kind with nonsmooth kernels, SIAM J. Numer. Anal., 15 (1978), no. 3, 511–522.

[21] S. Sabermahani and Y. Ordokhani, Fibonacci wavelets and Galerkin method to investigate fractional optimal control problems with bibliometric analysis, J. Vib. Control, 27 (2021), no. 15–16, 1778–1792.

[22] S. Sabermahani Y. Ordokhani, and S. A. Yousefi. Fibonacci wavelets and their applications for solving two classes of time-varying delay problems, Optim. Control Appl. Methods, 41 (2020), no. 2, 395–416.

[23] F. A. Shah, M. Irfan, K. S. Nisar, R. T. Matoog, and E. E. Mahmoud, Fibonacci wavelet method for solving time-fractional telegraph equations with Dirichlet boundary conditions, Results Phys., 24 (2021), Art. ID 104123, 10 pp.

[24] S. C. Shiralashetti and S. Kumbinarasaiah, CAS wavelets analytic solution and Genocchi polynomials numerical solutions for the integral and integro-differential equations, J. Interdiscip. Math., 22 (2019), no. 3, 201–218.

[25] S. C. Shiralashetti and L. Lamani, Fibonacci wavelet based numerical method for the solution of nonlinear Stratonovich Volterra integral equations, Sci. Afr., 10 (2020), Art. ID e00594, 13 pp.

[26] S. C. Shiralashetti and L. Lamani, A modern approach based on Bernstein polynomial multiwavelets to solve fredholm integral and system of fredholm integral equations, South East Asian J. Math. Math. Sci., 16 (2020), no. 3, 251–268.

[27] S. Sokhal and S.R. Verma, A Fourier wavelet series solution of partial differential equation through the separation of variables method, Appl. Math. Comput., 388 (2021), Paper No. 125480, 25 pp.

[28] H. M. Srivastava, F. A. Shah and N. A. Nayied, Fibonacci Wavelet Method for the Solution of the Non-Linear Hunter-Saxton Equation, Appl. Sci., 12 (2022), no. 15, Paper No. 7738, 17 pp.

[29] H. R. Thieme, A model for the spatial spread of an epidemic, J. Math. Biol., 4 (1977), no. 4, 337–351.

[30] F. Usta, Bernstein approximation technique for numerical solution of Volterra integral equations of the third kind, Comput. Appl. Math., 40 (2021), no. 5, Art. No. 161, 11 pp.

[31] P. Yadav, S. Jahan and K. S. Nisar, Solving fractional Bagley-Torvik equation by fractional order Fibonacci wavelet arising in fluid mechanics, Ain Shams Eng. J., 15 (2023), no. 1, Art. ID 102299, 8 pp.

[32] P. Yadav, S. Jahan and K. S. Nisar, Fibonacci wavelet collocation method for Fredholm integral equations of second kind, Qual. Theory Dyn. Syst., 22 (2023), no. 2, Art. No. 82, 17 pp.

[33] Z. B. Yang, X. F. Chen, Y. Xie, H. Zuo, H. H. Miao and X. W. Zhang, Wave motion analysis and modeling of membrane structures using the wavelet finite element method, Appl. Math. Model., 40 (2016), no. 3, 2407–2420.

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