Poincare Journal of Analysis & Applications
Vol. 10, No. 3 (2023), Special issue, 69-87

DOI: 10.46753/pjaa.2023.v010i03.005

OPERATORS ASSOCIATED WITH GABOR SYSTEMS IN $\mathcal{L}^2(\mathbb{R}, \mathbb{H})$

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Date of Receiving	:	10. 04. 2023
Date of Revision	:	26. 09. 2023
Date of Acceptance	:	07. 10. 2023

Abstract. In this paper, we introduce and study Gabor systems for quaternionic Hilbert space $\mathcal{L}^2(\mathbb{R}, \mathbb{H})$ and explicitly define their analysis and synthesis operators with the help of tensor product. We also provide some characterizations for Bessel systems and Gabor frames. Moreover, a necessary and sufficient condition for the boundedness of the synthesis operator with the help of a matrix valued function is also given. Further, we give the Walnut's representation for the Gabor frame operator in quaternionic setting by introducing the correlation functions. Finally, we give a necessary and sufficient condition for the invertibility and boundedness of the Gabor frame operator.

1. Introduction

An important task in signal processing is to investigate signals with the help of a set of basic building blocks that includes a fixed function which is shifted in time and frequency. Gabor systems, whose elements are generated by the time-frequency shifts of a fixed function of $\mathcal{L}^2(\mathbb{R})$, serve this purpose[5, 6, 7]. They are basically known for their applications in auditory signal processing, pseudodifferential operator analysis and uncertainity principle.

The generalization of the concept of Gabor systems to the quaternionic Hilbert space $\mathcal{L}^2(\mathbb{R}, \mathbb{H})$ has become an interesting and significant problem mainly due to its importance in signal processing. The non-commutativity of quaternions makes the construction and generalization of the results of Gabor systems to quaternionic Hilbert spaces, a bit difficult. In 2018, Cerejeiras et al. [3] studied Gabor systems for $\mathcal{L}^2(\mathbb{R}^2, \mathbb{H})$ by defining Gabor frame operator using two-sided windowed quaternionic Fourier transform. For detailed information about quaternionic fourier transforms, refer[1, 4]. Recently, Li [12] also studied the quaternionic Gabor frame operator for $\mathcal{L}^2(\mathbb{R}^2, \mathbb{H})$ through two-sided quaternionic Gabor Fourier transform which is almost a similar approach. Their

2010 *Mathematics Subject Classification.* 42C15, 42A38.
Key words and phrases. Gabor frames, frame operator, quaternions, Walnut's representation.
Communicated by: Firdous Ahmad Shah
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approach lacks the idea about the analysis and synthesis operators. This motivates us to study Gabor frames for $\mathcal{L}^2(\mathbb{R}, \mathbb{H})$ and define the Gabor frame operator through the analysis and synthesis operators, similar to the case of complex Hilbert spaces. We also prove various significant results concerning Gabor frames in $\mathcal{L}^2(\mathbb{R}, \mathbb{H})$. This paper gives a wider and abstract vision about the understanding of Gabor systems in $\mathcal{L}^2(\mathbb{R}, \mathbb{H})$.

Throughout this paper, $\mathbb{H}$ denotes the non-commutative field of quaternions, $\mathbb{Z}$ the set of all integers and $\mathfrak{H}$ a right quaternionic Hilbert space. By the term "right-linear operator" we mean $\mathbb{H}$-linear operator with scalars on the right hand side of the vectors.

Outline of the Paper: With a view of making the paper self-contained, we recall a few facts and results about quaternionic Hilbert spaces in the remaining part of this section. In section 2, we define Gabor frames in $\mathcal{L}^2(\mathbb{R}, \mathbb{H})$ and give some important characterizations of Bessel systems and Gabor frames. Also, we define the Gabor frame operator with the help of analysis and synthesis operators. Finally, we conclude our paper with the Walnut's representation and a characterization for the boundedness and invertibility of the Gabor frame operator, in section 3.

2. Preliminaries

Quaternions are a four dimensional non-commutative extension of the complex numbers over $\mathbb{R}$. The quaternionic algebra $\mathbb{H}$ contain elements of the form

$$\mathfrak{q} = \mathfrak{q}_0 + i\mathfrak{q}_1 + j\mathfrak{q}_2 + k\mathfrak{q}_3,\ \ \mathfrak{q}_0, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3 \in \mathbb{R}$$

where $i^2 = j^2 = k^2 = ijk = -1$. For any $\mathfrak{q} = \mathfrak{q}_0 + i\mathfrak{q}_1 + j\mathfrak{q}_2 + k\mathfrak{q}_3 \in \mathbb{H}$, $\mathfrak{q}_0$ is called the scalar (or real part) and $i\mathfrak{q}_1 + j\mathfrak{q}_2 + k\mathfrak{q}_3$ is called the imaginary (or vector part) of $\mathfrak{q}$. Further, it can be expressed as $\mathfrak{q} = a + v$, where a is the scalar part and v is the imaginary part of $\mathfrak{q}$ and its conjugate $\bar{\mathfrak{q}}$ is given by

$$\bar{\mathfrak{q}} = \mathfrak{q}_0 - i\mathfrak{q}_1 - j\mathfrak{q}_2 - k\mathfrak{q}_3.$$

This directs a norm of $\mathfrak{q} \in \mathbb{H}$ defined as

$$|\mathfrak{q}| = \sqrt{\mathfrak{q}\bar{\mathfrak{q}}} = \sqrt{\mathfrak{q}_0^2 + \mathfrak{q}_1^2 + \mathfrak{q}_2^2 + \mathfrak{q}_3^2}.$$

For basic definitions and information about quaternionic Hilbert spaces and related terms, one may refer [8, 15].

The quaternion's exponential function for $\mathfrak{q} = a + v \in \mathbb{H}$ is given by

$$e^{\mathfrak{q}} = e^a\Big(\cos(|v|) + \frac{v}{|v|}\sin(|v|)\Big).$$

A quaternion-valued function $\mathfrak{f} : \mathbb{R} \to \mathbb{H}$ can be expressed as

$$\mathfrak{f}(x) = \mathfrak{f}_0(x) + i\mathfrak{f}_1(x) + j\mathfrak{f}_2(x) + k\mathfrak{f}_3(x),$$

where $x \in \mathbb{R}$ and $\mathfrak{f}_n : \mathbb{R} \to \mathbb{R}$, $n = 0, 1, 2, 3$. The conjugate of a function $\mathfrak{f} : \mathbb{R} \to \mathbb{H}$ is defined as $\bar{\mathfrak{f}} : \mathbb{R} \to \mathbb{H}$ such that $\bar{\mathfrak{f}}(x) = \overline{\mathfrak{f}(x)}$ and the right scalar multiplication $\mathfrak{f}\mathfrak{q} : \mathbb{R} \to \mathbb{H}$ with $(\mathfrak{f}\mathfrak{q})(x) = \mathfrak{f}(x)\mathfrak{q}$, $\mathfrak{q} \in \mathbb{H}, x \in \mathbb{R}$. The right quaternionic vector space of square integrable quaternion-valued functions given by

$$\mathcal{L}^2(\mathbb{R}, \mathbb{H}) = \Big\{\mathfrak{f} : \mathbb{R} \to \mathbb{H} \ \Big|\ \int_{\mathbb{R}} |\mathfrak{f}(x)|^2 dx < \infty\Big\}$$

forms a right quaternionic Hilbert space under the quaternion-valued inner product

$$\langle \mathfrak{f}|\mathfrak{g}\rangle = \int_{\mathbb{R}} \overline{\mathfrak{f}(x)}\mathfrak{g}(x)dx,\ \ \mathfrak{f},\mathfrak{g}\in\mathcal{L}^2(\mathbb{R},\mathbb{H})$$

where, dx is the usual Lebesgue measure on $\mathbb{R}$.

The space of all quaternion-valued essentially bounded functions is given by

$$\mathcal{L}^\infty(\mathbb{R},\mathbb{H}) = \left\{f:\mathbb{R}\to\mathbb{H}\ \Big|\ \operatorname*{ess\,sup}_{x\in\mathbb{R}}|\mathfrak{f}(x)|<\infty\right\} \text{ with } \|\mathfrak{f}\|_\infty = \operatorname*{ess\,sup}_{x\in\mathbb{R}}|\mathfrak{f}(x)|.$$

Also the space $\ell^2_{\mathbb{H}}$ defined by

$$\ell^2_{\mathbb{H}} = \left\{\{\mathfrak{q}_n\}_{n\in\mathbb{Z}}\subset\mathbb{H}\Big|\sum_{n\in\mathbb{Z}}|\mathfrak{q}_n|^2<\infty\right\}$$

under the right multiplication by quaternions together with the inner product given by

$$\langle\{\mathfrak{p}_n\}_{n\in\mathbb{Z}}|\{\mathfrak{q}_n\}_{n\in\mathbb{Z}}\rangle \ = \ \sum_{n\in\mathbb{Z}}\overline{\mathfrak{p}_n}\mathfrak{q}_n,\ \ \{\mathfrak{p}_n\}_{n\in\mathbb{Z}},\{\mathfrak{q}_n\}_{n\in\mathbb{Z}}\in\ell^2_{\mathbb{H}},$$

forms a right quaternionic Hilbert space. In fact it is a two-sided quaternionic Hilbert space and hence $\ell^2_{\mathbb{H}}\otimes\ell^2_{\mathbb{H}}$ forms a right quaternionic Hilbert space under the inner product

$$\langle\mathfrak{p}\otimes\mathfrak{q}|\mathfrak{p}^{'}\otimes\mathfrak{q}^{'}\rangle = \left\langle\langle\mathfrak{p}^{'}|\mathfrak{p}\rangle\mathfrak{q}\Big|\mathfrak{q}^{'}\right\rangle,\ \ \mathfrak{p},\mathfrak{q},\mathfrak{p}^{'}\mathfrak{q}^{'}\in\ell^2_{\mathbb{H}}.$$

For details, refer [2, 14]. Also, one may define partial isometries on $\ell^2_{\mathbb{H}}$ with the help of the standard orthonormal basis $\{\mathfrak{e}_n\}_{n\in\mathbb{Z}}$ of $\ell^2_{\mathbb{H}}$ as $|\mathfrak{e}_m\rangle_{\ell^2_{\mathbb{H}}}:\ell^2_{\mathbb{H}}\to\ell^2_{\mathbb{H}}\otimes\ell^2_{\mathbb{H}}$ such that

$$|\mathfrak{e}_m\rangle_{\ell^2_{\mathbb{H}}}(\mathfrak{q}) = \mathfrak{e}_m\otimes\mathfrak{q},\ \ \mathfrak{q}\in\ell^2_{\mathbb{H}},\ m\in\mathbb{Z}.$$

And for $\mathfrak{p}=\{\mathfrak{p}_n\}_{n\in\mathbb{Z}},\mathfrak{q}=\{\mathfrak{q}_n\}_{n\in\mathbb{Z}}\in\ell^2_{\mathbb{H}}$ its adjoint ${}_{\ell^2_{\mathbb{H}}}\langle\mathfrak{e}_m|:\ell^2_{\mathbb{H}}\otimes\ell^2_{\mathbb{H}}\to\ell^2_{\mathbb{H}}$ is given by

$$\begin{aligned}{}_{\ell^2_{\mathbb{H}}}\langle\mathfrak{e}_m|(\mathfrak{p}\otimes\mathfrak{q}) &= \langle\mathfrak{e}_m|\mathfrak{p}\rangle\mathfrak{q}\\ &= \mathfrak{p}_m\mathfrak{q}\\ &= \{\mathfrak{p}_m\mathfrak{q}_n\}_{n\in\mathbb{Z}}.\end{aligned}$$

Definition 2.1 ([10])**.** A function $\mathfrak{g}\in\mathcal{L}^\infty(\mathbb{R},\mathbb{H})$ is said to be in *quaternionic Wiener space* $\mathcal{W}(\mathbb{R},\mathbb{H})$ if

$$\|\mathfrak{g}\|_{\mathcal{W}} := \sum_{n\in\mathbb{Z}}\operatorname*{ess\,sup}_{x\in[0,1]}|\mathfrak{g}(x+n)| = \sum_{n\in\mathbb{Z}}\|\mathfrak{g}.\chi_{[n,n+1]}\|_\infty<\infty.$$

It is easy to observe that functions in the quaternionic Wiener space are locally bounded and in $\mathcal{L}^1(\mathbb{R},\mathbb{H})$.

3. Gabor frames in Quaternionic Hilbert spaces

For $b,w\in\mathbb{R}$, define the translation operator $\mathcal{T}_b$ and the modulation operator E_w on $\mathcal{L}^2(\mathbb{R},\mathbb{H})$ as:

$$\mathcal{T}_b\mathfrak{f}(x)=\mathfrak{f}(x-b)\ \text{ and }\ E_w\mathfrak{f}(x)=\mathfrak{f}(x)e^{-2\pi jwx},\ \mathfrak{f}\in\mathcal{L}^2(\mathbb{R},\mathbb{H}),\ x\in\mathbb{R}.$$

One may observe that $\mathcal{T}_b$ and E_w are not commutative in nature. Indeed for $b,w\in\mathbb{R}$,

$$\mathcal{T}_bE_w\mathfrak{f}(x)=\mathfrak{f}(x-b)e^{-2\pi jw(x-b)}=E_w\mathcal{T}_b\mathfrak{f}(x)e^{2\pi jwb},\ x\in\mathbb{R}.$$

Thus $E_w\mathcal{T}_b\mathfrak{f}=(\mathcal{T}_bE_w\mathfrak{f})e^{-2\pi jwb},\ \mathfrak{f}\in\mathcal{L}^2(\mathbb{R},\mathbb{H})$.

Note that we will also let E_w denote the function $x\mapsto e^{-2\pi jwx}$ on $\mathbb{R}$.

Definition 3.1. Let $\mathfrak{g} \in \mathcal{L}^2(\mathbb{R}, \mathbb{H})$ be a non-zero function and $\alpha, \beta > 0$. Then, define the *Gabor system* generated by $\mathfrak{g}, \alpha, \beta$ as the set of time-frequency shifts given by

$$\mathcal{G}(\mathfrak{g}, \alpha, \beta) := \{\mathfrak{g}_{m,n} = E_{\beta m} \mathcal{T}_{\alpha n} \mathfrak{g} \mid m, n \in \mathbb{Z}\},$$

where $\mathfrak{g}$ is called the *window function* for the Gabor system $\mathcal{G}(\mathfrak{g}, \alpha, \beta)$.

Definition 3.2. The Gabor system $\mathcal{G}(\mathfrak{g}, \alpha, \beta)$ is said to be a *Gabor frame* for $\mathcal{L}^2(\mathbb{R}, \mathbb{H})$ if there exist two constants $0 < r_1 \leq r_2 < \infty$ such that

$$r_1 \|\mathfrak{f}\|^2 \leq \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} |\langle \mathfrak{g}_{m,n} | \mathfrak{f} \rangle|^2 \leq r_2 \|\mathfrak{f}\|^2, \ \mathfrak{f} \in \mathcal{L}^2(\mathbb{R}, \mathbb{H}). \tag{3.1}$$

The real constants r_1 and r_2 are known as lower and upper frame bounds for the Gabor frame $\mathcal{G}(\mathfrak{g}, \alpha, \beta)$ respectively and the inequality (3.1) is known as the Gabor frame inequality. The system $\mathcal{G}(\mathfrak{g}, \alpha, \beta)$ is said to be a *Bessel system* for $\mathcal{L}^2(\mathbb{R}, \mathbb{H})$ with Bessel bound r_2 if it satisfies the right hand side of the Gabor frame inequality. Also, a Gabor frame is known as Parseval Gabor frame if $r_1 = r_2 = 1$.

Lemma 3.3. *Let $\mathcal{G}(\mathfrak{g}, \alpha, \beta)$ be a Bessel system for $\mathcal{L}^2(\mathbb{R}, \mathbb{H})$ with Bessel bound r and $\{\mathfrak{e}_n\}_{n \in \mathbb{Z}}$ be the standard orthonormal basis of $\ell^2_{\mathbb{H}}$. Then for each $m \in \mathbb{Z}$, $\Phi_m : \mathcal{L}^2(\mathbb{R}, \mathbb{H}) \to \ell^2_{\mathbb{H}}$ given by*

$$\Phi_m(\mathfrak{f}) = \sum_{n \in \mathbb{Z}} \mathfrak{e}_n \langle \mathfrak{g}_{m,n} | \mathfrak{f} \rangle, \ \mathfrak{f} \in \mathcal{L}^2(\mathbb{R}, \mathbb{H}) \tag{3.2}$$

defines a bounded right-linear operator. Moreover, its adjoint $\Phi_m^* : \ell^2_{\mathbb{H}} \to \mathcal{L}^2(\mathbb{R}, \mathbb{H})$ is given by $\Phi_m^*(\mathfrak{q}) = \sum\limits_{n \in \mathbb{Z}} g_{m,n} \mathfrak{q}_n$, $\mathfrak{q} = \{\mathfrak{q}_n\}_{n \in \mathbb{Z}} \in \ell^2_{\mathbb{H}}$.

Proof. Since $\mathcal{G}(\mathfrak{g}, \alpha, \beta)$ is a Bessel system for $\mathcal{L}^2(\mathbb{R}, \mathbb{H})$, for $\mathfrak{f} \in \mathcal{L}^2(\mathbb{R}, \mathbb{H})$,

$$\|\Phi_m(\mathfrak{f})\|^2 = \sum_{n \in \mathbb{Z}} |\langle \mathfrak{g}_{m,n} | \mathfrak{f} \rangle|^2 \leq r \|\mathfrak{f}\|^2 < \infty.$$

Therefore, Φ_m is a well-defined bounded operator. Further for $\mathfrak{f} \in \mathcal{L}^2(\mathbb{R}, \mathbb{H})$ and $\mathfrak{q} = \{\mathfrak{q}_n\}_{n \in \mathbb{Z}} \in \ell^2_{\mathbb{H}}$, we compute

$$\begin{aligned}
\langle \mathfrak{f} | \Phi_m^*(\mathfrak{q}) \rangle_{\mathcal{L}^2(\mathbb{R}, \mathbb{H})} &= \langle \Phi_m(\mathfrak{f}) | \mathfrak{q} \rangle_{\ell^2_{\mathbb{H}}} \\
&= \Big\langle \sum_{n \in \mathbb{Z}} \mathfrak{e}_n \langle \mathfrak{g}_{m,n} | \mathfrak{f} \rangle \Big| \mathfrak{q} \Big\rangle_{\ell^2_{\mathbb{H}}} \\
&= \sum_{n \in \mathbb{Z}} \langle \mathfrak{f} | \mathfrak{g}_{m,n} \rangle \langle \mathfrak{e}_n | \mathfrak{q} \rangle \\
&= \Big\langle \mathfrak{f} \Big| \sum_{n \in \mathbb{Z}} \mathfrak{g}_{m,n} \mathfrak{q}_n \Big\rangle.
\end{aligned}$$

Thus, we obtain $\Phi_m^*(\mathfrak{q}) = \sum\limits_{n \in \mathbb{Z}} \mathfrak{g}_{m,n} \mathfrak{q}_n$, $\mathfrak{q} = \{\mathfrak{q}_n\}_{n \in \mathbb{Z}} \in \ell^2_{\mathbb{H}}$. $\square$

We now give a characterization for Bessel systems in the space $\mathcal{L}^2(\mathbb{R}, \mathbb{H})$ with the help of an operator.

Theorem 3.4. *Let $\mathcal{G}(\mathfrak{g},\alpha,\beta)$ be a Gabor system for $\mathcal{L}^2(\mathbb{R},\mathbb{H})$. Then, $\mathcal{G}(\mathfrak{g},\alpha,\beta)$ is a Bessel system for $\mathcal{L}^2(\mathbb{R},\mathbb{H})$ with Bessel bound r if and only if there exists a bounded right-linear operator $\mathcal{U}$ from $\ell^2_{\mathbb{H}}\otimes\ell^2_{\mathbb{H}}$ into $\mathcal{L}^2(\mathbb{R},\mathbb{H})$ with $\|\mathcal{U}\|\le\sqrt{r}$ such that*

$$\mathcal{U}(\mathfrak{p}\otimes\mathfrak{q}) \quad = \quad \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\mathfrak{g}_{m,n}(\mathfrak{p}_m\mathfrak{q}_n), \quad \mathfrak{p}=\{\mathfrak{p}_n\}_{n\in\mathbb{Z}},\mathfrak{q}=\{\mathfrak{q}_n\}_{n\in\mathbb{Z}}\in\ell^2_{\mathbb{H}}.$$

Proof. First suppose that $\mathcal{G}(\mathfrak{g},\alpha,\beta)$ is a Bessel system for $\mathcal{L}^2(\mathbb{R},\mathbb{H})$ with Bessel bound r. Define $\mathcal{V}:\mathcal{L}^2(\mathbb{R},\mathbb{H})\to\ell^2_{\mathbb{H}}\otimes\ell^2_{\mathbb{H}}$ as

$$\begin{aligned}\mathcal{V}(\mathfrak{f}) &= \sum_{m\in\mathbb{Z}}|\mathfrak{e}_m\rangle_{\ell^2_{\mathbb{H}}}\Phi_m(\mathfrak{f})\\ &= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\mathfrak{e}_m\otimes\mathfrak{e}_n\langle\mathfrak{g}_{m,n}|\mathfrak{f}\rangle,\ \ \mathfrak{f}\in\mathcal{L}^2(\mathbb{R},\mathbb{H})\end{aligned}$$

where Φ_m is as defined in (3.2). Then

$$\|\mathcal{V}(\mathfrak{f})\|^2 \quad = \quad \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}|\langle\mathfrak{g}_{m,n}|\mathfrak{f}\rangle|^2\le r\|\mathfrak{f}\|^2.$$

Take $\mathcal{U}=\mathcal{V}^*$. Then

$$\mathcal{U}(\mathfrak{p}\otimes\mathfrak{q})=\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\mathfrak{g}_{m,n}(\mathfrak{p}_m\mathfrak{q}_n), \quad \mathfrak{p}=\{\mathfrak{p}_n\}_{n\in\mathbb{Z}},\mathfrak{q}=\{\mathfrak{q}_n\}_{n\in\mathbb{Z}}\in\ell^2_{\mathbb{H}}.$$

Since $\mathcal{V}$ is bounded, so is $\mathcal{U}$ with $\|\mathcal{U}\|\le\sqrt{r}$.

Conversely, let $\mathcal{U}:\ell^2_{\mathbb{H}}\otimes\ell^2_{\mathbb{H}}\to\mathcal{L}^2(\mathbb{R},\mathbb{H})$ be a bounded right-linear operator with $\|\mathcal{U}\|\le\sqrt{r}$. Then, $\mathcal{U}^*$ is also bounded with the same norm. As $\mathcal{U}^*(\mathfrak{f})=\sum\limits_{m\in\mathbb{Z}}\sum\limits_{n\in\mathbb{Z}}\mathfrak{e}_m\otimes\mathfrak{e}_n\langle\mathfrak{g}_{m,n}|\mathfrak{f}\rangle$, we get

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}|\langle\mathfrak{g}_{m,n}|\mathfrak{f}\rangle|^2\le r\|\mathfrak{f}\|^2,\ \ \mathfrak{f}\in\mathcal{L}^2(\mathbb{R},\mathbb{H}).$$

Thus, $\mathcal{G}(\mathfrak{g},\alpha,\beta)$ is a Bessel system for $\mathcal{L}^2(\mathbb{R},\mathbb{H})$ with Bessel bound r. □

In the following result, we give a characterization for Gabor frames in $\mathcal{L}^2(\mathbb{R},\mathbb{H})$ as an extention of Theorem 3.4.

Theorem 3.5. *Let $\mathcal{G}(\mathfrak{g},\alpha,\beta)$ be a Gabor system for $\mathcal{L}^2(\mathbb{R},\mathbb{H})$. Then, $\mathcal{G}(\mathfrak{g},\alpha,\beta)$ is a Gabor frame for $\mathcal{L}^2(\mathbb{R},\mathbb{H})$ if and only if there exists a bounded right-linear operator $\mathcal{U}$ from $\ell^2_{\mathbb{H}}\otimes\ell^2_{\mathbb{H}}$ onto $\mathcal{L}^2(\mathbb{R},\mathbb{H})$ with $\|\mathcal{U}\|\le\sqrt{r}$ such that*

$$\mathcal{U}(\mathfrak{p}\otimes\mathfrak{q}) \quad = \quad \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\mathfrak{g}_{m,n}(\mathfrak{p}_m\mathfrak{q}_n), \quad \mathfrak{p}=\{\mathfrak{p}_n\}_{n\in\mathbb{Z}},\mathfrak{q}=\{\mathfrak{q}_n\}_{n\in\mathbb{Z}}\in\ell^2_{\mathbb{H}}. \tag{3.3}$$

Proof. Firstly, let $\mathcal{G}(\mathfrak{g},\alpha,\beta)$ be a Gabor frame for $\mathcal{L}^2(\mathbb{R},\mathbb{H})$. Then by Theorem 3.4, there exists a bounded right-linear operator $\mathcal{U}$ from $\ell^2_{\mathbb{H}}\otimes\ell^2_{\mathbb{H}}$ to $\mathcal{L}^2(\mathbb{R},\mathbb{H})$ satisfying (3.3). Observe that its adjoint $\mathcal{U}^*:\mathcal{L}^2(\mathbb{R},\mathbb{H})\to\ell^2_{\mathbb{H}}\otimes\ell^2_{\mathbb{H}}$ given by

$$\mathcal{U}^*(\mathfrak{f})=\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\mathfrak{e}_m\otimes\mathfrak{e}_n\langle\mathfrak{g}_{m,n}|\mathfrak{f}\rangle,\ \ \mathfrak{f}\in\mathcal{L}^2(\mathbb{R},\mathbb{H}),$$

is an injective operator with closed range. Hence, $\mathcal{U}$ is a surjective operator.

Conversely since $\mathcal{U}$ is surjective, $\mathcal{U}^*$ is injective and hence there exists a positive constant r_1 such that

$$r_1\|\mathfrak{f}\|^2 \leq \|\mathcal{U}^*(\mathfrak{f})\|^2 = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |\langle \mathfrak{g}_{m,n}|\mathfrak{f}\rangle|^2,\ \mathfrak{f} \in \mathcal{L}^2(\mathbb{R},\mathbb{H}).$$

Thus, again by Theorem 3.4, $\mathcal{G}(\mathfrak{g},\alpha,\beta)$ is a Gabor frame for $\mathcal{L}^2(\mathbb{R},\mathbb{H})$. □

3.1. Analysis and Synthesis operators. We now define the analysis and synthesis operators for the Bessel system $\mathcal{G}(\mathfrak{g},\alpha,\beta)$. The operator $\Phi_\mathcal{G} : \mathcal{L}^2(\mathbb{R},\mathbb{H}) \to \ell^2_\mathbb{H}\otimes\ell^2_\mathbb{H}$ defined by

$$\Phi_\mathcal{G}(\mathfrak{f}) \quad = \quad \sum_{m\in\mathbb{Z}} |\mathfrak{e}_m\rangle_{\ell^2_\mathbb{H}}\Phi_m(\mathfrak{f}) = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \mathfrak{e}_m\otimes\mathfrak{e}_n\langle\mathfrak{g}_{m,n}|\mathfrak{f}\rangle,\ \mathfrak{f}\in\mathcal{L}^2(\mathbb{R},\mathbb{H})$$

is called the *analysis operator* and the adjoint $\Phi^*_\mathcal{G} : \ell^2_\mathbb{H}\otimes\ell^2_\mathbb{H} \to \mathcal{L}^2(\mathbb{R},\mathbb{H})$ of $\Phi_\mathcal{G}$ given by

$$\Phi^*_\mathcal{G}(\mathfrak{p}\otimes\mathfrak{q}) \quad = \quad \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \mathfrak{g}_{m,n}(\mathfrak{p}_m\mathfrak{q}_n),\quad \mathfrak{p} = \{\mathfrak{p}_n\}_{n\in\mathbb{Z}}, \mathfrak{q} = \{\mathfrak{q}_n\}_{n\in\mathbb{Z}} \in \ell^2_\mathbb{H} \qquad (3.4)$$

is called the *synthesis operator* for $\mathcal{G}(\mathfrak{g},\alpha,\beta)$.

By composing the analysis operator $\Phi_\mathcal{G}$ and the synthesis operator $\Phi^*_\mathcal{G}$, we obtain the *Gabor frame operator* $\mathcal{S}_\mathcal{G} : \mathcal{L}^2(\mathbb{R},\mathbb{H}) \to \mathcal{L}^2(\mathbb{R},\mathbb{H})$ given by

$$\mathcal{S}_\mathcal{G}(\mathfrak{f}) = \Phi^*_\mathcal{G}\Phi_\mathcal{G}(\mathfrak{f}) = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \mathfrak{g}_{m,n}\langle\mathfrak{g}_{m,n}|\mathfrak{f}\rangle,\quad \mathfrak{f}\in\mathcal{L}^2(\mathbb{R},\mathbb{H}).$$

Remark 3.6. It can be easily verified that a Gabor system $\mathcal{G}(\mathfrak{g},\alpha,\beta)$ is a Parseval Gabor frame if and only if the analysis operator is an isometry. Indeed

$$\|\Phi_\mathcal{G}(\mathfrak{f})\|^2 = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |\langle\mathfrak{g}_{m,n}|\mathfrak{f}\rangle|^2 = \langle\mathcal{S}_\mathcal{G}(\mathfrak{f})|\mathfrak{f}\rangle,\ \mathfrak{f}\in\mathcal{L}^2(\mathbb{R},\mathbb{H})$$

and in view of the Gabor frame inequality the analysis operator of the Gabor frame $\mathcal{G}(\mathfrak{g},\alpha,\beta)$ is an injective operator.

Nextly, we prove some important properties of the Gabor frame operator.

Theorem 3.7. *Let $\mathcal{G}(\mathfrak{g},\alpha,\beta)$ be a Gabor frame for $\mathcal{L}^2(\mathbb{R},\mathbb{H})$ with the Gabor frame operator $\mathcal{S}_\mathcal{G}$. Then, $\mathcal{S}_\mathcal{G}$ is a bounded, right-linear, positive and invertible operator.*

Proof. $\mathcal{S}_\mathcal{G}$ is a bounded right-linear operator as a composition of two bounded right-linear operators. Also

$$0 \leq \|\Phi_\mathcal{G}(\mathfrak{f})\|^2 = \langle\mathcal{S}_\mathcal{G}(\mathfrak{f})|\mathfrak{f}\rangle,\ \mathfrak{f}\in\mathcal{L}^2(\mathbb{R},\mathbb{H}).$$

Therefore, $\mathcal{S}_\mathcal{G}$ is positive. Now from the Gabor frame inequality (3.1), we have $r_1\|\mathfrak{f}\|^2 \leq \langle\mathcal{S}_\mathcal{G}(\mathfrak{f})|\mathfrak{f}\rangle \leq r_2\|\mathfrak{f}\|^2$, $\mathfrak{f}\in\mathcal{L}^2(\mathbb{R},\mathbb{H})$ or $Ir_1 \leq \mathcal{S}_\mathcal{G} \leq Ir_2$. This gives

$$0 \leq I - \mathcal{S}_\mathcal{G}r_2^{-1} \leq I\Big(\frac{r_2-r_1}{r_2}\Big),$$

and hence

$$\|I - \mathcal{S}_\mathcal{G}r_2^{-1}\| = \sup_{\|\mathfrak{f}\|=1} |\langle I - \mathcal{S}_\mathcal{G}r_2^{-1}(\mathfrak{f})|\mathfrak{f}\rangle| \leq \frac{r_2-r_1}{r_2} < 1.$$

Thus, by (Proposition 3.6 [13]) $\mathcal{S}_\mathcal{G}$ is an invertible operator. □

Now, we define a matrix valued function with the help of a fixed function of $\mathcal{L}^2(\mathbb{R},\mathbb{H})$ that will be used in the next result.
Let $\mathfrak{g}\in\mathcal{L}^2(\mathbb{R},\mathbb{H})$ be a non-zero function, $\alpha,\beta>0$ and $P(x)=(P_{m,n}(x))_{m,n\in\mathbb{Z}},\ x\in\mathbb{R}$ be a matrix valued function whose entry in the m^{th}-row and n^{th}-column is defined as

$$P_{m,n}(x):=\mathfrak{g}\Big(x-\alpha n-\frac{m}{\beta}\Big),\ m,n\in\mathbb{Z},\ x\in\mathbb{R}.$$

One may verify that $P(x)$ is a bi-infinite matrix for almost all $x\in\mathbb{R}$. Let $P(x)^*$ be the conjugate transpose of $P(x)$. For each $x\in\mathbb{R}$, define the matrix

$$\Omega(x)=(\Omega_{m,n}(x))_{m,n\in\mathbb{Z}}:=P(x)^*P(x), \tag{3.5}$$

where

$$\Omega_{m,n}(x)=\sum_{r\in\mathbb{Z}}\overline{\mathfrak{g}\Big(x-\alpha m-\frac{r}{\beta}\Big)}\mathfrak{g}\Big(x-\alpha n-\frac{r}{\beta}\Big),\ m,n\in\mathbb{Z},x\in\mathbb{R}.$$

Since $\mathcal{T}_{\alpha m}\overline{\mathfrak{g}},\mathcal{T}_{\alpha n}\mathfrak{g}\in\mathcal{L}^2(\mathbb{R},\mathbb{H})$, by Cauchy-Schwartz inequality, $\mathcal{T}_{\alpha m}\overline{\mathfrak{g}}.\mathcal{T}_{\alpha n}\mathfrak{g}\in\mathcal{L}^1(\mathbb{R},\mathbb{H})$. Now

$$\begin{aligned}\int_{[0,\frac{1}{\beta}]}|\Omega_{m,n}(x)|dx &\leq \int_{[0,\frac{1}{\beta}]}\sum_{r\in\mathbb{Z}}\Big|\overline{\mathfrak{g}\Big(x-\alpha m-\frac{r}{\beta}\Big)}\mathfrak{g}\Big(x-\alpha n-\frac{r}{\beta}\Big)\Big|dx\\ &= \int_{\mathbb{R}}|\overline{\mathfrak{g}(x-\alpha m)}\mathfrak{g}(x-\alpha n)|dx<\infty. \end{aligned}\tag{3.6}$$

Thus, the series $\Omega_{m,n}(x)$ is absolutely convergent for almost all $x\in[0,\frac{1}{\beta}]$. Since $x\mapsto\Omega_{m,n}(x)$ is a $\frac{1}{\beta}\mathbb{Z}$-periodic function, we may conclude that $\Omega_{m,n}(x)$ is absolutely convergent for almost all $x\in\mathbb{R}$. Further for each $x\in\mathbb{R}$, the operator $\Omega(x)$ acts on the finite sequence $\mathfrak{q}=\{\mathfrak{q}_n\}_{n\in\mathbb{Z}}\in\ell^2_{\mathbb{H}}$ as

$$\Big(\Omega(x)\mathfrak{q}\Big)_m=\sum_{n\in\mathbb{Z}}\Omega_{m,n}(x)\mathfrak{q}_n=\sum_{n\in\mathbb{Z}}\sum_{r\in\mathbb{Z}}\overline{\mathfrak{g}\Big(x-\alpha m-\frac{r}{\beta}\Big)}\mathfrak{g}\Big(x-\alpha n-\frac{r}{\beta}\Big)\mathfrak{q}_n.$$

Also we compute

$$\begin{aligned}\langle\mathfrak{q}|\Omega(x)\mathfrak{q}\rangle &= \sum_{m\in\mathbb{Z}}\overline{\mathfrak{q}_m}\Big(\sum_{n\in\mathbb{Z}}\Omega_{m,n}(x)\mathfrak{q}_n\Big)\\ &= \sum_{m,n\in\mathbb{Z}}\overline{\mathfrak{q}_m}\sum_{r\in\mathbb{Z}}\overline{\mathfrak{g}\Big(x-\alpha m-\frac{r}{\beta}\Big)}\mathfrak{g}\Big(x-\alpha n-\frac{r}{\beta}\Big)\mathfrak{q}_n\\ &= \sum_{m,n\in\mathbb{Z}}\sum_{r\in\mathbb{Z}}\overline{\mathfrak{q}_m}\ \overline{\mathfrak{g}\Big(x-\alpha m-\frac{r}{\beta}\Big)}\mathfrak{g}\Big(x-\alpha n-\frac{r}{\beta}\Big)\mathfrak{q}_n\\ &= \sum_{r\in\mathbb{Z}}\Big[\Big(\sum_{m\in\mathbb{Z}}\overline{\mathfrak{q}_m}\ \overline{\mathfrak{g}\Big(x-\alpha m-\frac{r}{\beta}\Big)}\Big)\Big(\sum_{n\in\mathbb{Z}}\mathfrak{g}\Big(x-\alpha n-\frac{r}{\beta}\Big)\mathfrak{q}_n\Big)\Big]\\ &= \sum_{r\in\mathbb{Z}}\Big|\sum_{n\in\mathbb{Z}}\mathfrak{g}\Big(x-\alpha n-\frac{r}{\beta}\Big)\mathfrak{q}_n\Big|^2\geq 0. \end{aligned}$$

Thus for each $x\in\mathbb{R}$, $\Omega(x)$ is a positive operator in the sense that $\langle\mathfrak{q}|\Omega(x)\mathfrak{q}\rangle\geq 0$ for $\mathfrak{q}=\{\mathfrak{q}_n\}_{n\in\mathbb{Z}}\in\ell^2_{\mathbb{H}}$ with compact support .

In the next result, we prove the boundedness of the synthesis operator for the case when the window function belongs to the quaternionic Wiener space. For that, we give the following lemma which gives a quantitative estimate for the periodizations of functions and will be used in further results.

Lemma 3.8. *Let* $\mathfrak{g} \in \mathcal{W}(\mathbb{R}, \mathbb{H})$ *and* $\alpha > 0$. *Then*

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}} \sum_{n\in\mathbb{Z}} |\mathfrak{g}(x - \alpha n)| \leq \Big(\frac{1}{\alpha} + 1\Big) \|\mathfrak{g}\|_{\mathcal{W}}.$$

Proof. Proof follows on the similar lines as in [9], Lemma 6.1.2. □

Theorem 3.9. *Let* $\mathcal{G}(\mathfrak{g}, \alpha, \beta)$ *be a Gabor system for* $\mathcal{L}^2(\mathbb{R}, \mathbb{H})$ *where* $\mathfrak{g} \in \mathcal{W}(\mathbb{R}, \mathbb{H})$ *and* $\alpha, \beta > 0$. *Then, the synthesis operator* $\Phi^*_{\mathcal{G}} : \ell^2_{\mathbb{H}} \otimes \ell^2_{\mathbb{H}} \to \mathcal{L}^2(\mathbb{R}, \mathbb{H})$ *as defined in (3.4) is a bounded right-linear operator with operator norm* $\|\Phi^*_{\mathcal{G}}\|^2 \leq \Big(\frac{1}{\alpha} + 1\Big)\Big(\frac{1}{\beta} + 1\Big)\|\mathfrak{g}\|^2_{\mathcal{W}}$.

Proof. Let $\mathfrak{g} \in \mathcal{W}(\mathbb{R}, \mathbb{H})$, $\mathfrak{p} \otimes \mathfrak{q} \in \ell^2_{\mathbb{H}} \otimes \ell^2_{\mathbb{H}}$, where $\mathfrak{p} = \{\mathfrak{p}_n\}_{n\in\mathbb{Z}}$, $\mathfrak{q} = \{\mathfrak{q}_n\}_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{H}}$ have finite support. For each $n \in \mathbb{Z}$, define

$$\mathcal{K}_n(x) = \sum_{m\in\mathbb{Z}} e^{-2\pi j\beta mx}\mathfrak{p}_m\mathfrak{q}_n \ , \ x \in \mathbb{R}.$$

It is easy to verify that only finitely many $\mathcal{K}_n$'s are non-zero and each $\mathcal{K}_n$ is $\frac{1}{\beta}\mathbb{Z}$-periodic such that

$$\|\mathcal{K}_n\|^2_{\mathcal{L}^2\left([0,\frac{1}{\beta}],\mathbb{H}\right)} = \int_{[0,\frac{1}{\beta}]} |\mathcal{K}_n(x)|^2 dx. \tag{3.7}$$

Also,

$$\begin{aligned}
\|\mathcal{K}_n\|^2_{\mathcal{L}^2\left([0,\frac{1}{\beta}],\mathbb{H}\right)} &= \Big\langle \sum_{l\in\mathbb{Z}} e^{-2\pi j\beta l}\mathfrak{p}_l\mathfrak{q}_n \Big| \sum_{m\in\mathbb{Z}} e^{-2\pi j\beta m}\mathfrak{p}_m\mathfrak{q}_n \Big\rangle \\
&= \sum_{l,m\in\mathbb{Z}} \overline{\mathfrak{p}_l\mathfrak{q}_n} \langle e^{-2\pi j\beta l} | e^{-2\pi j\beta m}\rangle \mathfrak{p}_m\mathfrak{q}_n \\
&= \frac{1}{\beta} \sum_{m\in\mathbb{Z}} |\mathfrak{p}_m\mathfrak{q}_n|^2. \qquad (3.8)
\end{aligned}$$

From (3.7) and (3.8) we get,

$$\|\mathcal{K}_n\|^2_{\mathcal{L}^2\left([0,\frac{1}{\beta}],\mathbb{H}\right)} = \int_{[0,\frac{1}{\beta}]} |\mathcal{K}_n(x)|^2 dx = \frac{1}{\beta} \sum_{m\in\mathbb{Z}} |\mathfrak{p}_m\mathfrak{q}_n|^2.$$

Let $f_0 := \sum\limits_{n\in\mathbb{Z}} \mathcal{T}_{\alpha n}\mathfrak{g}.\mathcal{K}_n = \sum\limits_{m\in\mathbb{Z}} \sum\limits_{n\in\mathbb{Z}} E_{\beta m}\mathcal{T}_{\alpha n}\mathfrak{g}(\mathfrak{p}_m\mathfrak{q}_n) = \sum\limits_{m\in\mathbb{Z}} \sum\limits_{n\in\mathbb{Z}} \mathfrak{g}_{m,n}(\mathfrak{p}_m\mathfrak{q}_n) = \Phi^*_{\mathcal{G}}(\mathfrak{p} \otimes \mathfrak{q})$.
Since for $r \in \mathbb{Z}$, $\frac{r}{\beta} + [0, \frac{1}{\beta}]$ are disjoint except on the boundaries whose measure is zero,

therefore

$$\begin{aligned}
\|f_0\|^2 &= \sum_{l,n\in\mathbb{Z}} \int_{\mathbb{R}} \overline{\mathcal{K}_l(x)}\ \overline{\mathfrak{g}(x-\alpha l)}\mathfrak{g}(x-\alpha n)\mathcal{K}_n(x)dx \\
&= \sum_{l,n\in\mathbb{Z}} \sum_{r\in\mathbb{Z}} \int_{\frac{r}{\beta}+[0,\frac{1}{\beta}]} \overline{\mathcal{K}_l(x)}\ \overline{\mathfrak{g}(x-\alpha l)}\mathfrak{g}(x-\alpha n)\mathcal{K}_n(x)dx \qquad (3.9)\\
&= \sum_{l,n\in\mathbb{Z}} \sum_{r\in\mathbb{Z}} \int_{[0,\frac{1}{\beta}]} \overline{\mathcal{K}_l(x)}\ \overline{\mathfrak{g}\Big(x-\alpha l-\frac{r}{\beta}\Big)}\mathfrak{g}\Big(x-\alpha n-\frac{r}{\beta}\Big)\mathcal{K}_n(x)dx \\
&= \sum_{l,n\in\mathbb{Z}} \int_{[0,\frac{1}{\beta}]} \overline{\mathcal{K}_l(x)}\ \Big(\sum_{r\in\mathbb{Z}} \overline{\mathfrak{g}\Big(x-\alpha l-\frac{r}{\beta}\Big)}\mathfrak{g}\Big(x-\alpha n-\frac{r}{\beta}\Big)\Big)\mathcal{K}_n(x)dx.
\end{aligned}$$

Since $\mathcal{T}_{\alpha l}\bar{\mathfrak{g}}.\mathcal{T}_{\alpha n}\mathfrak{g} \in \mathcal{L}^1(\mathbb{R},\mathbb{H})$, thus its periodization belongs to $\mathcal{L}^1\big([0,\frac{1}{\beta}],\mathbb{H}\big)$ and hence everything is well-defined. Now, using Fubini's theorem [9] (Appendix A.13) and Lemma 3.8, we compute

$$\begin{aligned}
\sum_{l\in\mathbb{Z}} |\Omega_{l,n}(x)| &\leq \sum_{l\in\mathbb{Z}} \sum_{r\in\mathbb{Z}} \Big|\mathfrak{g}\Big(x-\alpha l-\frac{r}{\beta}\Big)\Big|\Big|\mathfrak{g}\Big(x-\alpha n-\frac{r}{\beta}\Big)\Big| \\
&= \sum_{r\in\mathbb{Z}} \Big(\sum_{l\in\mathbb{Z}} \Big|\mathfrak{g}\Big(x-\alpha l-\frac{r}{\beta}\Big)\Big|\Big)\Big|\mathfrak{g}\Big(x-\alpha n-\frac{r}{\beta}\Big)\Big| \\
&\leq \Big(\frac{1}{\alpha}+1\Big)\|\mathfrak{g}\|_{\mathcal{W}} \sum_{r\in\mathbb{Z}} \Big|\mathfrak{g}\Big(x-\alpha n-\frac{r}{\beta}\Big)\Big| \\
&\leq \Big(\frac{1}{\alpha}+1\Big)(1+\beta)\|\mathfrak{g}\|_{\mathcal{W}}^2 := M(\alpha,\beta,\mathfrak{g}).
\end{aligned}$$

Since $\Omega_{l,n}(x) = \overline{\Omega_{n,l}(x)}$, row sum can also be estimated. Therefore by Schur's test [9], the matrix $\Omega(x)$ in (3.5) defines a bounded operator on $\ell^2_{\mathbb{H}}$ such that

$$\|\Omega(x)\| \leq \Big(\frac{1}{\alpha}+1\Big)(1+\beta)\|\mathfrak{g}\|_{\mathcal{W}}^2 = M(\alpha,\beta,\mathfrak{g}),\ \ x\in\mathbb{R}.$$

Thus, for almost all $x \in \mathbb{R}$

$$0 \leq \sum_{l,n\in\mathbb{Z}} \overline{\mathcal{K}_l(x)}\Omega_{l,n}(x)\mathcal{K}_n(x) = \Big\langle \{\mathcal{K}_l(x)\}_{l\in\mathbb{Z}}\Big|\Omega(x)\{\mathcal{K}_n(x)\}_{n\in\mathbb{Z}}\Big\rangle \leq M(\alpha,\beta,\mathfrak{g}) \sum_{n\in\mathbb{Z}} |\mathcal{K}_n(x)|^2.$$

Therefore, we have

$$\begin{aligned}
\|f_0\|^2 &= \sum_{l,n\in\mathbb{Z}} \int_{[0,\frac{1}{\beta}]} \overline{\mathcal{K}_l(x)} \Big(\sum_{r\in\mathbb{Z}} \overline{\mathfrak{g}\Big(x-\alpha l-\frac{r}{\beta}\Big)} \mathfrak{g}\Big(x-\alpha n-\frac{r}{\beta}\Big)\Big)\mathcal{K}_n(x)dx \\
&= \int_{[0,\frac{1}{\beta}]} \sum_{l,n\in\mathbb{Z}} \overline{\mathcal{K}_l(x)}\ \Omega_{l,n}(x)\mathcal{K}_n(x)dx \qquad (3.10)\\
&\le M(\alpha,\beta,\mathfrak{g})\sum_{n\in\mathbb{Z}}\int_{[0,\frac{1}{\beta}]} |\mathcal{K}_n(x)|^2dx \qquad (3.11)\\
&= \Big(\frac{1}{\alpha}+1\Big)(1+\beta)\|\mathfrak{g}\|^2_{\mathcal{W}}\frac{1}{\beta}\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}|\mathfrak{p}_m\mathfrak{q}_n|^2\\
&= \Big(\frac{1}{\alpha}+1\Big)\Big(\frac{1}{\beta}+1\Big)\|\mathfrak{g}\|^2_{\mathcal{W}}\Big(\sum_{m\in\mathbb{Z}}|\mathfrak{p}_m|^2\Big)\Big(\sum_{n\in\mathbb{Z}}|\mathfrak{q}_n|^2\Big)\\
&= \Big(\frac{1}{\alpha}+1\Big)\Big(\frac{1}{\beta}+1\Big)\|\mathfrak{g}\|^2_{\mathcal{W}}\|\mathfrak{p}\otimes\mathfrak{q}\|^2.
\end{aligned}$$

Hence $\|\Phi^*_{\mathcal{G}}\|^2 \le \Big(\frac{1}{\alpha}+1\Big)\Big(\frac{1}{\beta}+1\Big)\|\mathfrak{g}\|^2_{\mathcal{W}}$. □

Corollary 3.10. Let $\mathcal{G}(\mathfrak{g},\alpha,\beta)$ be a Gabor system for $\mathcal{L}^2(\mathbb{R},\mathbb{H})$ where $\mathfrak{g}\in\mathcal{W}(\mathbb{R},\mathbb{H})$ and $\alpha,\beta>0$. Then, $\mathcal{G}(\mathfrak{g},\alpha,\beta)$ forms a Bessel system for $\mathcal{L}^2(\mathbb{R},\mathbb{H})$ with Bessel bound $\Big(\frac{1}{\alpha}+1\Big)\Big(\frac{1}{\beta}+1\Big)\|\mathfrak{g}\|^2_{\mathcal{W}}$.

Proof. Follows from Theorem 3.4 and Theorem 3.9. □

Example 3.11. Let $\mathfrak{g}:\mathbb{R}\to\mathbb{H}$ be defined as

$$\mathfrak{g}(x)=\begin{cases} 1+jx+kx^2 & 0\le x\le 1\\ 0 & elsewhere.\end{cases}$$

One may observe that $\mathfrak{g}\in\mathcal{W}(\mathbb{R},\mathbb{H})$ such that

$$\begin{aligned}
\|\mathfrak{g}\|_{\mathcal{W}}=\sum_{n\in\mathbb{Z}}\operatorname*{ess\,sup}_{x\in[0,1]}|\mathfrak{g}(x+n)| &= \operatorname*{ess\,sup}_{x\in[0,1]}|\mathfrak{g}(x)|+\operatorname*{ess\,sup}_{x\in[0,1]}|\mathfrak{g}(x+1)|+\operatorname*{ess\,sup}_{x\in[0,1]}|\mathfrak{g}(x-1)|\\
&= \operatorname*{ess\,sup}_{x\in[0,1]}(\sqrt{1+x^2+x^4})=\sqrt{3}.
\end{aligned}$$

Thus, for any $\alpha,\beta>0$, $\mathcal{G}(\mathfrak{g},\alpha,\beta)$ forms a Bessel system for $\mathcal{L}^2(\mathbb{R},\mathbb{H})$ with Bessel bound $3\Big(\frac{1}{\alpha}+1\Big)\Big(\frac{1}{\beta}+1\Big)$.

Corollary 3.12. Let $\mathcal{G}(\mathfrak{g},\alpha,\beta)$ be a Gabor system for $\mathcal{L}^2(\mathbb{R},\mathbb{H})$ where $\mathfrak{g}\in\mathcal{W}(\mathbb{R},\mathbb{H})$ and $\alpha,\beta>0$. Then, the analysis operator $\Phi_{\mathcal{G}}$ and the Gabor frame operator $\mathcal{S}_{\mathcal{G}}=\Phi^*_{\mathcal{G}}\Phi_{\mathcal{G}}$ for $\mathcal{G}(\mathfrak{g},\alpha,\beta)$ are bounded operators with norms

$$\|\Phi_{\mathcal{G}}\|^2\le\Big(\frac{1}{\alpha}+1\Big)\Big(\frac{1}{\beta}+1\Big)\|\mathfrak{g}\|^2_{\mathcal{W}}\quad\text{and}\quad\|\mathcal{S}_{\mathcal{G}}\|^2\le\Big(\frac{1}{\alpha}+1\Big)^2\Big(\frac{1}{\beta}+1\Big)^2\|\mathfrak{g}\|^4_{\mathcal{W}}.$$

Proof. Straight forward. □

Corollary 3.13. Let $\mathcal{G}(\mathfrak{g},\alpha,\beta)$ be a Gabor system for $\mathcal{L}^2(\mathbb{R},\mathbb{H})$, where $\mathfrak{g}\in\mathcal{W}(\mathbb{R},\mathbb{H})$ and $\alpha,\beta>0$. Then for each $m\in\mathbb{Z}$ and $\mathfrak{f}\in\mathcal{L}^2(\mathbb{R},\mathbb{H})$,$\{\langle\mathfrak{g}_{m,n}|\mathfrak{f}\rangle\}_{n\in\mathbb{Z}}\in\ell^2_{\mathbb{H}}$ and hence $\{\langle\mathfrak{g}_{m,n}|\mathfrak{f}\rangle\}_{m\in\mathbb{Z}}\otimes\{\langle\mathfrak{g}_{m,n}|\mathfrak{f}\rangle\}_{n\in\mathbb{Z}}\in\ell^2_{\mathbb{H}}\otimes\ell^2_{\mathbb{H}}$.

Proof. We know that

$$\sum_{n\in\mathbb{Z}}|\langle\mathfrak{g}_{m,n}|\mathfrak{f}\rangle|^2 \leq \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}|\langle\mathfrak{g}_{m,n}|\mathfrak{f}\rangle|^2 = \|\Phi_{\mathcal{G}}(\mathfrak{f})\|^2 \leq \Big(\frac{1}{\alpha}+1\Big)\Big(\frac{1}{\beta}+1\Big)\|\mathfrak{g}\|_{\mathcal{W}}^2\|\mathfrak{f}\|^2,$$

and hence we get the desired result. □

Next, we give a necessary and sufficient condition for the boundedness of the synthesis operator.

Theorem 3.14. *Let $\mathcal{G}(\mathfrak{g},\alpha,\beta)$ be a Gabor system for $\mathcal{L}^2(\mathbb{R},\mathbb{H})$. Then, the operator $\Phi_{\mathcal{G}}^*:\ell_{\mathbb{H}}^2\otimes\ell_{\mathbb{H}}^2\to\mathcal{L}^2(\mathbb{R},\mathbb{H})$ as defined in (3.4) is bounded if and only if*

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}}\|\Omega(x)\| \;<\; \infty, \tag{3.12}$$

where $\Omega(x)$ is as defined in (3.5).

Proof. First suppose that $\operatorname*{ess\,sup}_{x\in\mathbb{R}}\|\Omega(x)\| \;<\; \infty$. Thus, there exists a constant $C>0$ such that $\|\Omega(x)\| \;<\; C$, for almost all $x\in\mathbb{R}$. Let $\mathfrak{p}=\{\mathfrak{p}_n\}_{n\in\mathbb{Z}},\mathfrak{q}=\{\mathfrak{q}_n\}_{n\in\mathbb{Z}}\in\ell_{\mathbb{H}}^2$ have finite support. Now, as in Theorem 3.9, we may define for each $n\in\mathbb{Z}$, $\mathcal{K}_n(x)=\sum_{m\in\mathbb{Z}}e^{-2\pi j\beta mx}\mathfrak{p}_m\mathfrak{q}_n$ and $f_0=\sum_{n\in\mathbb{Z}}\mathcal{T}_{\alpha n}\mathfrak{g}.\mathcal{K}_n=\Phi_{\mathcal{G}}^*(\mathfrak{p}\otimes\mathfrak{q})$.
From (3.10) we conclude that

$$\|\Phi_{\mathcal{G}}^*(\mathfrak{p}\otimes\mathfrak{q})\|^2=\|f_0\|^2\leq\frac{C}{\beta}\|\mathfrak{p}\otimes\mathfrak{q}\|^2.$$

Conversely, let $\Phi_{\mathcal{G}}^*$ be unbounded. Then for every $N\geq 1$, there exist $\mathfrak{p}^N=\{\mathfrak{p}_n^N\}_{n\in\mathbb{Z}}$ and $\mathfrak{q}^N=\{\mathfrak{q}_n^N\}_{n\in\mathbb{Z}}\in\ell_{\mathbb{H}}^2$ with finite support such that

$$\|\Phi_{\mathcal{G}}^*(\mathfrak{p}^N\otimes\mathfrak{q}^N)\|^2 \;>\; N\|\mathfrak{p}^N\otimes\mathfrak{q}^N\|^2.$$

Define for $n\in\mathbb{Z}$,

$$\mathcal{K}_n^N(x)=\sum_{m\in\mathbb{Z}}e^{-2\pi j\beta mx}\mathfrak{p}_m^N\mathfrak{q}_n^N.$$

Thus, by (3.10) we get

$$\int_{[0,\frac{1}{\beta}]}\sum_{l,n\in\mathbb{Z}}\overline{\mathcal{K}_l^N(x)}\;\Omega_{l,n}(x)\mathcal{K}_n^N(x)dx>N\beta\sum_{n\in\mathbb{Z}}\int_{[0,\frac{1}{\beta}]}|\mathcal{K}_n^N(x)|^2dx.$$

Since integrands on the both sides are non-negative, there exists a set $A^{(N)}\subset[0,\frac{1}{\beta}]$ of positive measure such that

$$\sum_{l,n\in\mathbb{Z}}\overline{\mathcal{K}_l^N(x)}\;\Omega_{l,n}(x)\mathcal{K}_n^N(x)>N\beta\sum_{n\in\mathbb{Z}}|\mathcal{K}_n^N(x)|^2,\; x\in A^{(N)}.$$

This implies

$$\|\{\mathcal{K}_n^N(x)\}_{n\in\mathbb{Z}}\|\|\Omega(x)\{\mathcal{K}_n^N(x)\}_{n\in\mathbb{Z}}\|\geq\sum_{l,n\in\mathbb{Z}}\overline{\mathcal{K}_l^N(x)}\;\Omega_{l,n}(x)\mathcal{K}_n^N(x)>N\beta\sum_{n\in\mathbb{Z}}|\mathcal{K}_n^N(x)|^2.$$

Thus we have, $\|\Omega(x)\|>N\beta,\; x\in A^{(N)}$ which contradicts (3.12) and hence, we get the desired result. □

Corollary 3.15. Let $\mathcal{G}(\mathfrak{g},\alpha,\beta)$ be a Gabor system for $\mathcal{L}^2(\mathbb{R},\mathbb{H})$. Then, $\mathcal{G}(\mathfrak{g},\alpha,\beta)$ is a Bessel system for $\mathcal{L}^2(\mathbb{R},\mathbb{H})$ if and only if

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}} \|\Omega(x)\| \; < \; \infty,$$

where $\Omega(x)$ is as defined in (3.5).

Proof. Follows from Theorem 3.4 and Theorem 3.14. □

Theorem 3.16. *Let $\mathcal{G}(\mathfrak{g},\alpha,\beta)$ be a Gabor system for $\mathcal{L}^2(\mathbb{R},\mathbb{H})$, where $\mathfrak{g}\in\mathcal{W}(\mathbb{R},\mathbb{H})$ with $\alpha,\beta>0$ and*

$$A := \frac{1}{\beta}\inf_{x\in[0,\alpha]}\Big(\sum_{n\in\mathbb{Z}}|\mathfrak{g}(x-\alpha n)|^2 - \sum_{l\neq 0}\Big|\sum_{n\in\mathbb{Z}}\mathfrak{g}(x-\alpha n)\overline{\mathfrak{g}\Big(x-\alpha n-\frac{l}{\beta}\Big)}\Big|\Big) > 0.$$

Then, $\mathcal{G}(\mathfrak{g},\alpha,\beta)$ forms a frame for $\mathcal{L}^2(\mathbb{R},\mathbb{H})$ with lower and upper frame bounds A and B where $B := \Big(\frac{1}{\alpha}+1\Big)\Big(\frac{1}{\beta}+1\Big)\|\mathfrak{g}\|^2_{\mathcal{W}}$.

Proof. Let $\mathfrak{f}\in\mathcal{L}^2(\mathbb{R},\mathbb{H})$ be a continuous function with compact support and $F_n(x) := \sum_{l\in\mathbb{Z}}\overline{\mathfrak{g}(x-\alpha n-\frac{l}{\beta})\mathfrak{f}(x-\frac{l}{\beta})}$, $n\in\mathbb{Z}, x\in\mathbb{R}$. Note that the series on the right hand side converges for almost all $x\in\mathbb{R}$ due to periodicity and

$$\begin{aligned}\langle\mathfrak{g}_{m,n}|\mathfrak{f}\rangle = \int_{\mathbb{R}}\overline{E_{\beta m}T_{\alpha n}\mathfrak{g}(x)}\mathfrak{f}(x)dx &= \sum_{l\in\mathbb{Z}}\int_{[0,\frac{1}{\beta}]}e^{2\pi j\beta m x}\overline{\mathfrak{g}\Big(x-\alpha n-\frac{l}{\beta}\Big)}\mathfrak{f}\Big(x-\frac{l}{\beta}\Big)dx\\ &= \int_{[0,\frac{1}{\beta}]}e^{2\pi j\beta m x}F_n(x).\end{aligned}$$

Now,

$$\begin{aligned}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}|\langle\mathfrak{g}_{m,n}|\mathfrak{f}\rangle|^2 &= \frac{1}{\beta}\sum_{n\in\mathbb{Z}}\int_{[0,\frac{1}{\beta}]}|F_n(x)|^2dx\\ &= \frac{1}{\beta}\sum_{n\in\mathbb{Z}}\int_{[0,\frac{1}{\beta}]}\sum_{l\in\mathbb{Z}}\overline{\mathfrak{f}\Big(x-\frac{l}{\beta}\Big)}\mathfrak{g}\Big(x-\alpha n-\frac{l}{\beta}\Big)F_n(x)dx\\ &= \frac{1}{\beta}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}\overline{\mathfrak{f}(x)}\mathfrak{g}(x-\alpha n)F_n(x)dx\\ &= \frac{1}{\beta}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}\overline{\mathfrak{f}(x)}\mathfrak{g}(x-\alpha n)\sum_{l\in\mathbb{Z}}\overline{\mathfrak{g}\Big(x-\alpha n-\frac{l}{\beta}\Big)}\mathfrak{f}\Big(x-\frac{l}{\beta}\Big)dx\\ &= \frac{1}{\beta}\int_{\mathbb{R}}|\mathfrak{f}(x)|^2\sum_{n\in\mathbb{Z}}|\mathfrak{g}(x-\alpha n)|^2dx\\ &\quad+\frac{1}{\beta}\sum_{l\neq 0}\int_{\mathbb{R}}\overline{\mathfrak{f}(x)}\sum_{n\in\mathbb{Z}}\mathfrak{g}(x-\alpha n)\overline{\mathfrak{g}\Big(x-\alpha n-\frac{l}{\beta}\Big)}\mathfrak{f}\Big(x-\frac{l}{\beta}\Big)dx\\ &\geq \frac{1}{\beta}\int_{\mathbb{R}}|\mathfrak{f}(x)|^2\Big(\sum_{n\in\mathbb{Z}}|\mathfrak{g}(x-\alpha n)|^2-\sum_{l\neq 0}\Big|\sum_{n\in\mathbb{Z}}\mathfrak{g}(x-\alpha n)\overline{\mathfrak{g}\Big(x-\alpha n-\frac{l}{\beta}\Big)}\Big|\Big)dx\\ &\geq A\|\mathfrak{f}\|^2.\end{aligned}$$

Thus, by the above arguments and Corollary 3.10, $\mathcal{G}(\mathfrak{g},\alpha,\beta)$ forms a frame for $\mathcal{L}^2(\mathbb{R},\mathbb{H})$. $\square$

Example 3.17. Let $\alpha=\beta=1$ and $\mathfrak{g}$ be such that

$$\mathfrak{g}=\begin{cases} j(x+2) & 0<x\leq 1,\\ k(\frac{x+1}{2}) & 1<x\leq 2,\\ 0 & \textit{elsewhere}.\end{cases}$$

Consider for $x\in(0,1]$

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\mathfrak{g}(x-n)\overline{g(x-n-l)} &= \begin{cases} \mathfrak{g}(x)\overline{\mathfrak{g}(x+1)} & l=-1\\ |\mathfrak{g}(x)|^2+|\mathfrak{g}(x+1)|^2 & l=0\\ \mathfrak{g}(x+1)\overline{\mathfrak{g}(x)} & l=1\\ 0 & \textit{elsewhere}\end{cases}\\ &= \begin{cases} -i\frac{(x+2)^2}{2} & l=-1\\ \frac{5}{4}(x+2)^2 & l=0\\ i\frac{(x+2)^2}{2} & l=1\\ 0 & \textit{elsewhere}.\end{cases}\end{aligned}$$

Thus,

$$\sum_{n\in\mathbb{Z}}|\mathfrak{g}(x-n)|^2=\frac{5}{4}(x+2)^2,\ x\in(0,1].$$

Also

$$\sum_{l\neq 0}\Big|\sum_{n\in\mathbb{Z}}\mathfrak{g}(x-n)\overline{g(x-n-l)}\Big|=(x+2)^2,\ x\in(0,1],$$

and hence

$$\inf_{x\in(0,1]}\Big(\sum_{n\in\mathbb{Z}}|\mathfrak{g}(x-n)|^2-\sum_{l\neq 0}\Big|\sum_{n\in\mathbb{Z}}\mathfrak{g}(x-n)\overline{\mathfrak{g}\Big(x-n-\frac{l}{\beta}\Big)}\Big|\Big)=\inf_{x\in(0,1]}\frac{1}{4}(x+2)^2=1.$$

Moreover, $\mathfrak{g}\in\mathcal{W}(\mathbb{R},\mathbb{H})$ with

$$\|\mathfrak{g}\|_{\mathcal{W}}=\operatorname*{ess\,sup}_{x\in(0,1]}|\mathfrak{g}(x)|+\operatorname*{ess\,sup}_{x\in(0,1]}|\mathfrak{g}(x+1)|=\frac{9}{2}.$$

Therefore, by Corollary 3.10 and Theorem 3.16, $\mathcal{G}(\mathfrak{g},1,1)$ forms a frame for $\mathcal{L}^2(\mathbb{R},\mathbb{H})$ with lower and upper frame bounds 1 and 81 respectively.

4. Walnut's Representation for the Gabor Frame Operator

To give the Walnut's representation of the Gabor frame operator, we firstly need to define some correlation functions with the help of a given window function $\mathfrak{g}\in\mathcal{L}^2(\mathbb{R},\mathbb{H})$.

Definition 4.1. Let $\mathfrak{g}\in\mathcal{L}^2(\mathbb{R},\mathbb{H})$ and $\alpha,\ \beta>0$ be given. The *correlation functions* for $\mathfrak{g}$ are defined as

$$\mathring{G}_m(x)=\sum_{n\in\mathbb{Z}}\mathfrak{g}(x-\alpha n)\overline{\mathfrak{g}\Big(x-\alpha n-\frac{m}{\beta}\Big)}=\sum_{n\in\mathbb{Z}}(\mathfrak{g}.\mathcal{T}_{\frac{m}{\beta}}\bar{\mathfrak{g}})(x-\alpha n),\ x\in\mathbb{R},m\in\mathbb{Z}.$$

For each $m \in \mathbb{Z}$, $\mathring{G}_m$ is a periodization of $\mathfrak{g}.\mathcal{T}_{\frac{m}{\beta}}\bar{\mathfrak{g}}$ with period $\alpha\mathbb{Z}$ and hence in view of (3.6), we conclude that $\mathring{G}_m \in \mathcal{L}^1([0,\alpha],\mathbb{H})$.

In the following result we prove the boundedness of the correlation functions for the case when the window function $\mathfrak{g}$ belongs to the Wiener space.

Lemma 4.2. *Let $\mathfrak{g} \in \mathcal{W}(\mathbb{R},\mathbb{H})$ and $\alpha,\ \beta > 0$. Then for each $m \in \mathbb{Z}$, $\mathring{G}_m \in \mathcal{L}^\infty(\mathbb{R},\mathbb{H})$. Further,*

$$\sum_{m\in\mathbb{Z}} \|\mathring{G}_m\|_\infty \leq \Big(\frac{1}{\alpha}+1\Big)(2\beta+2)\|\mathfrak{g}\|^2_{\mathcal{W}}. \tag{4.1}$$

Proof. Since $\mathfrak{g} \in \mathcal{W}(\mathbb{R},\mathbb{H})$, for each $m \in \mathbb{Z}$ we have $\mathfrak{g}.\mathcal{T}_{\frac{m}{\beta}}\bar{\mathfrak{g}} \in \mathcal{W}(\mathbb{R},\mathbb{H})$. Now by applying Lemma 3.8 on $\mathfrak{g}.\mathcal{T}_{\frac{m}{\beta}}\bar{\mathfrak{g}}$, we get

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}} \sum_{n\in\mathbb{Z}} |(\mathfrak{g}.\mathcal{T}_{\frac{m}{\beta}}\bar{\mathfrak{g}})(x-\alpha n)| \leq \Big(\frac{1}{\alpha}+1\Big)\|\mathfrak{g}.\mathcal{T}_{\frac{m}{\beta}}\bar{\mathfrak{g}}\|_{\mathcal{W}}.$$

Thus,

$$\|\mathring{G}_m\|_\infty \leq \Big(\frac{1}{\alpha}+1\Big)\|\mathfrak{g}.\mathcal{T}_{\frac{m}{\beta}}\bar{\mathfrak{g}}\|_{\mathcal{W}},\ m\in\mathbb{Z}.$$

This gives

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} \|\mathring{G}_m\|_\infty &\leq \Big(\frac{1}{\alpha}+1\Big)\sum_{m\in\mathbb{Z}}\|\mathfrak{g}.\mathcal{T}_{\frac{m}{\beta}}\bar{\mathfrak{g}}\|_{\mathcal{W}} \\
&= \Big(\frac{1}{\alpha}+1\Big)\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\|(\mathfrak{g}.\mathcal{T}_n\chi_{[0,1]}).(\mathcal{T}_{\frac{m}{\beta}}\bar{\mathfrak{g}}.\mathcal{T}_n\chi_{[0,1]})\|_\infty \\
&\leq \Big(\frac{1}{\alpha}+1\Big)\sum_{n\in\mathbb{Z}}\|\mathfrak{g}.\mathcal{T}_n\chi_{[0,1]}\|_\infty\Big(\sum_{m\in\mathbb{Z}}\|\mathcal{T}_{\frac{m}{\beta}}\bar{\mathfrak{g}}.\mathcal{T}_n\chi_{[0,1]}\|_\infty\Big).
\end{aligned} \tag{4.2}$$

Also for each $m \in \mathbb{Z}$, we compute the inner sum

$$\|\mathcal{T}_{\frac{m}{\beta}}\bar{\mathfrak{g}}.\mathcal{T}_n\chi_{[0,1]}\|_\infty = \operatorname*{ess\,sup}_{x\in\mathbb{R}}|\mathcal{T}_{\frac{m}{\beta}}\bar{\mathfrak{g}}.\mathcal{T}_n\chi_{[0,1]}(x)| = \operatorname*{ess\,sup}_{x\in[0,1]+n-\frac{m}{\beta}}|\mathfrak{g}(x)| \leq \sum_{k\in A_m}\|\mathfrak{g}.\mathcal{T}_k\chi_{[0,1]}\|_\infty,$$

where $A_m = \{k \in \mathbb{Z} \mid [0,1]+n-\frac{m}{\beta} \cap [0,1]+k \neq \phi\}$. Since each $k \in \mathbb{Z}$ belongs to at most $(2\beta+2)$ of the A_m's, we get

$$\sum_{m\in\mathbb{Z}}\|\mathcal{T}_{\frac{m}{\beta}}\bar{\mathfrak{g}}.\mathcal{T}_n\chi_{[0,1]}\|_\infty \leq (2\beta+2)\sum_{k\in\mathbb{Z}}\|\mathfrak{g}.\mathcal{T}_k\chi_{[0,1]}\|_\infty = (2\beta+2)\|\mathfrak{g}\|_{\mathcal{W}}. \tag{4.3}$$

Combining (4.2) and (4.3) we obtain

$$\sum_{m\in\mathbb{Z}}\|\mathring{G}_m\|_\infty \leq \Big(\frac{1}{\alpha}+1\Big)(2\beta+2)\|\mathfrak{g}\|^2_{\mathcal{W}}. \qquad \square$$

We now give an important structure theorem, the Walnut's representation for the Gabor frame operator, that enables the Gabor frame operator to be expressible in terms of the correlation functions.

Theorem 4.3. *Let $\mathcal{G}(\mathfrak{g},\alpha,\beta)$ be a Gabor system for $\mathcal{L}^2(\mathbb{R},\mathbb{H})$ where $\mathfrak{g}\in\mathcal{W}(\mathbb{R},\mathbb{H})$ and $\alpha,\beta>0$. Then, the Gabor frame operator $\mathcal{S}_{\mathcal{G}}$ for $\mathcal{G}(\mathfrak{g},\alpha,\beta)$ can be expressed as*

$$\mathcal{S}_{\mathcal{G}}(\mathfrak{f})=\frac{1}{\beta}\sum_{m\in\mathbb{Z}}\mathring{G}_m.\mathcal{T}_{\frac{m}{\beta}}\mathfrak{f},\ \ \mathfrak{f}\in\mathcal{L}^2(\mathbb{R},\mathbb{H}).$$

Moreover, $\mathcal{S}_{\mathcal{G}}$ is bounded on $\mathcal{L}^2(\mathbb{R},\mathbb{H})$ with the operator norm

$$\|\mathcal{S}_{\mathcal{G}}\|\leq 2\Big(1+\frac{1}{\alpha}\Big)\Big(1+\frac{1}{\beta}\Big)\|\mathfrak{g}\|_{\mathcal{W}}^2.$$

Proof. For each $\mathfrak{f}\in\mathcal{L}^2(\mathbb{R},\mathbb{H})$, we have

$$\mathcal{S}_{\mathcal{G}}\mathfrak{f}(x)=\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\mathfrak{g}(x-\alpha n)e^{-2\pi j\beta mx}\langle\mathfrak{g}_{m,n}|\mathfrak{f}\rangle,\ \ x\in\mathbb{R}. \tag{4.4}$$

For each $n\in\mathbb{Z}$, define

$$\mathcal{K}_n(x)=\sum_{m\in\mathbb{Z}}\langle\mathfrak{f}|\mathfrak{g}_{m,n}\rangle e^{2\pi j\beta mx},\ \ x\in\mathbb{R}.$$

From Corollary 3.13 we know that, $\{\langle\mathfrak{f}|\mathfrak{g}_{m,n}\rangle\}_{m\in\mathbb{Z}}\in\ell^2_{\mathbb{H}},\ \mathfrak{f}\in\mathcal{L}^2(\mathbb{R},\mathbb{H})$. Therefore, each $\mathcal{K}_n$ is $\frac{1}{\beta}\mathbb{Z}$- periodic and belongs to $\mathcal{L}^2([0,\frac{1}{\beta}],\mathbb{H})$ with norm

$$\|\mathcal{K}_n\|^2_{\mathcal{L}^2([0,\frac{1}{\beta}],\mathbb{H})}=\frac{1}{\beta}\sum_{m\in\mathbb{Z}}|\langle\mathfrak{f}|\mathfrak{g}_{m,n}\rangle|^2.$$

Also,

$$\langle\mathfrak{f}|\mathfrak{g}_{m,n}\rangle=\int_{\mathbb{R}}\overline{\mathfrak{f}(x)}\mathfrak{g}(x-\alpha n)e^{-2\pi j\beta mx}dx=\int_{\mathbb{R}}(\bar{\mathfrak{f}}.\mathcal{T}_{\alpha n}\mathfrak{g})(x)e^{-2\pi j\beta mx}dx=(\widehat{\bar{\mathfrak{f}}.\mathcal{T}_{\alpha n}\mathfrak{g}})(\beta m).$$

By restricting the Poisson summation formula [11] on $\{0\}\times\mathbb{R}\simeq\mathbb{R}$ and applying on $\mathcal{K}_n$, we write

$$\mathcal{K}_n(x)=\frac{1}{\beta}\sum_{m\in\mathbb{Z}}(\bar{\mathfrak{f}}.\mathcal{T}_{\alpha n}\mathfrak{g})\Big(x-\frac{m}{\beta}\Big). \tag{4.5}$$

Let $\mathfrak{f}$ be bounded with compact support. Then, (4.5) is valid almost everywhere on $[0,\frac{1}{\beta}]$ and the series is absolutely convergent almost everywhere. From (4.4) and (4.5), we obtain

$$\begin{aligned}\mathcal{S}_{\mathcal{G}}\mathfrak{f}(x)&=\sum_{n\in\mathbb{Z}}\mathcal{T}_{\alpha n}\mathfrak{g}(x)\overline{\mathcal{K}_n(x)}\\&=\sum_{n\in\mathbb{Z}}\mathcal{T}_{\alpha n}\mathfrak{g}(x)\Big(\frac{1}{\beta}\sum_{m\in\mathbb{Z}}(\mathcal{T}_{\alpha n}\bar{\mathfrak{g}}.\mathfrak{f})\Big(x-\frac{m}{\beta}\Big)\Big),\ \ x\in\mathbb{R}.\end{aligned}$$

Since $\mathfrak{f}$ has compact support, for each $x\in\mathbb{R}$, the sum over m is finite and hence the order of summation can be interchanged. We obtain

$$\begin{aligned}\mathcal{S}_{\mathcal{G}}\mathfrak{f}(x)&=\frac{1}{\beta}\sum_{m\in\mathbb{Z}}\Big(\sum_{n\in\mathbb{Z}}\mathfrak{g}(x-\alpha n)\overline{\mathfrak{g}\Big(x-\alpha n-\frac{m}{\beta}\Big)}\Big)\mathfrak{f}\Big(x-\frac{m}{\beta}\Big)\\&=\frac{1}{\beta}\sum_{m\in\mathbb{Z}}\mathring{G}_m(x)\mathcal{T}_{\frac{m}{\beta}}\mathfrak{f}(x),\ \ x\in\mathbb{R}.\end{aligned}$$

The above equality extends to the whole $\mathcal{L}^2(\mathbb{R},\mathbb{H})$ by density and hence,

$$\mathcal{S}_{\mathcal{G}}(\mathfrak{f}) = \frac{1}{\beta}\sum_{m\in\mathbb{Z}} \mathring{G}_m.\mathcal{T}_{\frac{m}{\beta}}\mathfrak{f},\ \mathfrak{f}\in\mathcal{L}^2(\mathbb{R},\mathbb{H}). \tag{4.6}$$

Now from (4.1) and (4.6), we obtain

$$\begin{aligned}\|\mathcal{S}_{\mathcal{G}}(\mathfrak{f})\| &\le \frac{1}{\beta}\sum_{m\in\mathbb{Z}}\|\mathring{G}_m.\mathcal{T}_{\frac{m}{\beta}}\mathfrak{f}\|\\ &\le \frac{1}{\beta}\sum_{m\in\mathbb{Z}}\|\mathring{G}_m\|_\infty\|\mathcal{T}_{\frac{m}{\beta}}\mathfrak{f}\|_2\\ &= \frac{1}{\beta}\sum_{m\in\mathbb{Z}}\|\mathring{G}_m\|_\infty\|\mathfrak{f}\|_2\\ &\le 2\Big(1+\frac{1}{\alpha}\Big)\Big(1+\frac{1}{\beta}\Big)\|\mathfrak{g}\|^2_{\mathcal{W}}\|\mathfrak{f}\|_2,\ \mathfrak{f}\in\mathcal{L}^2(\mathbb{R},\mathbb{H})\end{aligned}$$

and hence we complete the proof. □

Next, we consider a sesquilinear form on $\mathcal{L}^2(\mathbb{R},\mathbb{H})\times\mathcal{L}^2(\mathbb{R},\mathbb{H})$, $(\mathfrak{h},\mathfrak{f})\to\langle\mathfrak{h}|\mathcal{S}_{\mathcal{G}}\mathfrak{f}\rangle$, $\mathfrak{f},\mathfrak{h}\in\mathcal{L}^2(\mathbb{R},\mathbb{H})$. A matrix representation of the Gabor frame operator can be derived using this sesquilinear form. Let $\mathfrak{g}\in\mathcal{W}(\mathbb{R},\mathbb{H})$ and $\alpha,\beta>0$. Define for each $l,m\in\mathbb{Z}$,

$$G_{l,m}(x) = \sum_{n\in\mathbb{Z}}\mathfrak{g}\Big(x-\alpha n-\frac{l}{\beta}\Big)\overline{\mathfrak{g}\Big(x-\alpha n-\frac{m}{\beta}\Big)},\ x\in\mathbb{R}.$$

One may observe that

$$G_{0,m}(x) = \mathring{G}_m(x) \ \text{ and } \ G_{l,l+m}(x) = \mathring{G}_m\Big(x-\frac{l}{\beta}\Big),\ x\in\mathbb{R}. \tag{4.7}$$

Now, we obtain a matrix representation of the Gabor frame operator using the above defined sesquilinear form.

Theorem 4.4. *Let $\mathcal{G}(\mathfrak{g},\alpha,\beta)$ be a Gabor system for $\mathcal{L}^2(\mathbb{R},\mathbb{H})$ with the Gabor frame operator $\mathcal{S}_{\mathcal{G}}$ where $\mathfrak{g}\in\mathcal{W}(\mathbb{R},\mathbb{H})$ and $\alpha,\beta>0$. Then for $\mathfrak{f},\mathfrak{h}\in\mathcal{L}^2(\mathbb{R},\mathbb{H})$,*

$$\langle\mathfrak{h}|\mathcal{S}_{\mathcal{G}}\mathfrak{f}\rangle = \sum_{l,m\in\mathbb{Z}}\frac{1}{\beta}\int_{[0,\frac{1}{\beta}]}\overline{\mathcal{T}_{\frac{l}{\beta}}\mathfrak{h}(x)}G_{l,m}(x)\mathcal{T}_{\frac{m}{\beta}}\mathfrak{f}(x)dx. \tag{4.8}$$

Proof. Let us first assume that $\mathfrak{f},\mathfrak{h}$ are bounded functions with compact support. Then for each $x\in\mathbb{R}$, the sequences $\{\mathcal{T}_{\frac{l}{\beta}}\mathfrak{f}(x)\big|l\in\mathbb{Z}\}$ and $\{\mathcal{T}_{\frac{l}{\beta}}\mathfrak{h}(x)\big|l\in\mathbb{Z}\}$ have finite support. From (4.6) and the periodization trick (as used in (3.9)) with period $\frac{1}{\beta}\mathbb{Z}$, we compute

$$\begin{aligned}\langle\mathfrak{h}|\mathcal{S}_{\mathcal{G}}\mathfrak{f}\rangle &= \frac{1}{\beta}\int_{\mathbb{R}}\overline{\mathfrak{h}(x)}\sum_{n\in\mathbb{Z}}\mathring{G}_n(x)\mathcal{T}_{\frac{n}{\beta}}\mathfrak{f}(x)dx\\ &= \frac{1}{\beta}\int_{[0,\frac{1}{\beta}]}\sum_{l\in\mathbb{Z}}\overline{\mathfrak{h}\Big(x-\frac{l}{\beta}\Big)}\sum_{n\in\mathbb{Z}}\mathring{G}_n\Big(x-\frac{l}{\beta}\Big)\mathfrak{f}\Big(x-\frac{l}{\beta}-\frac{n}{\beta}\Big)dx.\end{aligned}$$

Since $\mathfrak{f},\mathfrak{h}$ are bounded with compact support, the sum over n and l are finite and hence the order of summation and integration can be interchanged. Substituting $m=l+n$

and using (4.7) we obtain

$$\langle \mathfrak{h}|\mathcal{S}_{\mathcal{G}}\mathfrak{f}\rangle = \sum_{l,m\in\mathbb{Z}} \frac{1}{\beta}\int_{[0,\frac{1}{\beta}]} \overline{\mathcal{T}_{\frac{l}{\beta}}\mathfrak{h}(x)}G_{l,m}(x)\mathcal{T}_{\frac{m}{\beta}}\mathfrak{f}(x)dx. \tag{4.9}$$

For the extension of (4.9) to $\mathcal{L}^2(\mathbb{R},\mathbb{H})$, we consider the operator $G(x)$, $x\in\mathbb{R}$ acting on finite sequences $\mathfrak{q}=\{\mathfrak{q}_n\}_{n\in\mathbb{Z}}\in\ell^2_{\mathbb{H}}$ by matrix multiplication as

$$\Big(G(x)\mathfrak{q}\Big)_l = \sum_{m\in\mathbb{Z}} G_{l,m}(x)\mathfrak{q}_m. \tag{4.10}$$

Since $\mathfrak{g}\in\mathcal{W}(\mathbb{R},\mathbb{H})$, for each $x\in\mathbb{R}$ the operator $G(x)$ satisfies the condition of Schur's test. Indeed using Lemma 3.8, we obtain

$$\begin{aligned}\sum_{l\in\mathbb{Z}}|G_{l,m}(x)| &\le \sum_{l\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\Big|\mathfrak{g}\Big(x-\alpha n-\frac{l}{\beta}\Big)\Big|\Big|\overline{\mathfrak{g}\Big(x-\alpha n-\frac{m}{\beta}\Big)}\Big|\\ &\le \sum_{n\in\mathbb{Z}}\Big[\sum_{l\in\mathbb{Z}}\Big|\mathfrak{g}\Big(x-\alpha n-\frac{l}{\beta}\Big)\Big|\Big]\Big|\mathfrak{g}\Big(x-\alpha n-\frac{m}{\beta}\Big)\Big|\\ &\le \Big(\frac{1}{\alpha}+1\Big)(1+\beta)\|\mathfrak{g}\|^2_{\mathcal{W}}.\end{aligned}$$

A similar estimate holds for $\sum_{m\in\mathbb{Z}}|G_{l,m}(x)|$ and hence for each $x\in\mathbb{R}$, $G(x)$ defines a bounded operator on $\ell^2_{\mathbb{H}}$. Also if $\mathfrak{f}$ and $\mathfrak{h}\in\mathcal{L}^2(\mathbb{R},\mathbb{H})$, then both the sequences $\{\mathcal{T}_{\frac{l}{\beta}}\mathfrak{f}(x)\big|l\in\mathbb{Z}\}$ and $\{\mathcal{T}_{\frac{l}{\beta}}\mathfrak{h}(x)\big|l\in\mathbb{Z}\}$ belongs to $\ell^2_{\mathbb{H}}$ for almost all $x\in\mathbb{R}$. Therefore, the matrix representation holds for arbitrary $\mathfrak{f},\mathfrak{h}\in\mathcal{L}^2(\mathbb{R},\mathbb{H})$. □

Remark 4.5. One may observe that if $\mathfrak{g}\in\mathcal{L}^2(\mathbb{R},\mathbb{H})$ only, then $\mathcal{T}_{\frac{l}{\beta}}\mathfrak{g}.\mathcal{T}_{\frac{m}{\beta}}\bar{\mathfrak{g}}\in L^1(\mathbb{R},\mathbb{H})$ and hence $G_{l,m}\in\mathcal{L}^1([0,\alpha],\mathbb{H})$, $l,m\in\mathbb{Z}$. Therefore, (4.8) holds only for bounded functions with compact support.

Remark 4.6. For each $x\in\mathbb{R}$, the operator $G(x)$ defines a positive operator, i.e., $\langle\mathfrak{q}|G(x)\mathfrak{q}\rangle\ge 0$ for $\mathfrak{q}=\{\mathfrak{q}_n\}_{n\in\mathbb{Z}}\in\ell^2_{\mathbb{H}}$ with compact support. Indeed,

$$\begin{aligned}\langle\mathfrak{q}|G(x)\mathfrak{q}\rangle &= \sum_{l\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\overline{\mathfrak{q}_l}\,G_{l,m}(x)\mathfrak{q}_m\\ &= \sum_{l,m\in\mathbb{Z}}\overline{\mathfrak{q}_l}\sum_{n\in\mathbb{Z}}\mathfrak{g}\Big(x-\alpha n-\frac{l}{\beta}\Big)\overline{\mathfrak{g}\Big(x-\alpha n-\frac{m}{\beta}\Big)}\mathfrak{q}_m\\ &= \sum_{n\in\mathbb{Z}}\sum_{l,m\in\mathbb{Z}}\overline{\mathfrak{q}_l}\,\mathfrak{g}\Big(x-\alpha n-\frac{l}{\beta}\Big)\overline{\mathfrak{g}\Big(x-\alpha n-\frac{m}{\beta}\Big)}\mathfrak{q}_m\\ &= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\Big|\mathfrak{g}\Big(x-\alpha n-\frac{m}{\beta}\Big)\Big|^2|\mathfrak{q}_m|^2 \ \ge 0.\end{aligned}$$

In the next result, we give a characterization for the invertibility and boundedness of the Gabor frame operator, in terms of the above defined operator $G(x)$ for almost all $x\in\mathbb{R}$.

Theorem 4.7. *Let $\mathcal{G}(\mathfrak{g},\alpha,\beta)$ be a Gabor system for $\mathcal{L}^2(\mathbb{R},\mathbb{H})$ with the Gabor frame operator $\mathcal{S}_{\mathcal{G}}$ where $\mathfrak{g}\in\mathcal{W}(\mathbb{R},\mathbb{H})$. Then,*

(a) *$\mathcal{S}_{\mathcal{G}}$ is invertible over $\mathcal{L}^2(\mathbb{R},\mathbb{H})$ if and only if $G(x) \geq Ir$, for almost all $x \in \mathbb{R}$ for some positive constant r where $G(x)$ is as defined in (4.10).*

(b) *$\mathcal{S}_{\mathcal{G}}$ is a bounded operator on $\mathcal{L}^2(\mathbb{R},\mathbb{H})$ if and only if $G(x) \leq Ir$, for almost all $x \in \mathbb{R}$ for some positive constant r.*

Proof. (a) Let $G(x) \geq Ir$, for almost all $x \in \mathbb{R}$ for some positive constant r. Then, for every bounded function $\mathfrak{f}$ with compact support we have

$$\sum_{l\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \overline{\mathcal{T}_{\frac{l}{\beta}}\mathfrak{f}(x)}G_{l,m}(x)\mathcal{T}_{\frac{m}{\beta}}\mathfrak{f}(x) \geq r\sum_{l\in\mathbb{Z}}|\mathcal{T}_{\frac{l}{\beta}}\mathfrak{f}(x)|^2.$$

Integrating over $[0,\frac{1}{\beta}]$ and using (4.8) we obtain

$$\langle \mathfrak{f}|\mathcal{S}_{\mathcal{G}}\mathfrak{f}\rangle \geq \frac{r}{\beta}\int_{[0,\frac{1}{\beta}]}\sum_{l\in\mathbb{Z}}|\mathcal{T}_{\frac{l}{\beta}}\mathfrak{f}(x)|^2dx = \frac{r}{\beta}\|\mathfrak{f}\|^2.$$

By density, this extends to $\mathcal{L}^2(\mathbb{R},\mathbb{H})$ and hence $\mathcal{S}_{\mathcal{G}}$ is invertible over $\mathcal{L}^2(\mathbb{R},\mathbb{H})$.

Conversely, assume that $\mathcal{S}_{\mathcal{G}}$ is not invertible. Then, there exists a sequence of bounded functions $\{\mathfrak{f}_k\}_{k\in\mathbb{N}}$ with compact support such that

$$\langle \mathfrak{f}_k|\mathcal{S}_{\mathcal{G}}\mathfrak{f}_k\rangle < \frac{1}{k}\|\mathfrak{f}_k\|^2,\ k\in\mathbb{N}.$$

Using (4.8) we obtain

$$\sum_{l,m\in\mathbb{Z}}\frac{1}{\beta}\int_{[0,\frac{1}{\beta}]}\overline{\mathcal{T}_{\frac{l}{\beta}}\mathfrak{f}_k(x)}G_{l,m}(x)\mathcal{T}_{\frac{m}{\beta}}\mathfrak{f}_k(x)dx \quad < \quad \frac{1}{k}\int_{[0,\frac{1}{\beta}]}\sum_{l\in\mathbb{Z}}|\mathcal{T}_{\frac{l}{\beta}}\mathfrak{f}_k(x)|^2dx.$$

This implies

$$0 \quad < \quad \int_{[0,\frac{1}{\beta}]}\Big(\frac{1}{k}\sum_{l\in\mathbb{Z}}|\mathcal{T}_{\frac{l}{\beta}}\mathfrak{f}_k(x)|^2 - \frac{1}{\beta}\sum_{l,m\in\mathbb{Z}}\overline{\mathcal{T}_{\frac{l}{\beta}}\mathfrak{f}_k(x)}G_{l,m}(x)\mathcal{T}_{\frac{m}{\beta}}\mathfrak{f}_k(x)\Big)dx.$$

Thus, for each $k\in\mathbb{N}$, there exists a set $A_k \subset [0,\frac{1}{\beta}]$ of positive measure such that

$$\frac{1}{k}\sum_{l\in\mathbb{Z}}\Big|\mathfrak{f}_k\Big(x-\frac{l}{\beta}\Big)\Big|^2 - \frac{1}{\beta}\sum_{l,m\in\mathbb{Z}}\overline{\mathfrak{f}_k\Big(x-\frac{l}{\beta}\Big)}G_{l,m}(x)\mathfrak{f}_k\Big(x-\frac{m}{\beta}\Big) > 0,\ x\in A_k.$$

Therefore $\inf\limits_{\{\mathfrak{q}:\|\mathfrak{q}\|\leq 1\}}\langle\mathfrak{q}|G(x)\mathfrak{q}\rangle < \frac{\beta}{k}$, $x\in A_k$. Hence, $G(x)\geq Ir$ cannot hold for almost all $x\in\mathbb{R}$.

(b) Proof follows on the same lines. □

Acknowledgements

The research of the first author is supported by CSIR (File No: 09/045(1679)/2019-EMR-I).

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