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SOME PROPERTIES OF GENERALIZED FRAMES FOR OPERATORS

VARINDER KUMAR† AND SAPNA MALHOTRA

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Abstract. In this paper, we study some interesting properties of K-g-frames. We establish a condition on operator K that makes a K-g-frame a g-frame. Also, we obtain condition under which sum of two K-g-frames turns out to be a K-g-frame. Further, a relationship between dual of sum of two K-g-frames and sum of duals of these K-g-frames has been established. We have also obtained conditions on operators U and V such that$\{\Lambda_j U + \Gamma_j V\}_{j \in J}$ becomes a K-g-frame. Next, the notion of $2K$-frames has been extended to $2K$-g-frames and it was observed that every $2K$-g-frame turns out to be a K-g-frame but converse fails to hold. We establish a condition on operator K such that a K-g-frame becomes a $2K$-g-frame. We then prove a necessary condition under which the sum of a K-g-frame and $2K$-g-frame turns out to be a K-g-frame. Also, a result that relates the resolution of identity operator and $2K$-g-frames has been proved. Finally, we prove a characterization for a g-Bessel sequence to be a $2K$-g-frame.

1. Introduction

Duffin and Schaeffer [11] during their study of the Nonharmonic Fourier series introduced the concept of frames for Hilbert spaces in 1952. The concept of frames was reintroduced in the year 1986, by Daubechies, Grossman and Meyer [6]. As frames form a redundant system, thereby have abundant applications in pure and applied mathematics like filter bank theory, image processing, sigma-delta quantization, capacity of transmission channel, coding theory, data transmission techonology to name a few.

The notion of frame theory has been improved and generalized by many authors. The concept of g-Riesz bases and g-frames was coined by Sun [23]. While trying to generalize Banach frames Kaushik *et. al* [18, 19, 20] introduced the concept of fusion Banach frames in Banach spaces and studied various riveting properties of fusion Banach frames. To study the atomic decomposition systems in Hilbert spaces, L. Găvruţa [14] introduced

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†Corresponding author.

K-frames in Hilbert spaces. In [7, 12, 13, 15, 22, 25, 26] various characterization of K-frames are studied. With the deepening of research in K-frames and g -frames, Zhou *et. al* [27, 28] in 2013, introduced the notion of K-g-frames, which was restricted to the range of bounded linear operator K. It was observed that K-g-frames are generalization of g-frames that have better advantages in practical applications than g-frames. In recent years, K-g-frames have emerged as one of the centers of interest in frame theory. Several properties of K-g-frames have been studied in [1, 2, 10, 16, 17, 24].

In this paper, we study some properties of K-g-frames. Also, we discuss the conditions, under which the sum of two K-g-frames is again a K-g-frame. The concept of $2K$-frames was introduced in [21], which have the dynamism of being theoretically foundational as well as have practical applications. We extend the notion of $2K$-frames to $2K$-g-frames which possess the flexibility and some properties of $2K$-frames and K-g-frames. We examine the conditions under which the sum of a $2K$-g-frame and a K-g-frame is a K-g-frame. Besides this, we illustrate our assertions with the help of examples. Further, we discuss the relation between the resolution of identity operator and $2K$-g-frames. At last, we prove a characterization for a g-Bessel sequence to be a $2K$-g-frame.

Throughout this paper, H will denote a seperable Hilbert space and $B(H_1, H_2)$ denotes the collection of all bounded linear operators from H_1 to H_2, where H_1 and H_2 are seperable Hilbert spaces. In particular, $B(H)$ is the collection of all bounded linear operators from H to H. T^* denote the adjoint of the operator T. $\{H_j : j \in J\}$ is a sequence of closed subspaces of H, where J is a countable indexing set. The sequence space

$$l^2\left(\{H_j\}_{j\in J}\right) = \left\{ \{x_j\}_{j\in J} : x_j \in H_j, j \in J, \sum_{j\in J} \|x_j\|^2 < \infty \right\}$$

with the inner product given by $\langle \{x_j\}_{j\in J}, \{y_j\}_{j\in J}\rangle = \sum_{j\in J}\langle x_j, y_j\rangle_{H_j}$ is a seperable Hilbert space.

2. Preliminaries

In this section, we give some definitions and results which will help us to lay foundation for the main results.

Definition 2.1. [5] Let H be a seperable Hilbert space, a sequence $\{f_j\}_{j\in J}$ in H is said to be a frame for H, if there exist constants $0 < A \le B < \infty$ such that

$$A\|x\|^2 \le \sum_{j\in J} |\langle x, f_j\rangle|^2 \le B\|x\|^2, \quad \forall \quad x \in H. \tag{2.1}$$

The constants A and B are called lower and upper frame bounds respectively. If $A = B$, then the frame is said to be a tight frame. If $A = B = 1$, then the frame is said to be a Parseval's frame. If the right hand inequality of (2.1) holds, then $\{f_j\}_{j\in J}$ is known as a Bessel sequence.

Definition 2.2. [13] Let $K \in B(H)$. A sequence $\{f_j\}_{j\in J}$ in H is said to be a K-frame for H, if there exist constants $0 < A \le B < \infty$ such that

$$A\|K^*x\|^2 \le \sum_{j\in J} |\langle x, f_j\rangle|^2 \le B\|x\|^2, \quad \forall \quad x \in H. \tag{2.2}$$

The constants A and B are called lower and upper bound of K-frame. In particular, if $A\|K^*x\|^2 = \sum_{j\in J} |\langle x, f_j\rangle|^2$, for all $x \in H$, then the K-frame is called a A-tight K-frame for H. If $A = 1$, then it is called a Parseval K-frame for H.

Remark 2.3. From equation (2.2), it is clear that every K-frame is a Bessel sequence.

Definition 2.4. [23] A sequence $\{\Lambda_j \in B(H, H_j) : j \in J\}$ is said to be a g-frame for H with respect to $\{H_j\}_{j\in J}$, if there exist constants $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{j\in J} \|\Lambda_j x\|^2 \leq B\|x\|^2, \quad \forall \quad x \in H. \tag{2.3}$$

We call the constants A and B the lower and upper frame bounds of g-frame, respectively. The g-frame $\{\Lambda_j\}_{j\in J}$ is said to be tight, if $A = B$ and a Parseval g-frame, if $A = B = 1$. If only the right hand inequality in (2.3) holds, then $\{\Lambda_j\}_{j\in J}$ is called a g-Bessel sequence for H with respect to $\{H_j\}_{j\in J}$.

Definition 2.5. [27] Let $K \in B(H)$ and $\{\Lambda_j \in B(H, H_j), j \in J\}$. Then $\{\Lambda_j\}_{j\in J}$ is called a K-g-frame for H with respect to $\{H_j\}_{j\in J}$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\|K^*x\|^2 \leq \sum_{j\in J} \|\Lambda_j x\|^2 \leq B\|x\|^2, \quad \forall \quad x \in H. \tag{2.4}$$

The constants A and B are called the lower and upper frame bounds of K-g-frame, respectively.

Remark 2.6. For $K = I$, K-g-frame is a g-frame for H with respect to $\{H_j\}_{j\in J}$.

Remark 2.7. Every K-g-frame is a g-Bessel sequence for H with respect to $\{H_j\}_{j\in J}$. Thus, the bounded linear operator $T_\Lambda : l^2(\{H_j\}_{j\in J}) \to H$ defined by $T_\Lambda(\{x_j\}_{j\in J}) = \sum_{j\in J} \Lambda_j^* x_j$, for all $\{x_j\}_{j\in J} \in l^2(\{H_j\}_{j\in J})$ is a well defined bounded linear operator and the adjoint operator $T_\Lambda^* : H \to l^2(\{H_j\}_{j\in J})$ given by $T_\Lambda^* x = \{\Lambda_j x\}_{j\in J}$, for all $x \in H$ is also a bounded linear operator. The operator $S_\Lambda : H \to H$ defined by $S_\Lambda x = T_\Lambda T_\Lambda^* x = T_\Lambda(\{\Lambda_j x\}_{j\in J}) = \sum_{j\in J} \Lambda_j^* \Lambda_j x$, for all $x \in H$, is also bounded and linear. The operators T_Λ, T_Λ^* and S_Λ are called the pre-frame operator (synthesis operator), analysis operator and frame operator, respectively, of K-g-frame $\{\Lambda_j\}_{j\in J}$.

Definition 2.8. [4] Let $K \in B(H)$. Suppose $\{\Lambda_j \in B(H, H_j) : j \in J\}$ and $\{\Theta_j \in B(H, H_j) : j \in J\}$ are two K-g frames. Then $\{\Theta_j\}_{j\in J}$ is a K-g-dual of $\{\Lambda_j\}_{j\in J}$ if $T_\Lambda T_\Theta^* = I_{R(K)}$.

Theorem 2.9. [8] *(Douglas Factorization Theorem) Let $T_1 \in B(H_1, H)$ and $T_2 \in B(H_2, H)$. Then the following are equivalent:*

(1) $R(T_1) \subset R(T_2)$;
(2) there exists $\lambda > 0$ such that $T_1 T_1^* \leq \lambda T_2 T_2^*$;
(3) there exists $X \in B(H_1, H_2)$ such that $T_1 = T_2 X$.

Theorem 2.10. [16] *Let $K \in B(H)$. If $\{\Lambda_j\}_{j\in J}$ is a g-frame for H with respect to $\{H_j\}_{j\in J}$, then $\{\Lambda_j K^*\}_{j\in J}$ is a K-g frame for H with respect to $\{H_j\}_{j\in J}$.*

Definition 2.11. [9] Let $K \in B(H)$. Then K is said to be bounded below, if there exists a constant $C > 0$ such that $\|Kx\| \geq C\|x\|$, for all $x \in H$.

Definition 2.12. [3] A family of bounded operators $\{\Gamma_j\}_{j\in J}$ on H is called a resolution of identity on H, if for all $x \in H$, we have $x = \sum_{j\in J} \Gamma_j(x)$ and the series converges unconditionally for all $x \in H$.

3. Properties of K-g-frames for Hilbert Spaces

In this section, we give several properties of K-g-frames. Further, we discuss the conditions under which the sum of a K-g-frame and a g-Bessel sequence is a K-g-frame.

Proposition 3.1. Let $K_1, K_2 \in B(H)$ and $\{\Lambda_j \in B(H, H_j) : j \in J\}$ be a K_1-g frame with bounds A and B. Then $\{\Lambda_j K_2^*\}_{j\in J}$ is a K_2K_1-g-frame for H with respect to $\{H_j\}_{j\in J}$.

Proof. Replacing x by K_2^*x, in the definition of K_1-g-frame, we obtain

$$A\| \left(K_2K_1\right)^* x\|^2 \leq \sum_{j\in J} \|\Lambda_j K_2^* x\|^2 \leq B\|K_2^*\|^2\|x\|^2 \quad \forall \quad x \in H.$$

Hence, $\{\Lambda_j K_2^*\}_{j\in J}$ is a K_2K_1-g-frame for H with respect to $\{H_j\}_{j\in J}$. □

Corollary 3.2. Let $K \in B(H)$ and $\{\Lambda_j \in B(H, H_j) : j \in J\}$ be a K-g-frame with bounds A and B. Then $\{\Lambda_j(K^n)^*\}_{j\in J}$ is a K^{n+1}-g-frame for H with respect to $\{H_j\}_{j\in J}$.

Theorem 3.3. *Let $K \in B(H)$ be bounded below and $\{\Lambda_j \in B(H, H_j) : j \in J\}$ be such that $\{\Lambda_j K^{-1}\}_{j\in J}$ is a g-frame for $R(K)$. Then $\{\Lambda_j K^*\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$.*

Proof. Let $\{\Lambda_j K^{-1}\}_{j\in J}$ be a g-frame for $R(K)$, then there exists constants $0 < A \leq B < \infty$ such that $A\|x\|^2 \leq \sum_{j\in J} \|\Lambda_j K^{-1}x\|^2 \leq B\|x\|^2$, for all $x \in R(K)$. As K is a bounded below, therefore there exists $C > 0$ such that $C\|x\|^2 \leq \|Kx\|^2$, for all $x \in H$. Therefore, $AC\|x\|^2 \leq \sum_{j\in J} \|\Lambda_j x\|^2 \leq B\|K\|^2\|x\|^2$, for all $x \in H$. Hence, $\{\Lambda_j\}_{j\in J}$ is a g- frame for H with respect to $\{H_j\}_{j\in J}$. By Theorem 2.10, $\{\Lambda_j K^*\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$. □

Theorem 3.4. *Let $K_1, K_2 \in B(H)$ and $\{\Lambda_j\}_{j\in J}$ be a K_i-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds A_i and $B_i, (i = 1, 2)$, respectively. Let $B = max(B_1, B_2)$. Then the following hold:*
(i) For $\alpha \neq 0, \{\Lambda_j\}_{j\in J}$ is a αK_i-g-frame $(i = 1, 2)$ for H with respect to $\{H_j\}_{j\in J}$.
(ii) $\{\Lambda_j\}_{j\in J}$ is a $(K_1 + K_2)$-g-frame for H with respect to $\{H_j\}_{j\in J}$.
(iii)$\{\Lambda_j\}_{j\in J}$ is a K_1K_2-g-frame for H with respect to $\{H_j\}_{j\in J}$.

Proof. (i) For $i = 1, 2$, $\{\Lambda_j\}_{j\in J}$ is a K_i-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds A_i and B_i, then we have

$$\frac{A_i}{|\alpha|^2}\|\alpha K_i^* x\|^2 \leq \sum_{j\in J} \|\Lambda_j x\|^2 \leq B_i\|x\|^2, \quad \forall \quad x \in H.$$

Thus, $\{\Lambda_j\}_{j\in J}$ is a αK_i-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds $\frac{A_i}{|\alpha|^2}$ and B_i.

(ii) Since $\{\Lambda_j\}_{j\in J}$ is a K_i-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds A_i and $B_i, (i = 1, 2)$, we have

$$\frac{1}{\frac{1}{A_1} + \frac{1}{A_2}}\|(K_1 + K_2)^* x\|^2 \le \sum_{j\in J} \|\Lambda_j x\|^2 \le \; B\|x\|^2, \quad \forall \quad x \in H.$$

Thus, $\{\Lambda_j\}_{j\in J}$ is a $(K_1 + K_2)$-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds $\frac{1}{\frac{1}{A_1}+\frac{1}{A_2}}$ and B.

(iii) For any $x \in H$,

$$A_1\|(K_1K_2)^* x\|^2 \le \|K_2^*\|^2 \sum_{j\in J} \|\Lambda_j x\|^2 \le \|K_2^*\|^2 B\|x\|^2 \quad \forall \quad x \in H.$$

$$\frac{A_1}{\|K_2^*\|^2}\|(K_1K_2)^* x\|^2 \le \sum_{j\in J} \|\Lambda_j x\|^2 \le B\|x\|^2, \quad \forall \quad x \in H.$$

Therefore, $\{\Lambda_j\}_{j\in J}$ is a K_1K_2-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds $\frac{A_1}{\|K_2^*\|^2}$ and B. □

Corollary 3.5. For $i = 1, 2, .., n$, let $K_i \in B(H)$ and $\{\Lambda_j\}$ be a K_i- g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds A_i and B_i . Let $B = max(B_i : i = 1, 2, ..., n)$. Then the following hold:

(i) $\{\Lambda_j\}_{j\in J}$ is a $\sum_{i=1}^n K_i$-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds $\frac{1}{\sum_{i=1}^n \frac{1}{A_i}}$ and B.

(ii) $\{\Lambda_j\}_{j\in J}$ is a $\prod_{i=1}^n K_i$-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds $\frac{A_1}{\prod_{l=1}^{n-1}\|K_l^*\|^2}$ and B.

A K-g-frame for a Hilbert space H is not essentially a g-frame. We give the following example to illustrate the existence of such a K-g frame.

Example 3.6. Suppose that $H = \mathbf{R}^4, J = \{1, 2, 3, 4\}$. Let $\{e_j\}_{j\in J}$ be an orthonormal basis of H and $H_j = \overline{span}\{e_j\}$. Define $K : H \to H$ as $Kx = \langle x, e_1\rangle e_2 + \langle x, e_2\rangle e_2 + \langle x, e_2\rangle e_3 + \langle x, e_4\rangle e_4$; and for $1 \le j \le 4$, $\Lambda_j : H \to H_j$ as $\Lambda_1 x = (\langle x, e_2\rangle + \langle x, e_3\rangle)e_1$; $\Lambda_2 x = \langle x, e_2\rangle e_2$; $\Lambda_3 x = \langle x, e_2\rangle e_3$; $\Lambda_4 x = \langle x, e_4\rangle e_4$.

Then, it can easily be verified that for any $x = (a_1, a_2, a_3, a_4) \in H$, $\|K^* x\|^2 \le \sum_{j=1}^4 \|\Lambda_j x\|^2 \le 4\|x\|^2$. Thus, $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$ but is not a g-frame for H with respect to $\{H_j\}_{j\in J}$ as for $x = e_1, \|x\|^2 = 1$ and $\sum_{j=1}^4 \|\Lambda_j x\|^2 = 0$.

So, it is natural to ask for a condition on operator K under which a K-g-frame for H with respect to $\{H_j\}_{j\in J}$ becomes a g-frame for H with respect to $\{H_j\}_{j\in J}$.

Theorem 3.7. *Let $K \in B(H)$ be such that K^* is bounded below and $\{\Lambda_j \in B(H, H_j) : j \in J\}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$. Then $\{\Lambda_j\}_{j\in J}$ is a g-frame for H with respect to $\{H_j\}_{j\in J}$.*

Proof. Since K^* is bounded below, therefore there exists $C > 0$ such that $C\|x\|^2 \le \|K^* x\|^2$, for all $x \in H$. By the definition of K-g-frame, there exists constants $0 < A \le B < \infty$ such that $A\|K^* x\|^2 \le \sum_{j\in J} \|\Lambda_j x\|^2 \le B\|x\|^2$, for all $x \in H$. It follows that $AC\|x\|^2 \le \sum_{j\in J} \|\Lambda_j x\|^2 \le B\|x\|^2$, for all $x \in H$. Therefore, $\{\Lambda_j\}_{j\in J}$ is a g-frame for H with respect to $\{H_j\}_{j\in J}$. □

We may wonder, does there exist a $K \in B(H)$, for which a g-frame becomes a K-g-frame. The answer to this question is given in our next theorem.

Theorem 3.8. *Let $\{\Lambda_j \in B(H, H_j) : j \in J\}$ be a g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds A and B respectively. Then $\{\Lambda_j\}_{j\in J}$ is a S_Λ-g-frame for H with respect to $\{H_j\}_{j\in J}$, where S_Λ is the g-frame operator of $\{\Lambda_j\}_{j\in J}$.*

Proof. Since $\{\Lambda_j\}_{j\in J}$ is a g-frame, therefore the pre-frame operator T_Λ is surjective. Also, S_Λ being invertible is surjective. Thus, $R(T_\Lambda) = H = R(S_\Lambda)$. Therefore by Theorem 2.9, there exists $\lambda > 0$ such that $\lambda\|S_\Lambda^* x\|^2 \leq \|T^* x\|^2$. Hence, $\lambda\|S_\Lambda^* x\|^2 \leq \sum_{j\in J}\|\Lambda_j x\|^2 \leq B\|x\|^2$. □

Next, we observe that the sum of two K-g frames for a Hilbert space H with respect to $\{H_j\}_{j\in J}$ need not be a K-g frame for H with respect to $\{H_j\}_{j\in J}$ as is illustrated in the following example.

Example 3.9. Consider the Hilbert space H, $\{e_j\}_{j\in J}$, subspaces $\{H_j\}_{j\in J}$ and operator K as in Example 3.6. For $1 \leq j \leq 4$, define $\Lambda_j : H \to H_j$ as $\Lambda_1 x = (\langle x, e_2\rangle + \langle x, e_3\rangle)e_1; \Lambda_2 x = (\langle x, e_3\rangle - \langle x, e_1\rangle)e_2; \Lambda_3 x = \langle x, e_2\rangle e_3; \Lambda_4 x = \langle x, e_4\rangle e_4$ and $\Gamma_j = -\Lambda_j$. For any $x = (a_1, a_2, a_3, a_4) \in H, \|K^* x\|^2 \leq \sum_{j=1}^4 \|\Lambda_j x\|^2 \leq 4\|x\|^2$ and $\|K^* x\|^2 \leq \sum_{j=1}^4 \|\Gamma_j x\|^2 \leq 4\|x\|^2$. Therefore, $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are K-g-frames for H with respect to $\{H_j\}_{j\in J}$. But, for $x = e_4, \|K^* x\|^2 = 1$ and $\sum_{j=1}^4 \|(\Lambda_j + \Gamma_j)x\|^2 = 0$. Hence, $\{\Lambda_j + \Gamma_j\}_{j\in J}$ is not a K-g frame for H with respect to $\{H_j\}_{j\in J}$.

Recall that sum of two g-Bessel sequences is again a g-Bessel sequence whereas sum of two K-g-frames is not a K-g-frame. So, we want to determine the conditions under which the sum of two K-g-frames for H is again a K-g-frame for H. In this direction, we first obtain a condition under which the sum of a K-g-frame $\{\Lambda_j \in B(H, H_j) : j \in J\}$ and a g-Bessel sequence $\{\Gamma_j \in B(H, H_j) : j \in J\}$ becomes a K-g-frame for H with respect to $\{H_j\}_{j\in J}$.

Theorem 3.10. *Let $K \in B(H)$ and $\{\Lambda_j \in B(H, H_j) : j \in J\}$ be a K-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds A and B. Let $\{\Gamma_j \in B(H, H_j) : j \in J\}$ be a g-Bessel sequence with pre-frame operator T_Γ. If there exists a positive constant β satisfying $A \geq 2\beta$ and $\|T_\Gamma^* x\|^2 \leq \beta\|K^* x\|^2$, for all $x \in H$, then $\{\Lambda_j + \Gamma_j\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$.*

Proof. It follows from the definition of K-g-frames, that $\{\Lambda_j \in B(H, H_j) : j \in J\}$ is a g-Bessel sequence. As sum of two g Bessel sequences is again a g-Bessel sequence. It follows that $\{\Lambda_j + \Gamma_j\}_{j\in J}$ is a g-Bessel sequence. Also, for $x \in H$,

$$\sum_{j\in J}\|\Lambda_j x\|^2 \leq 2\left(\sum_{j\in J}\|\Gamma_j x\|^2 + \sum_{j\in J}\|(\Lambda_j + \Gamma_j)x\|^2\right).$$

Thus,

$$\begin{aligned} 2\sum_{j\in J}\|(\Lambda_j + \Gamma_j)x\|^2 &\geq A\|K^* x\|^2 - 2\|T_\Gamma^* x\|^2. \\ &\geq (A - 2\beta)\|K^* x\|^2. \end{aligned}$$

$$\frac{1}{2}(A-2\beta)\|K^*x\|^2 \leq \sum_{j\in J} \|(\Lambda_j+\Gamma_j)x\|^2.$$

It follows that $\{\Lambda_j+\Gamma_j\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$. $\square$

Since every K-g-frame is a g-Bessel sequence. Therefore, we have the following corollary that follows from Theorem 3.10.

Corollary 3.11. Let $K \in B(H)$ and $\{\Lambda_j \in B(H,H_j) : j \in J\}$ and $\{\Gamma_j \in B(H,H_j) : j \in J\}$ be two K-g-frames for H with respect to $\{H_j\}_{j\in J}$ with bounds A_1, B_1 and A_2, B_2, respectively. Let $A = min(A_1, A_2)$ and $B = max(B_1, B_2)$. If T_Γ be the pre-frame operator of $\{\Gamma_j \in B(H,H_j) : j \in J\}$ satisying $\|T_\Gamma^* x\|^2 \leq \beta\|K^*x\|^2$, for all $x \in H$, where β is a positive constant such that $A \geq 2\beta$. Then $\{\Lambda_j+\Gamma_j\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$.

In the next result, we establish a relationship between the K-g-dual of the sum $\{\Lambda_j+\Gamma_j\}_{j\in J}$ and K-g-duals of $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$.

Theorem 3.12. *Let $K \in B(H)$ and $\{\Lambda_j \in B(H,H_j) : j \in J\}$ and $\{\Gamma_j \in B(H,H_j) : j \in J\}$ be two K-g-frames for H with respect to $\{H_j\}_{j\in J}$ such that $\{\Lambda_j+\Gamma_j\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$. Let $\{\Theta_j\}_{j\in J}$ and $\{\Psi_j\}_{j\in J}$ be the K-g-dual frames of $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ with pre-frame operators T_Θ and T_Ψ, respectively. If $\|T_\Psi^* x\|^2 \leq \beta\|K^*x\|^2$, for all $x \in H$, where β is a positive constant satisfying $A \geq 2\beta$ and $\sum_{j\in J} \Gamma_j^*\Theta_j x = \sum_{j\in J} \Lambda_j^*\Psi_j x = 0$, for all $x \in H$. Then $\{\frac{1}{2}\Theta_j + \frac{1}{2}\Psi_j\}$ is a K-g-dual of $\{\Lambda_j+\Gamma_j\}_{j\in J}$.*

Proof. By Theorem 3.4 and Corollary 3.11, it follows that $\{\frac{1}{2}\Theta_j + \frac{1}{2}\Psi_j\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$. For any $y \in R(K)$, $T_{\Lambda+\Gamma}T^*_{(\frac{\Theta}{2}+\frac{\Psi}{2})}y = \sum_{j\in J} (\Lambda_j+\Gamma_j)^* \left(\frac{\Theta_j}{2}+\frac{\Psi_j}{2}\right) y = y$. Therefore, $\{\frac{1}{2}\Theta_j + \frac{1}{2}\Psi_j\}$ is a K-g-dual of $\{\Lambda_j + \mu_j\}_{j\in J}$. $\square$

In the next result, we observe that a K-g-frame gives rise to another K-g-frame.

Theorem 3.13. *Let $K \in B(H)$ and $\{\Lambda_j \in B(H,H_j) : j \in J\}$ be a K-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds A and B. If $U \in B(H)$ is bounded below and satisfy $UK^* = K^*U$, then $\{\Lambda_j U\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$.*

Proof. Since U is bounded below, there exists some $C > 0$ such that $C\|x\|^2 \leq \|Ux\|^2$, for all $x \in H$. As $UK^* = K^*U$, we have $\sum_{j\in J} \|\Lambda_j Ux\|^2 \geq A\|K^*Ux\|^2 \geq AC\|K^*x\|^2$, for all $x \in H$. Also, $\sum_{j\in J} \|\Lambda_j Ux\|^2 \leq B\|Ux\|^2 \leq \|U\|^2\|x\|^2$, for all $x \in H$. Therefore, $\{\Lambda_j U\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$. $\square$

If $K \in B(H)$ and $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are two K-g-frames, we look for the conditions on the operators $U, V \in B(H)$ that makes $\{\Lambda_j U + \Gamma_j V\}_{j\in J}$ a K-g-frame.

Theorem 3.14. *Let $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ be two K-g-frames of H with respect to $\{H_j\}_{j\in J}$ with frame bounds A_1, B_1 and A_2, B_2 respectively. Let T_Λ and T_Γ be the pre-frame operators of $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$. Let $U, V \in B(H)$ satisfy $UK^* = K^*U$ and $\|Ux\| \geq \|x\|$. If there exists a positive constant β such that $\beta\|K^*x\|^2 \geq \|T_\Gamma^* Vx\|^2$, for all $x \in H$ and $A_1 > 2\beta$, then $\{\Lambda_j U + \Gamma_j V\}_{j\in J}$ a K-g-frame.*

Proof. By definition of K-g-frame, it follows that $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are g-Bessel sequences. Since $U,V\in B(H)$, therefore $\{\Lambda_j U+\Gamma_j V\}_{j\in J}$ is a g-Bessel sequence. Thus, for any $x\in H$,

$$\begin{aligned}2\sum_{j\in J}\|(\Lambda_j Ux+\Gamma_j Vx)\|^2 &\geq \sum_{j\in J}\|\Lambda_j Ux\|^2-2\sum_{j\in J}\|\Gamma_j Vx\|^2.\\ &\geq A_1\|UK^*x\|^2-2\|T_\Gamma^* Vx\|^2.\\ &\geq (A_1-2\beta)\|K^*x\|^2.\end{aligned}$$

Hence, it follows that $\{\Lambda_j U+\Gamma_j V\}_{j\in J}$ a K-g-frame for H with respect to $\{H_j\}_{j\in J}$. □

In the next theorem, we see that if $T_\Lambda T_\Gamma^*=0$, then we just need U to satisfy a certain condition that will make $\{\Lambda_j U+\Gamma_j V\}_{j\in J}$ a K-g frame without any restriction on V.

Theorem 3.15. *Let $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ be two K-g frames of H with respect to $\{H_j\}_{j\in J}$ with frame bounds A_1,B_1 and A_2,B_2 respectively. Let T_Λ and T_Γ be the pre-frame operators of $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$such that $T_\Lambda T_\Gamma^*=0$. Let $U,V\in B(H)$ satisfy $UK^*=K^*U$ and $\|Ux\|\geq\|x\|$, then $\{\Lambda_j U+\Gamma_j V\}_{j\in J}$ a K-g-frame of H with respect to $\{H_j\}_{j\in J}$.*

Proof. By definition of K-g-frame, it follows that $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are g-Bessel sequences. As $U,V\in B(H)$, therefore $\{\Lambda_j U+\Gamma_j V\}_{j\in J}$ is a g-Bessel sequence. As $T_\Lambda T_\Gamma^*=0$, it follows that, $\sum_{j\in J}\Lambda_j^*\Gamma_j x=0$, for all $x\in H$. Thus, for $x\in H$,

$$\begin{aligned}\sum_{j\in J}\|(\Lambda_j U+\Gamma_j V)x\|^2 &= \sum_{j\in J}\|\Lambda_j Ux\|^2+\sum_{j\in J}\|\Gamma_j Vx\|^2\\ &\geq A_1\|K^*Ux\|^2\geq A_1\|K^*x\|^2.\end{aligned}$$

Therefore, $\{\Lambda_j U+\Gamma_j V\}_{j\in J}$ is a K-g-frame. □

Corollary 3.16. Let $K\in B(H)$ and $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ be two K-g-frames of H with respect to $\{H_j\}_{j\in J}$ with frame bounds A_1,B_1 and A_2,B_2, respectively. Let T_Λ and T_Γ be the corresponding pre-frame operators such that $T_\Lambda T_\Gamma^*=0$. If $U,V\in B(H)$ be such that $U^*K\in B(H)$, then $\{\Lambda_j U+\Gamma_j V\}_{j\in J}$ is a U^*K-g-frame for H with respect to $\{H_j\}_{j\in J}$.

Proof. From Theorem 3.15, it follows that $\{\Lambda_j U+\Gamma_j V\}_{j\in J}$ is a g-Bessel sequence for H with respect to $\{H_j\}_{j\in J}$ and for any $x\in H$, $\sum_{j\in J}\|\Lambda_j U+\Gamma_j V\|^2\geq\|K^*Ux\|^2=\|(U^*K)^2x\|^2$. Thus, $\{\Lambda_j U+\Gamma_j V\}_{j\in J}$ is a U^*K-g-frame for H with respect to $\{H_j\}_{j\in J}$. □

4. $2K$-g-frames for Hilbert Space

In this section, we introduce the notion of $2K$-g-frames.The relation between $2K$-g-frames and K-g-frames are studied. Further, we see how resolution of identity operator on H give rise to $2K$-g-frames. Finally, a characterisation for a g-Bessel sequence to be a $2K$-g-frame is proved.

Definition 4.1. Let $K \in B(H)$. A sequence $\{\Lambda_j \in B(H, H_j) : j \in J\}$ is said to be a $2K$-g-frame for H with respect to $\{H_j\}_{j\in J}$, if there exist constants $0 < A \leq B < \infty$ such that

$$A\|K^*x\|^2 \leq \sum_{j\in J} \|\Lambda_j x\|^2 \leq B\|K^*x\|^2, \quad \forall \quad x \in H.$$

It can easily be observed that every $2K$-g-frame is a K-g-frame but converse is not true. The sequence of operators $\{\Lambda_j\}_{j\in J}$ defined in Example 3.9 is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$ but is not a $2K$-g-frame for H with respect to $\{H_j\}_{j\in J}$ as there does not exist any constant $B > 0$ satisfying $\|K^*x\|^2 \leq \sum_{j=1}^{4} \|\Lambda_j x\|^2 \leq B\|K^*x\|^2$, for all $x \in H$.

Next we find a condition on $K \in B(H)$, so that a K-g-frame becomes a $2K$-g-frame.

Theorem 4.2. *Let $K \in B(H)$ such that $K^*$is bounded below. Then every K-g-frame $\{\Lambda_j : \Lambda_j \in B(H, H_j) : j \in J\}$ for H with respect to $\{H_j\}_{j\in J}$ is a $2K$-g-frame.*

Proof. Since K^* is bounded below, there exists a constant $C > 0$ such that $C\|x\|^2 \leq \|K^*x\|^2$, for all $x \in H$. Also since $\{\Lambda_j\}_{j\in j}$ is a K-g frame, there exists constants $0 < A \leq B < \infty$ such that $A\|K^*x\|^2 \leq \sum_{j\in J} \|\Lambda_j x\|^2 \leq B\|x\|^2$, for all $x \in H$. So, $A\|K^*x\|^2 \leq \sum_{j\in J} \|\Lambda_j x\|^2 \leq \frac{B}{C}\|K^*x\|^2$, for all $x \in H$. Hence, $\{\Lambda_j\}_{j\in J}$ is a $2K$-g frame for H with respect to $\{H_j\}_{j\in J}$. □

Also, it can be easily observed that the sum of two $2K$-g-frames is a $2K$-g- frame. We know that sum of two K-g-frames need not be a K-g-frame, Next, we check the sum of a K-g-frame and a $2K$-g- frame.

Example 4.3. Consider the Hilbert space H, $\{e_j\}_{j\in J}, \{H_j\}_{j\in J}, \{\Lambda_j\}_{j\in J}$ and operator K as defined in Example 3.9. For $1 \leq j \leq 4$, define $\Gamma_j : H \to H_j$as $\Gamma_1 x = \langle x, e_2\rangle e_1; \Gamma_2 x = (\langle x, e_2\rangle + \langle x, e_3\rangle)e_2; \Gamma_3 x = \langle x, e_2\rangle e_3; \Gamma_4 x = -\langle x, e_4\rangle e_4$. Then, $\|K^*x\|^2 \leq \sum_{j=1}^{4} \|\Lambda_j x\|^2 \leq 4\|x\|^2$ and $\|K^*x\|^2 \leq \sum_{j=1}^{4} \|\Gamma_j x\|^2 \leq 4\|K^*x\|^2$, for all $x \in H$. For $x = e_4, \|K^*x\|^2 = 1$ and $\sum_{j=1}^{4} \|(\Lambda_j + \Gamma_j)x\|^2 = 0$.

From above example we see that, $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ is a $2K$-g-frame for H with respect to $\{H_j\}_{j\in J}$, but $\{\Lambda_j + \Gamma_j\}_{j\in J}$ is not a K-g-frame for H, as there does not exist any constant $A > 0$ satisfying $A\|K^*x\|^2 \leq \sum_{j=1}^{4} \|(\Lambda_j + \Gamma_j)x\|^2$, for all $x \in H$.

It is natural to look for a condition under which the sum of a K-g-frame and a $2K$-g-frame turns out to be a K-g-frame.

Theorem 4.4. *Let $K \in B(H)$ and $\{\Lambda_j \in B(H, H_j) : j \in J\}$ be a K-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds A_1 and B_1 . Let $\{\Gamma_j \in B(H, H_j) : j \in J\}$ be a $2K$-g-frame with bounds A_2 and B_2 such that $0 < 2B_2 < A_1$. Then $\{\Lambda_j + \Gamma_j\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$.*

Proof. Since $\{\Lambda_j\}_{j\in J}$ is a K-g frame for H with bounds A_1 and B_1 and $\{\Gamma_j\}_{j\in J}$ is a $2K$-g frame for H with bounds A_2 and B_2. So for any $x \in H$, we have $\sum_{j\in J} \|(\Lambda_j + \Gamma_j)x\|^2 \leq 2\sum_{j\in J} \|\Lambda_j x\|^2 + 2\sum_{j\in J} \|\Gamma_j x\|^2 \leq 2(B_1 + B_2\|K^*\|^2)\|x\|^2$. It follows that $\{\Lambda_j + \Gamma_j\}_{j\in J}$ is a g-Bessel sequence. Also, $\sum_{j\in J} \|\Lambda_j x\|^2 \leq 2\sum_{j\in J} \|(\Lambda_j + \Gamma_j)x\|^2 + 2\sum_{j\in J} \|\Gamma_j x\|^2$. Thus $\frac{(A_1 - 2B_2)}{2}\|K^*x\|^2 \leq \sum_{j\in J} \|(\Lambda_j x + \Gamma_j)x\|^2 \leq 2(B_1 + B_2\|K^*\|^2)\|x\|^2$. Hence, $\{\Lambda_j + \Gamma_j\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$. □

Remark 4.5. One may note that the conditions on the bounds in the above theorem cannot be relaxed. In Example 4.3, we have $A_1 = 1$ and $B_2 = 4$, so $2B_2$ is not less than A_1 and $\{\Lambda_j + \Gamma_j\}_{j\in J}$ is not a K-g-frame.

Next, we give a result that relates the resolution of identity operator and $2K$-g-frames.

Theorem 4.6. *Let $K \in B(H)$ and $\{\Lambda_j \in B(H, H_j) : j \in J\}$ be a g-Bessel sequence with Bessel bound B such that $\{\Lambda_j^*\Lambda_j\}_{j\in J}$ is a resolution of identity operator on H. Then $\{\Lambda_j K^*\}_{j\in J}$ is $2K$-g-frame for H.*

Proof. As $\{\Lambda_j^*\Lambda_j\}_{j\in J}$ is a resolution of identity operator on H, therefore $x = \sum_{j\in J}\Lambda_j^*\Lambda_j x$, for all $x \in H$. Replacing x by K^*x in the definition of g-Bessel sequence, $\sum_{j\in J}\|\Lambda_j K^*x\|^2 \leq B\|K^*x\|^2$, for all $x \in H$. For any $x \in H$, $\|K^*x\|^4 = |\sum_{j\in J}\langle\Lambda_j K^*x, \Lambda_j K^*x\rangle|^2 \leq \sum_{j\in J}\|\Lambda_j K^*x\|^2 \sum_{j\in J}\|\Lambda_j K^*x\|^2 \leq B\|K^*x\|^2\sum_{j\in J}\|\Lambda_j x\|^2$. Thus, $(1/B)\|K^*x\|^2 \leq \sum_{j\in J}\|\Lambda_j K^*x\|^2 \leq B\|K^*x\|^2$ for all $x \in H$. □

In the next result, we establish a relationship between a tight g-frame and a $2K$-g-frame.

Theorem 4.7. *Let $\{\Lambda_j \in B(H, H_j) : j \in J\}$ be a tight g-frame with frame bound A. If there exists $I \subset J$ such that $\sum_{j\in I}\|\Lambda_j\|^2 < A$, then $\{\Lambda_j K^*\}_{j\in J\setminus I}$ is a $2K$-g-frame for H with respect to $\{H_j\}_{j\in J}$.*

Proof. As $\{\Lambda_j\}_{j\in J}$ is a tight g-frame for H with respect to $\{H_j\}_{j\in J}$, therefore $A\|x\|^2 = \sum_{j\in J}\|\Lambda_j x\|^2$, for all $x \in H$. In particular, $A\|K^*x\|^2 = \sum_{j\in J}\|\Lambda_j K^*x\|^2 \leq \sum_{j\in J\setminus I}\|\Lambda_j K^*x\|^2 + \|K^*x\|^2\sum_{j\in I}\|\Lambda_j\|^2$. Thus,

$$\{A - \sum_{j\in I}\|\Lambda_j\|^2\}\|K^*x\|^2 \leq \sum_{j\in J\setminus I}\|\Lambda_j K^*x\|^2 \leq \sum_{j\in J}\|\Lambda_j K^*x\|^2 = A\|K^*x\|^2.$$

Hence $\{\Lambda_j K^*\}_{j\in J\setminus I}$ is a $2K$-g-frame for H with respect to $\{H_j\}_{j\in J}$. □

Finally, we give a characterization for a g-Bessel sequence to be a $2K$-g-frame.

Theorem 4.8. *Let $K \in B(H)$ and $\{\Lambda_j\}_{j\in J}$ be a g-Bessel sequence for H with respect to $\{H_j\}_{j\in J}$ and T_Λ be the corresponding pre-frame operator. Then $\{\Lambda_j\}_{j\in J}$ is a $2K$-g-frame for H with respect to $\{H_j\}_{j\in J}$ if and only if $R(K) = R(T_\Lambda)$.*

Proof. Since T_Λ is the pre-frame operator of $\{\Lambda_j\}_{j\in J}$, therefore $\|T_\Lambda^*x\|^2 = \sum_{j\in J}\|\Lambda_j x\|^2$. By definition of $2K$-g-frame, there exist constants $0 < A \leq B < \infty$ such that $A\|K^*x\|^2 \leq \sum_{j\in J}\|\Lambda_j x\|^2 \leq B\|K^*x\|^2$. Consequently, $AKK^* \leq T_\Lambda T_\Lambda^* \leq BKK^*$ Therefore, by Theorem 2.9 we have $R(K) = R(T_\Lambda)$. Conversely, let us assume that $R(K) = R(T_\Lambda)$. Then by Theorem 2.9 , there exists constants $A, B > 0$ such that $AKK^* \leq T_\Lambda T_\Lambda^* \leq BKK^*$. Thus, for all $x \in H, A\|K^*x\|^2 \leq \sum_{j\in J}\|\Lambda_j x\|^2 \leq B\|K^*x\|^2$. Hence $\{\Lambda_j\}_{j\in J}$ is a $2K$-g-frame for H with respect to $\{H_j\}_{j\in J}$. □

Conclusion

While working on K-g-frames it was obderved that if the bounded linear operator K^* is injective and has a closed range, then for a K-g-frame $\{\Lambda_j\}_{j\in J}$, for all $x \in H$ we have

$A\|K^*x\|^2 \leq \sum_{j\in I} \|\Lambda_j x\|^2 \leq B\|K^*x\|^2$, which gave us the motivation to define $2K$-g-frame. The inference that has been highlighted in the paper is the construction of a new K-g-frame from the sum of a K-g-frame and a $2K$-g-frame under certain necessary conditions. Using the results and certain concepts of this paper, the reader can study the equalities and inequalities related to $2K$-g-frames and might be able to obtain some new stability results for generalised frames for Hilbert spaces. Interested readers can also think about the dual and generalised duals of $2K$-g-frames.

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(Varinder Kumar) Department of Mathematics, Shaheed Bhagat Singh College, University of Delhi, Delhi-110017, India
E-mail address: varinder.kumar@sbs.du.ac.in

(Sapna Malhotra) Department of Mathematics, Gargi College, University of Delhi, Delhi-110049, India
E-mail address: sapna.malhotra@gargi.du.ac.in