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INJECTIVITY PROBLEM VIA g-FRAMES

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Abstract. The main issues in quantum detection theory are the injectivity and state estimation problems, which recently linked and partially studied via frame theory on Hilbert spaces. Inspired by some recent researches and works on the quantum detection problem by (discrete) frames, continuous frames and fusion frames, in this note we examine the quantum detection problem for generalized frames (g-frames). In the theory of g-frames, instead of using the frame element itself, we use a family of bounded operators.We characterize injectivity problem by g-frames and with the help of these characterizations, we give some equivalent conditions for the injectivity problems. We answer the injectivity problems both for trace and Hilbert-Schmidt classes of operators.

1. Introduction

One of the problems that is inspired by engineering applications in signal processing and information theory is quantum detection problem which has a variety of applications in optical communications, including the detection of coherent light signals such as radio, radar and laser signals. Quantum detection theory is a reformulation, in quantum mechanical terms, of statistical decision theory. The quantum detection problem can be split as follows: the injectivity problem and the state estimation problem. In recent years the injectivity linked to frame theory, and it is known that the goal of the injectivity problem is to classify frames that are injective with respect to self-adjoint Hilbert-Schmidt class of operators. Recently, the theory of frames has been studied from the perspective of theoretical physics [34] and the quantum detection and injectivity problem studied in the point of view of continuous and fusion frames and in this paper, we consider the Hilbert space g-frame version's of a quantum detection problem.

Let us first recall some backgrounds and basics about frames and the quantum detection problem.

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Throughout this paper, $\mathcal{H}$ is a separable Hilbert spaces, $\mathcal{H}^n$ a n-dimensional Hilbert space, $\mathbb{I}$ an index set which can be finite or countable, $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$ a family of Hilbert spaces, $\mathcal{B}(\mathcal{H})$ the Banach space of bounded operators on $\mathcal{H}$, $\mathcal{B}(\mathcal{H})^+$ positive operators on $\mathcal{H}$, $\mathcal{S}_p$ or $\mathcal{S}_p(\mathcal{H})$ the Banach space of Schatten p-class of operators on $\mathcal{H}$, $\mathcal{S}_1$ trace class of operators, $\mathcal{S}_2$ Hilbert-Schmidt class of operators of operators on $\mathcal{H}$, $\langle .,.\rangle_{HS}$ inner product on $\mathcal{S}_2$, $\mathbb{S}(\mathcal{H})$ the set of states on $\mathcal{H}$.

Recall that if $T \in \mathcal{B}(\mathcal{H})$ is a compact operator on a separable Hilbert space $\mathcal{H}$, then there exist orthonormal sets $\{e_n\}$ and $\{\sigma_n\}$ in $\mathcal{H}$ such that

$$Tx = \sum_n \lambda_n \langle x, e_n\rangle \sigma_n,$$

for $x \in \mathcal{H}$, where λ_n is the n-th singular value of T and $\{\lambda_n\} \in c_0$. Given $0 < p < \infty$, the Schatten p-class of $\mathcal{H}$ [32], denoted $\mathcal{S}_p$, is the space of all compact operators T on $\mathcal{H}$ with the singular value sequence $\{\lambda_n\}$ belonging to ℓ^p. It is known that, $\mathcal{S}_p$ is a Banach space with the norm

$$\|T\|_p = [\sum_n |\lambda_n|^p]^{\frac{1}{p}}. \tag{1.1}$$

$\mathcal{S}_1$ is called the *trace class* of $\mathcal{H}$ and $\mathcal{S}_2$ is called the *Hilbert-Schmidt class.* Theorem 1.4.6. of [36] shows that $T \in \mathcal{S}_p$ if and only if $T^p \in \mathcal{S}_1$. Moreover $\|T\|_p^p = \|T^p\|_1$. Also, $T \in \mathcal{S}_p$ if and only if $|T|^p = (T^*T)^{\frac{p}{2}} \in \mathcal{S}_1$ if and only if $T^*T \in \mathcal{S}_{\frac{p}{2}}$. Moreover, $\|T\|_p^p = \|T^*\|_p^p = \||T|\|_p^p = \||T|^p\|_1 = \|T^*T\|_{\frac{p}{2}}$.

It is proved that $\mathcal{S}_p$ is a two sided $*$-ideal of $\mathcal{B}(\mathcal{H})$. Further, for $T \in \mathcal{S}_p$ one has $\|T\|_p = \|T^*\|_p, \|T\| \leq \|T\|_p$ and if $T' \in \mathcal{B}(\mathcal{H})$ then $\|T'T\|_p \leq \|T'\|\|T\|_p$ and $\|TT'\|_p \leq \|T'\|\|T\|_p$. For more information about these operators see [16, 26, 29, 32, 36].

More precisely, for a given orthonormal basis $\{e_i\}$ the operator T is a Hilbert-Schmidt operator if $\|T\|_2 := (\sum_i \|Te_i\|^2)^{\frac{1}{2}} < \infty$ and it is trace class operator if $\|T\|_1 := \sum_i \langle |T|e_i, e_i\rangle < \infty$, in this case the trace of T is $tr(T) = \sum_i \langle Te_i, e_i\rangle$ which is finite and independent of choosing the orthonormal bases. It is known that trace induces an inner product by $\langle T, S\rangle_{HS} = tr(TS^*)$ for the Hilbert -Schmidt operators T, S, [26].

Let $Sym(\mathcal{H}) := \{T : T \in \mathcal{B}(\mathcal{H}), T = T^*\}$ denote the real Banach space of self-adjoint operators on $\mathcal{H}$ and the real cone $Sym^+(\mathcal{H}) := \{T = T^* \geq 0\}$.

1.1. Positive Operator-Valued Measures (POVM) and Quantum Detection Problem. Quantum theory tries to predict the probability of observing outcomes from a sequence of measurements of the system in an unknown state. This process is called quantum state tomography [27]. The outcome statistics are described by a positive operator- valued measure (POVM) [5, 13, 22, 23, 28, 33]. In fact, quantum measurement extracts the transmitted information from received quantum signals and therefore performs an important role in quantum communications. The simplest quantum measurement is the projection-valued measure (PVM), also called standard measurement or von Neumann measurement, where elementary projectors are usually used. Sometimes the positive-operator valued measure (POVM) is more efficient of obtaining information about the state of a quantum system than a standard measurement. Let X denote a set of outcomes (e.g. this could be a finite or infinite subset of $\mathbb{Z}^d$ or $\mathbb{R}^d$) and β a sigma algebra of subsets of X.

Definition 1.1. A positive operator-valued measure (POVM) is a function $\Pi : \beta \to Sym^+(\mathcal{H})$ satisfying:

1. $\Pi(\phi) = 0$.
2. (Countably additive) For every at most countable disjoint family $\{V_i\} \subset \beta$ and $x, y \in \mathcal{H}$ we have

$$\langle \Pi(\bigcup V_i)x, y\rangle = \sum_i \langle \Pi(V_i)x, y\rangle.$$

3. $\Pi(X) = I$ (the identity operator).

A quantum system is defined as a von Neumann algebra $\mathcal{A}$ of operators acting on $\mathcal{H}$. The set of states on $\mathcal{H}$ is $\mathbb{S}(\mathcal{H}) := \{t \in \mathcal{S}_1, T = T^* \geq 0, tr(T) = 1\}$. It is known that [4], the set of quantum states are in one-to-one correspondence with the linear functionals on $\mathcal{A}$ of the form:

$$\rho : \mathcal{A} \to \mathbb{C}, \quad for \quad S \in \mathbb{S}(\mathcal{H}), \rho(T) = tr(ST), \forall T \in \mathcal{A}.$$

Let $L(\beta, \mathbb{R})$ denote the set of real-valued bounded functions defined on β. Given a POVM Π associated to a von Neumann algebra $\mathcal{A}$, the quantum detection problem asks if there is a unique quantum state $\rho \in \mathbb{S}(\mathcal{A})$ compatible with the set of quantum measurements performed by the POVM Π? The quantum detection problem asks two questions:

(1) Is the following map injective $M : \mathbb{S}(\mathcal{A}) \to L(\beta, \mathbb{R}), M(\rho)(U) = \rho(\Pi(U))$? (Injectivity Problem)

(2) Range analysis, or state estimation: Assume M is injective. Then, given a map $p \in L(\beta, \mathbb{R})$, determine if p is in the range of M, hence is of the form $p = M(\rho)$ for some unique $\rho \in \mathbb{S}(\mathcal{A})$.

POVMs have a natural, and very valuable, subset which comes from Hilbert space frame theory [9, 10, 11]. For a background on frame POVMs, we recommend [2, 15, 18, 24].

1.2. Frames and g-frames in Hilbert spaces. Hilbert space frames have been traditionally used in signal processing because of their resilience to additive noise, resilience to quantization, the numerical stability of reconstruction and their ability to capture important signal characteristics. Today, frame theory is an exciting, dynamic and fast paced subject with applications to a wide variety of areas in mathematics and engineering, including sampling theory, operator theory, harmonic analysis, nonlinear sparse approximation, pseudodifferential operators, wavelet theory, wireless communication, data transmission with erasures, filter banks, signal processing, image processing, geophysics, quantum computing, sensor networks, and more.

Frames in Hilbert spaces were first introduced by Duffin and Schaeffer to deal with nonharmonic Fourier series in 1952 [14] and widely studied from 1986 since the great work of Daubechies, Grossmann and Meyer [12]. Frames are basis- like building blocks that span a vector space but allow for linear dependency, which is useful for reducing noise, find sparse representations, spherical codes, compressed sensing, signal processing, wavelet analysis and etc., see [9].

Definition 1.2. A countable family of elements $\{f_i\}_{i\in\mathbb{I}}$ in $\mathcal{H}$ is a frame for $\mathcal{H}$, if there exist constants $A, B > 0$ such that:

$$A\|x\|^2 \leq \sum_{i\in\mathbb{I}} |\langle x, f_i\rangle|^2 \leq B\|x\|^2, \quad \forall x \in \mathcal{H}. \tag{1.2}$$

The numbers A and B are celled the lower and upper frame bounds, respectively. The frame $\{f_i\}_{i\in\mathbb{I}}$ is called tight, if $A = B$ and is called Parseval, if $A = B = 1$. Also the sequence $\{f_i\}_{i\in\mathbb{I}}$ is called Bessel sequence, if the upper inequality in (1.2) holds. A Riesz basis for $\mathcal{H}$ is a family of the form $\{Ve_i\}_{i\in\mathbb{I}}$, where $\{e_i\}_{i\in\mathbb{I}}$ is an orthonormal basis for $\mathcal{H}$ and the operator $V : \mathcal{H} \longrightarrow \mathcal{H}$ is a bounded and invertible operator.

For the Bessel sequence $\{f_i\}_{i\in\mathbb{I}}$, the synthesis operator $T : \ell^2(\mathbb{I}) \longrightarrow \mathcal{H}$ is defined by $T\{c_i\} = \sum_{i\in\mathbb{I}} c_i f_i$, for all $\{c_i\}_{i\in\mathbb{I}} \in \ell^2$. Its adjoint operator $T^* : \mathcal{H} \to \ell^2(\mathbb{I})$ called analysis operator $T^*(x) = \{\langle x, f_i\rangle\}_{i\in\mathbb{I}}$, for all $x \in \mathcal{H}$. The operator $S : \mathcal{H} \longrightarrow \mathcal{H}$, which is defined by $S(x) = TT^*(x) = \sum_{i\in\mathbb{I}} \langle x, f_i\rangle f_i$, for all $x \in \mathcal{H}$, is called the frame operator. For a frame $\{f_i\}_{i\in\mathbb{I}}$, the operator T is onto, T^* is one to one and S is positive, self-adjoint and invertible.

A dual of the Bessel sequence $\{f_i\}_{i\in\mathbb{I}} \subseteq \mathcal{H}$ is a Bessel sequence $\{g_i\}_{i\in\mathbb{I}}$ in $\mathcal{H}$, such that

$$x = \sum_{i\in\mathbb{I}} \langle x, g_i\rangle f_i, \quad x \in \mathcal{H}.$$

For a frame $\{f_i\}_{i\in\mathbb{I}}$, the sequence $\{S^{-1}f_i\}_{i\in\mathbb{I}}$ is a frame which is called the canonical dual of $\{f_i\}_{i\in\mathbb{I}}$. For a more complete treatment of frame theory we recommend the excellent book of Christensen [11], the tutorials of Casazza [6, 7], the Memoir of Han and Larson [17] and the review papers [21, 20].

Over the years, various extensions of the frame theory have been investigated. Several of these are contained as special cases of the elegant theory for g-frames that was introduced by W. Sun in [35]. For example, one can consider: bounded quasi-projectors, fusion frames, pseudo-frames, oblique frames, outer frames and etc. For more studies on g-frames, see [25, 30].

Definition 1.3. We call $\Lambda := \{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) \ : \ i \in \mathbb{I}\}$ a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$, or simply, a g-frame for $\mathcal{H}$, if there exist two positive constants A, B such that

$$A\|f\|^2 \leq \sum_{i\in\mathbb{I}} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad f \in \mathcal{H}.$$

The positive numbers A and B are called the lower and upper g-frame bounds, respectively. We call Λ a tight g-frame if $A = B$ and we call it a Parseval g-frame if $A = B = 1$. If only the second inequality holds, we call it a g-Bessel sequence. If Λ is a g-frame, then the g-frame operator S_Λ is defined by

$$S_\Lambda f = \sum_{i\in I} \Lambda_i^* \Lambda_i f, \quad f \in \mathcal{H}$$

which is a bounded, positive and invertible operator such that

$$AI \leq S_\Lambda \leq BI$$

and for each $f \in \mathcal{H}$, we have

$$f = S_\Lambda S_\Lambda^{-1} f = S_\Lambda^{-1} S_\Lambda f = \sum_{i\in\mathbb{I}} S_\Lambda^{-1} \Lambda_i^* \Lambda_i f = \sum_{i\in\mathbb{I}} \Lambda_i^* \Lambda_i S_\Lambda^{-1} f.$$

The canonical dual g-frame for Λ is defined by $\{\Lambda_i S_\Lambda^{-1}\}_{i\in\mathbb{I}}$ with bounds $\frac{1}{B}, \frac{1}{A}$. In other words, $\{\Lambda_i S_\Lambda^{-1}\}_{i\in\mathbb{I}}$ and $\{\Lambda_i\}_{i\in\Lambda}$ are dual g-frames with respect to each other.

Note that, supposing $\mathcal{H}_i = W_i$ and $\Lambda_i = \pi_{W_i}$, the g-frame Λ is a fusion frame (frame of subspaces), [8]. Also, given frame $\{f_i\}$, letting $\mathcal{H}_i := \mathbb{C}$ and $\Lambda_i f := \langle f, f_i\rangle$ for all $f \in \mathcal{H}$ and $i \in \mathbb{I}$, $\{\Lambda_i\}$ is a g-frame.

2. Injectivity Problem via frames

The quantum detection problem with discrete frame coefficient measurements was recently settled by Botelho-Andrade et al. for both finite and infinite dimensional Hilbert spaces in [4, 5], where the characterization was given in terms of spanning properties of some derived sequences from the frame vectors. It is natural to study this problem and the results for frame's extensions. For continuous frames it has been investigated by Deguang Han and et.al. [19] and for fusion frames by three authors of this manuscript [31]. The notion of injectivity can be linked to the concept of norm and phase retrieval of frames [4, 1]. In this note, we will study and review the injectivity problem for g-frames.

Definition 2.1. A family of vectors $\mathcal{X} = \{x_k\}_{k\in\mathbb{I}}$ in a Hilbert space $\mathcal{H}$ is said to be injective if given a Hilbert- Schmidt self-adjoint operator T satisfying $\langle Tx_k, x_k\rangle = 0$ for all $k \in \mathbb{I}$, then $T = 0$.

It is known [3, 4] that if a family of vectors gives injectivity in a Hilbert space $\mathcal{H}^n$, then it is a frame for $\mathcal{H}^n$ and in case $\{x_k\}_{k\in\mathbb{I}}$ a frame for $\mathcal{H}$ which gives injectivity. If F is a bounded invertible operator on $\mathcal{H}$, then $\{Fx_k\}_{k\in\mathbb{I}}$ also gives injectivity.

Definition 2.2. Let $\Lambda = \{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) \ : \ i \in \mathbb{I}\}$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathbb{I}}$, where the bounded operators $\Lambda_i, i \in \mathbb{I}$ are Hilbert-Schmidt operators. Λ is said to be injective if given a Hilbert- Schmidt self-adjoint operator T satisfying $\langle \Lambda_i T, \Lambda_i\rangle_{HS} = 0$ for all $i \in \mathbb{I}$, then $T = 0$.

The following lemma shows that given an injective frame the induced g-frame is also injective.

Lemma 2.3. *Let $\{f_i\}$ be an injective family. Assume $\mathcal{H}_i := \mathbb{C}$ and $\Lambda_i f := \langle f, f_i\rangle$ for all $f \in \mathcal{H}$ and $i \in \mathbb{I}$. Then $\{\Lambda_i\}$ is an injective g-frame.*

Proof. Let T be a Hilbert-Schmidt, self- adjoint operator. Given any orthonormal basis $\{e_k\}_{k\in\mathbb{I}}$, we have

$$\begin{aligned}
\langle \Lambda_i T, \Lambda_i\rangle_{HS} &= tr(\Lambda_i^*\Lambda_i T) = \sum_{k\in\mathbb{I}} \langle \Lambda_i^*\Lambda_i T e_k, e_k\rangle \\
&= \sum_{k\in\mathbb{I}} \langle \Lambda_i T e_k, \Lambda_i e_k\rangle = \sum_{k\in\mathbb{I}} \langle \langle T e_k, f_i\rangle, \langle e_k, f_i\rangle\rangle \\
&= \sum_{k\in\mathbb{I}} \langle f_i, e_k\rangle\langle e_k, T^* f_i\rangle = \langle f_i, T^* f_i\rangle \\
&= \langle T f_i, f_i\rangle = 0 \Rightarrow T = 0.
\end{aligned}$$

$\square$

The following is an elementary example of injective g-frames.

Example 2.4. Let $P_i : \mathbb{R}^2 \to \mathbb{R}^2$, $i = 1, 2$ and $P_1\binom{x}{y} := \binom{x}{0}$, $P_1\binom{x}{y} := \binom{0}{y}$, $x, y \in \mathbb{R}$. Given the Hilbert-Schmidt and self-adjoint matrix $T = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ for $a \in \mathbb{R}$, we have

$$\langle P_i T, P_i \rangle_{HS} = tr(P_i^* P_i T) = \langle P_i T \begin{pmatrix} 1 \\ 0 \end{pmatrix}, P_i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle + \langle P_i T \begin{pmatrix} 0 \\ 1 \end{pmatrix}, P_i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle = a + a.$$

So $\langle P_i T, P_i \rangle_{HS} = 2a = 0 \Rightarrow a = 0 \Rightarrow T = 0$, which means $\{P_1, P_2\}$ is an injective g-frame.

Similar to discrete frames, the following proposition shows that an injective g-frame preserves the injectivity under action of an invertible operator.

Proposition 2.5. Let $\Lambda = \{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) \ : \ i \in \mathbb{I}\}$ be an injective g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$ for $\mathcal{H}$ and U be a bounded invertible operators on $\mathcal{H}$, then $\{\Lambda_i U\}$ is an injective g-frame.

Proof. It follows from [25] that $\{\Lambda_i U\}$ is a g-frame for $\mathcal{H}$. Suppose T is a Hilbert-Schmidt, self-adjoint operator on $\mathcal{H}$ and $\langle \Lambda_i UT, \Lambda_i U \rangle_{\mathcal{HS}} = 0$ for all $i \in \mathbb{I}$.

$$\langle \Lambda_i UT, \Lambda_i U \rangle_{HS} = 0 \Rightarrow tr(\Lambda_i UT(\Lambda_i U)^*) = 0 \Rightarrow tr[\Lambda_i (UTU^*) \Lambda_i^*] = 0, \quad \forall i \in \mathbb{I}.$$

The injectivity of $\{\Lambda_i\}$ implies that $UTU^* = 0$. Since U is invertible so $T = 0$. □

Immediately, one can say:

Corollary 2.6. If $\{\Lambda_i\}_{i \in \mathbb{I}}$ is an injective g-frame for $\mathcal{H}$ then its canonical dual $\{\Lambda_i S_\Lambda^{-1}\}_{i \in \mathbb{I}}$ is also injective.

Theorem 2.7. *Let $\Lambda = \{\Lambda_i : \mathcal{H} \to \mathcal{H}_i, i \in \mathbb{I}\}$ be g-frame of Hilbert-Schmidt operators for $\mathcal{H}$. The following statements are equivalent.*

(1) Whenever T_1, T_2 are Hilbert-Schmidt, positive, self-adjoint and $\langle \Lambda_i T_1, \Lambda_i \rangle_{HS} = \langle \Lambda_i T_2, \Lambda_i \rangle_{HS}$ for all $i \in \mathbb{I}$, then $T_1 = T_2$.

(2) Whenever T_1, T_2 are Hilbert-Schmidt, self-adjoint and $\langle \Lambda_i T_1, \Lambda_i \rangle_{HS} = \langle \Lambda_i T_2, \Lambda_i \rangle_{HS}$ for all $i \in \mathbb{I}$, then $T_1 = T_2$.

(3) $\Lambda = \{\Lambda_i : \mathcal{H} \to \mathcal{H}_i, i \in \mathbb{I}\}$ is injective.

Proof. $1 \to 2$ Assume $T = T_1 - T_2$, then T is also a Hilbert-Schmidt self-adjoint operator. Let $\{e_j\}_{j=1}^\infty$ be an ONB for $\mathcal{H}$ and let $\{u_j\}_{j=1}^\infty$ be an eigenbasis for T with respective eigenvalues $\{\alpha_j\}_{j=1}^\infty$ (Spectral Theorem of compact operators). Define the operators U and D on $\mathcal{H}$ by $Ue_j = u_j$ and $De_j = \alpha_j e_j$ for $j = 1, 2, ...$. Then U is unitary operator. The operator D is Hilbert-Schmidt self-adjoint operator (because $\alpha_j \in \mathbb{R}$ for $j = 1, 2, ...$) and $UDU^*(u_j) = UDe_j = U(\alpha_j e_j) = \alpha_j Ue_j = \alpha_j u_j = Tu_j$, for $j = 1, 2, ...$ So $T = UDU^*$. Now, let $\beta_j = |\alpha_j|$, $\gamma_j = |\alpha_j| - \alpha_j, j = 1, 2, ...$ be non-negative numbers then $\alpha_j = \beta_j - \gamma_j$. Let D_1, D_2 be the operators defined by

$$D_1 e_j = \beta_j e_j, D_2 e_j = \gamma_j e_j \text{ for } j = 1, 2, ...$$

Note that since T is Hilbert-Schmidt, $\sum_{j=1}^\infty \alpha_j^2$ converges. (T is compact $\Rightarrow$ $Tu_j = \alpha_j u_j \Rightarrow \sum_{j=1}^\infty \alpha_j^2 \leq \infty$). Hence D_1, D_2 are Hilbert-Schmidt, positive, self-adjoint and we have

$$T = UDU^* = U(D_1 - D_2)U^* = UD_1U^* - UD_2U^*.$$

Moreover UD_1U^* and UD_2U^* are Hilbert-Schmidt, positive and self-adjoint operators. Since

$$\begin{aligned}
0 = \langle \Lambda_i T, \Lambda_i \rangle_{HS} &= \langle \Lambda_i (UD_1U^* - UD_2U^*), \Lambda_i \rangle_{HS} \\
&= tr(\Lambda_i (UD_1U^* - UD_2U^*)\Lambda_i^*) \\
&= tr((\Lambda_i (UD_1U^*)\Lambda_i^*) - (\Lambda_i (UD_2U^*)\Lambda_i^*)) \\
&= tr(\Lambda_i (UD_1U^*)\Lambda_i^*) - tr(\Lambda_i (UD_2U^*)\Lambda_i^*) \\
&\Rightarrow \langle \Lambda_i (UD_1U^*), \Lambda_i \rangle_{HS} = \langle \Lambda_i (UD_2U^*), \Lambda_i \rangle_{HS} \\
&\Rightarrow UD_1U^* = UD_2U^*
\end{aligned}$$

Then $T = 0$ and Hence $T_1 = T_2$.

$2 \to 3$ Let T_1 and T_2 be any Hilbert-Schmidt, self-adjoint operators such that

$$\langle \Lambda_i T_1, \Lambda_i \rangle_{HS} = \langle \Lambda_i T_2, \Lambda_i \rangle_{HS} \Rightarrow T_1 = T_2$$

or equivalently

$$tr(\Lambda_i (T_1 - T_2)\Lambda_i^*) = 0 \Rightarrow T_1 = T_2.$$

Now, without loss of generality assume that for the Hilbert-Schmidt, self adjoint operator T, we have $\langle \Lambda_i T, \Lambda_i \rangle_{HS} = tr(\Lambda_i T \Lambda_i^*) = 0$. Letting $T_1 = T$ and $T_2 = 0$, we get $T = 0$ and so $\Lambda = \{\Lambda_i : \mathcal{H} \to \mathcal{H}_i, i \in \mathbb{I}\}$ gives injectivity.

$3 \to 1$ Assume T_1, T_2 are Hilbert-Schmidt, positive, self-adjoint and $\langle \Lambda_i T_1, \Lambda_i \rangle_{HS} = \langle \Lambda_i T_2, \Lambda_i \rangle_{HS}$ for all $i \in \mathbb{I}$,
then

$$\begin{aligned}
tr(\Lambda_i^* \Lambda_i T_1) = tr(\Lambda_i^* \Lambda_i T_2) &\Rightarrow tr(\Lambda_i^* \Lambda_i T_1 - \Lambda_i^* \Lambda_i T_2) = tr(\Lambda_i^* \Lambda_i (T_1 - T_2)) \\
&\Rightarrow \langle \Lambda_i (T_1 - T_2), \Lambda_i \rangle_{HS} = 0.
\end{aligned}$$

the injectivity of Λ shows that $T_1 - T_2 = 0$ which means $T_1 = T_2$.

□

The following theorem characterizes the injectivity problem in terms of trace class operators.

Theorem 2.8. *Let $\Lambda := \{\Lambda_i : \mathcal{H} \to \mathcal{H}_i, i \in \mathbb{I}\}$ be a g-frame for $\mathcal{H}$ of Hilbert-Schmidt operators, then the following are equivalent:*

(1) Whenever T, S are trace class, self-adjoint, positive operators and

$$tr(\Lambda_i T \Lambda_i^*) = tr(\Lambda_i S \Lambda_i^*)$$

for all $i \in \mathbb{I}$ then $T = S$.

(2) Whenever T, S are trace class, self-adjoint operators and

$$tr(\Lambda_i T \Lambda_i^*) = tr(\Lambda_i S \Lambda_i^*)$$

for all $i \in \mathbb{I}$ then $T = S$.

(3) Whenever T is trace class, self-adjoint operator and $tr(\Lambda_i T \Lambda_i^*) = 0$ for all $i \in \mathbb{I}$, then $T = 0$.

Proof. At the first, note that for the Hilbert-Schmidt operators $T_1, T_2 \in \mathcal{S}_2$, the operator $T_1 T_2^* \in \mathcal{S}_1$.

$1 \to 2$ Let $T, S \in \mathcal{S}_1$ be self-adjoint operators of trace one and $tr(T) = tr(S) = 1$. Getting $R = T - S$, the operator R belong to trace zero self-adjoint operators and $tr(R) = tr(T) - tr(s) = 1 - 1 = 0$. Let $\{e_j\}_{j=1}^{\infty}$be an ortonormal basis for $\mathcal{H}$ and let $\{v_j\}_{j=1}^{\infty}$ be an eigenbasis for R with respect eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ (spectral theorem of compact operators). Then $\sum_{j=1}^{\infty} \lambda_j = 0$. Define the operators V and D on $\mathcal{H}$ by $Ve_j = v_j$ and $De_j = \lambda_j e_j$ for $j = 1, 2, \ldots$. Then V is an unitary operator and D is a trace class self-adjoint operator of trace zero and $R = VDV^*$. Now define the non-negative numbers

$$r_1 = \frac{1 + |\lambda_1|}{A}, \quad s_1 = \frac{1 + |\lambda_1| - \lambda_1}{A}, \quad r_j = \frac{|\lambda_j|}{A}, \quad s_j = \frac{|\lambda_j| - \lambda_j}{A}$$

where $A = 1 + \sum_{j=1}^{\infty} |\lambda_j| = 1 + \sum_{j=1}^{\infty} |\lambda_j| - \sum_{j=1}^{\infty} \lambda_j > 0$. Then $\lambda_j = r_j - s_j$ for all j. Let D_1, D_2 be operators defined by $D_1 e_j = r_j e_j, D_2 e_j = s_j e_j$ for $j = 1, 2, \ldots$, then D_1, D_2 are trace class positive self-adjoint of trace one and we have $R = VDV^*$, $V(D_1 - D_2)V^* = VD_1V^* - VD_2V^*$. Moreover VD_1V^*, VD_2V^* are trace class positive self-adjoint operators of trace one, since

$$\begin{aligned} 0 &= tr(\Lambda_i^* \Lambda_i R) \\ &= tr(\Lambda_i^* \Lambda_i (VDV^*)) \\ &= tr(\Lambda_i^* \Lambda_i V(D_1 - D_2)V^*) \\ &= tr(\Lambda_i^* \Lambda_i ((VD_1V^*) - (VD_2V^*))) \\ &= tr(\Lambda_i^* \Lambda_i (VD_1V^*) - tr(\Lambda_i^* \Lambda_i (VD_2V^*)) \end{aligned}$$

so $tr(\Lambda_i^* \Lambda_i (VD_1V^*)) = tr(\Lambda_i^* \Lambda_i (VD_2V^*))$ which implies $VD_1V^* = VD_2V^*$ and so $R = 0$, $T = S$.

$2 \to 3$ Let T be any trace class operators of trace zero such that $tr(\Lambda_i T \Lambda_i^*) = 0$ for all $i \in \mathbb{I}$. Define an operator S on $\mathcal{H}$ by $Se_1 = e_{1,}, Se_j = 0$ for $j = 2, 3, \ldots$, then S and $T + S$ are trace class, self-adjoint operators of trace one and $tr(T + S) = tr(T) + tr(S) = 0 + 1 = 1$. Finally, $tr(\Lambda_i^* \Lambda_i (T + S)) = tr(\Lambda_i^* \Lambda_i S)$ implies that $T + S = S$ which means $T = 0$.

$3 \to 1$ Let T, S are trace class, positive self-adjoint operators of trace one such that $tr(\Lambda_i^* \Lambda_i T) = tr(\Lambda_i^* \Lambda_i S)$ for all $i \in \mathbb{I}$.

$$tr(\Lambda_i^* \Lambda_i T) = tr(\Lambda_i^* \Lambda_i S) \Rightarrow tr(\Lambda_i^* \Lambda_i (T - S) = 0 \Rightarrow tr(T - S) = tr(T) - tr(S) = 0$$

therefor $T = S$.

□

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