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COMPUTABILITY OF THE TRANSLATION OPERATOR

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Abstract. Gabor systems play an important role in various practical problems related to signal and image processing. In this paper, we study the computability of Gabor frames in L^2. Computable version of some existing results related to Gabor systems are obtained. Also, computability of operators associated with Gabor frames is discussed and the computability of Zak transform is defined. Finally, we discuss the existence of the computable dual frames associated with the Gabor Systems.

1. Introduction

Frames are generalizations of orthonormal basis in Hilbert spaces. The notion of frames was introduced in 1952 by Duffin and Schaeffer [8]. Basis in a Hilbert space H allows every $f \in H$ to be represented as a unique expansion in terms of the basis elements. However, the condition of uniqueness of expansion is very restrictive in nature. Frames are particularly important because they provide the desired flexibility. The redundant nature of frames is a desirable property in many practical problems. Gabor frames play a very important role in signal analysis. They are generated by modulations and translations of one single function.

In this paper, we extend the notion of computability to Gabor frames. We first study the computability of Translation, Modulation and Dilation operators. We then deal with the computable Gabor frames and computable versions of some of the related results. The computability of the Zak Transform, an important tool for analyzing Gabor frames, is also discussed.

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2. Preliminaries

In this section, we briefly summarize some notions from computable analysis as presented in [17]. Computable Analysis is the Turing machine based approach to computability in analysis. Pioneering work in this field has been done by Turing [16], Grzegorczyk [9], Lacombe[13], Banach and Mazur[1], Pour-El and Richards[15], Kreitz and Weihrauch[11] and many others. The basic idea of the representation based approach to computable analysis is to represent infinite objects like real numbers, functions or sets by infinite strings over some alphabet Σ (which at least contains the symbols 0 and 1). Thus, a representation of a set X is a surjective function $\delta :\subseteq \Sigma^\omega \to X$ where Σ^ω denotes the set of infinite sequences over Σ and the inclusion symbol indicates that the mapping might be partial. Here, (X, δ) is called a represented space.

A function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ is said to be computable if there exists some Turing machine, which computes infinitely long and transforms each sequence p, written on the input tape, into the corresponding sequence $F(p)$, written on one way output tape. Between two represented spaces, we define the notion of a computable function.

Definition 2.1. [6] Let (X, δ), (Y, δ') be represented spaces. A function $f :\subseteq X \to Y$ is called (δ, δ')-computable if there exists a computable function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $\delta' F(p) = f\delta(p)$, for all $p \in dom(f\delta)$.

We simply call, a function f computable, if the represented spaces are clear from the context. For comparing two representations δ, $\delta^{'}$ of a set X, we have the notion of reducibility of representations. δ is called reducible to $\delta^{'}$, $\delta \leq \delta^{'}$ (in symbols), if there exists a computable function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $\delta(p) = \delta^{'} F(p)$, for all $p \in dom(\delta)$. This is equivalent to the fact that the identity $I : X \to X$ is $(\delta, \delta^{'})$- computable. If $\delta \leq \delta^{'}$ and $\delta^{'} \leq \delta$, then δ and $\delta^{'}$ are called computably equivalent. Analogous to the notion of computability, we can define the notion of $(\delta, \delta^{'})$-continuity, by substituting a continuous function F in the Definition 2.1. On Σ^ω, we use the Cantor topology, which is simply the product topology of the discrete topology on Σ.

Given a represented space (X, δ), a computable sequence is defined as a computable function $f : \mathbb{N} \to X$ where we assume that $\mathbb{N}$ is represented by $\delta_{\mathbb{N}}(1^n 0^\omega) = n$ and a point $x \in X$ is called computable if there is a constant computable sequence with value x. The notion of $(\delta, \delta^{'})$-continuity agrees with the ordinary topological notion of continuity, as long as, we are dealing with admissible representations.

A representation δ of a topological space X is called admissible if δ is continuous and if the identity $I : X \to X$ is (δ', δ)-continuous for any continuous representation δ' of X. If δ, δ', are admissible representation of topological spaces X,Y, then a function $f :\subseteq X \to Y$ is (δ, δ') continuous if and only if it is sequentially continuous [2, 3].

Given two represented spaces (X, δ), (Y, δ'), there is a canonical representation $[\delta \to \delta']$ of the set of (δ, δ')-continuous functions $f : X \to Y$. If δ and δ' are admissible representations of sequential topological spaces X and Y respectively, then $[\delta \to \delta']$ is actually a representation of the set $C(X, Y)$ of continuous functions $f : X \to Y$. The function space representation can be characterized by the fact that it admits evaluation and type conversion.

Definition 2.2 (Evaluation and Type Conversion)**.** Let δ_i be a representation of X_i, $i = 1, 2, 3$.

(a) **Evaluation-**$ev : C(X_1, X_2) \times X_1 \to X_2$ defined by $(f, x) \to f(x)$ is $([[\delta_1 \to \delta_2], \delta_1], \delta_2)$-computable.
(b) **Type Conversion-**For any total function $f : X_1 \times X_2 \to X_3$, define a function $T(f) : X_1 \to X_3^{X_2}$ by $T(f)(x)(y) = f(x, y)$.Then f is $((\delta_1, \delta_2), \delta_3)$ -computable iff T(f) is $(\delta_1, [\delta_2 \to \delta_3])$ - computable.
Here, $X_3^{X_2}$ denotes the set of all functions $f : X_2 \to X_3$.

If (X, δ), (Y, δ') are admissibly represented sequential topological spaces, then, in the following, we will always assume that $C(X, Y)$ is represented by $[\delta \to \delta']$. It follows by evaluation and type conversion that the computable points in $(C(X, Y), [\delta \to \delta'])$ are just the (δ, δ')-computable functions $f :\subseteq X \to Y$ [17]. For a represented space (X, δ), we assume that the set of sequences $X^{\mathbb{N}}$ is represented by $\delta^{\mathbb{N}} \equiv [\delta_{\mathbb{N}} \to \delta]$. The computable points in $(X^{\mathbb{N}}, \delta^{\mathbb{N}})$ are just the computable sequences in (X, δ).
The notion of computable metric space was introduced by Lacombe [**?**]. However, we state the following definition given by Brattka [5].

Definition 2.3. [5] A tuple (X, d, α) is called a *computable metric space* if

(1) (X, d) is a metric space.
(2) $\alpha : \mathbb{N} \to X$ is a sequence which is a dense in X.
(3) $d \circ (\alpha \times \alpha) : \mathbb{N}^2 \to \mathbb{R}$ is a computable (double) sequence in $\mathbb{R}$.

Given a computable metric space (X, d, α), its Cauchy Representation $\delta_X :\subseteq \Sigma^\omega \to X$ is defined as

$$\delta_X(01^{n_0+1}01^{n_1+1}01^{n_2+1}...) := \lim_{i\to\infty} \alpha(n_i),$$

for all $n_i \in \mathbb{N}$ such that $(\alpha(n_i))_{i\in\mathbb{N}}$ converges and

$$d(\alpha(n_i), \alpha(n_j)) < 2^{-i}, \text{ for all } j > i.$$

In the following, we assume that computable metric spaces are represented by their Cauchy representation. All Cauchy representations are admissible with respect to the corresponding metric topology.

An example of a computable metric space is $(\mathbb{R}, d_{\mathbb{R}}, \alpha_{\mathbb{R}})$ with the Euclidean metric given by $d_{\mathbb{R}}(x, y) = \|x - y\|$ and a standard numbering of a dense subset $Q \subseteq \mathbb{R}$ as $\alpha_{\mathbb{R}}\langle i, j, k\rangle = (i-j)/(k+1)$. Here, the bijective Cantor pairing function $\langle \cdot, \cdot \rangle : \mathbb{N}^2 \to \mathbb{N}$ is defined as $\langle i, j\rangle = j + (i+j)(i+j+1)/2$ and this definition can be extended inductively to finite tuples. It is known that the Cantor pairing function and the projections of its inverse are computable. In the following, we assume that $\mathbb{R}$ is endowed with the Cauchy representation $\delta_{\mathbb{R}}$ induced by the computable metric space given above.
Brattka gave the following definition of a computable normed linear space.

Definition 2.4. [5] A space $(X, \|\cdot\|, e)$ is called a *computable normed space* if

(1) $\|\cdot\| : X \to \mathbb{R}$ is a norm on X.
(2) The linear span of $e : \mathbb{N} \to X$ is dense in X.
(3) (X, d, α_e) with $d(x, y) =\| x - y \|$ and $\alpha_e\langle k, \langle n_0, ..., n_k\rangle\rangle = \sum_{i=0}^{k} \alpha_{\mathbb{F}}(n_i)e_i$ is a computable metric space with Cauchy representation δ_X.

Here, $\alpha_{\mathbb{F}}$ is a standard numbering of $\mathbb{Q}_{\mathbb{F}}$ where $\mathbb{Q}_{\mathbb{F}}$= $\mathbb{Q}$ in case of $\mathbb{F} = \mathbb{R}$ and $\mathbb{Q}_{\mathbb{F}} = \mathbb{Q}[i]$ in case of $\mathbb{F} = \mathbb{C}$.

It was observed that computable normed space is automatically a computable vector space, i.e., the linear operations are all computable. If the underlying space $(X, \|\cdot\|)$ is a Banach space then $(X, \|\cdot\|, e)$ is called a *computable Banach space*. If X is a computable Banach space, then a sequence $(e_i)_{i\in\mathbb{N}}$ is said to be a *computable basis* if it is a Schauder Basis that is computable in X.

We always assume that computable normed spaces are represented by their Cauchy representations, which are admissible with respect to norm topology. Two computable Banach spaces with the same underlying set are called computably equivalent if the corresponding Cauchy representations are computably equivalent.

Brattka gave the following definition of computable Hilbert spaces.

Definition 2.5. [4] A *computable Hilbert space* $(H, \langle\cdot\rangle, e)$ is a separable Hilbert space $(H, \langle . \rangle)$ together with a fundamental sequence $e : \mathbb{N} \to H$ such that the induced normed space is a computable normed space.

The standard dual space representation of the dual space $H^{'}$ is given by

$$\delta_{H'}\langle p, q\rangle = f \Leftrightarrow [\delta_H \to \delta_{\mathbb{F}}](p) = f \text{ and } \delta_{\mathbb{R}}(q) = \|f\|$$

for all $f \in H^{'}$. A computable isomorphism $T : H \to H$ is an isomorphism such that T as well as T^{-1} are computable.

A sequence $(f_i)_{i\in\mathbb{N}}$ of elements in a Hilbert space H is called a *frame* for H if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i\in\mathbb{N}} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \text{ for all } f \in H.$$

If only the upper inequality holds in above, then $(f_i)_{i\in\mathbb{N}}$ is said to be a *Bessel sequence* in H. The numbers A and B are called a lower and upper frame bound for the frame $(f_i)_{i\in\mathbb{N}}$. The *synthesis operator* T given by $T((c_i)) = \sum_{i=1}^{\infty} c_i f_i, (c_i)_{i\in\mathbb{N}} \in l^2$ associated with a frame (f_i) is a bounded operator from l^2 onto H. The adjoint operator $T^* : H \to l^2$ given by $T^*(f) = (\langle f, f_i\rangle)_{i\in\mathbb{N}}$ is called the *analysis operator*. The *frame operator* $S : H \to H$ given by $S(f) = TT^*(f) = \sum_{i\in\mathbb{N}}\langle f, f_i\rangle f_i$ associated with the frame (f_i) is a bounded, invertible and positive operator mapping H onto itself. The *frame decomposition*, stated below, shows that every element in H has a representation as a superposition of the frame elements.

$$f = \sum_{i\in\mathbb{N}}\langle f, S^{-1}f_i\rangle f_i = \sum_{i\in\mathbb{N}}\langle f, f_i\rangle S^{-1}f_i$$

for all $f \in H$. The series converge unconditionally for all $f \in H$. Thus, it is natural to view a frame as some kind of generalized basis. A sequence $(g_i)_{i\in\mathbb{N}}$ in H such that $f = \sum\langle f, g_i\rangle f_i$ for all $f \in H$, is called a dual of $(f_i)_{i\in\mathbb{N}}$. If (g_i) is a frame then it is called a **dual frame** of (f_i).

We denote the set of step functions on $\mathbb{R}$ with rational values by $RSF = \{\sum\limits_{j=0}^{n} c_j\chi_{(a_j,b_j)}, n \in \mathbb{N}, a_j, b_j \in \mathbb{Q}, c_j \in \mathbb{Q}^2, a_j \leq b_j, 0 \leq j \leq n\}$. Here, we identify the set $\mathbb{Q} + \iota\mathbb{Q}$ with $\mathbb{Q}^2$. The set RSF is a countable dense subset of L_p for $1 \leq p \leq \infty$. Let $\alpha_{RSF} :\subseteq \Sigma^* \to RSF$ denote some notation of the set RSF with recursive enumerable domain such that the mappings which assign to a step function $\sum\limits_{j=0}^{n} c_j\chi_{(a_j,b_j)}$, the end points a_j, b_j respectively, the values c_j are computable. The space $(L_p, \|.\|, RSF, \alpha_{RSF})$ is a computable metric space for any computable p with $1 \le p \le \infty$. See [12] for details.

3. Computability of Operators associated with Gabor Systems

Given $f : \mathbb{R} \to \mathbb{C}$, we define the translation of f by $a \in \mathbb{R}$ as $T_a f(x) = f(x-a)$. The following result shows that for a computable real number a, T_a is a computable map.

Theorem 3.1. *The map $\mathbb{R} \mapsto C(L^2(\mathbb{R}), L^2(\mathbb{R}))$ defined as $\alpha \to T_\alpha$ is $(\delta_{\mathbb{R}}, [\delta_{L^2(\mathbb{R})} \to \delta_{L^2(\mathbb{R})}])$ computable, where $T_\alpha : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the translation operator given by $T_\alpha f(x) = f(x-\alpha)$.*

Proof. For a $\delta_{\mathbb{R}}$ name of $\alpha \in \mathbb{R}$ and $\delta_{L^2(\mathbb{R})}$ name of $f \in L^2(\mathbb{R})$, which contains a coded list of names of elements $\sum\limits_{j=0}^{n} c_j \chi_{(a_j,b_j)} \in$ RSF such that for all $n \in \mathbb{N}$,

$$\|f - \sum_{j=0}^{n} c_j \chi_{(a_j,b_j)}\|_{L^2(\mathbb{R})} < 2^{-n-1},$$

W e have

$$\begin{aligned}
\|T_\alpha f - \sum_{j=0}^{n} c_j \chi_{(a_j+\alpha,b_j+\alpha)}\|^2_{L^2(\mathbb{R})} &= \int_{-\infty}^{\infty} |T_\alpha f(x) - \sum_{j=0}^{n} c_j \chi_{(a_j+\alpha,b_j+\alpha)}(x)|^2 \; dx \\
&= \int_{-\infty}^{\infty} |f(u) - \sum_{j=0}^{n} c_j \chi_{(a_j,b_j)}(u)|^2 \; du \\
&= \|f - \sum_{j=0}^{n} c_j \chi_{(a_j,b_j)}\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Thus, we get $\|T_\alpha f - \sum\limits_{j=0}^{n} c_j \chi_{(a_j+\alpha,b_j+\alpha)}\| < 2^{-n-1}$. Let $C = \sum\limits_{j=0}^{n} \sqrt{2}\ |c_j|$. For given $\delta_{\mathbb{R}}$ name of $\alpha \in \mathbb{R}$, we can get $\alpha_n \in \mathbb{Q}$ such that for all $n \in \mathbb{N}$, $|\alpha - \alpha_n| < 2^{-n}$. Choose $k \in \mathbb{N}$ such that $2^{-k} < \frac{2^{-n-1}}{C}$ and $2^{-k} < \min\limits_{j=0}^{n}\{|b_j - a_j|\}$.

Since $\|\chi_{(a_j+\alpha,b_j+\alpha)} - \chi_{(a_j+\alpha_{2k},b_j+\alpha_{2k})}\|^2 = 2|\alpha_{2k} - \alpha|$, we have

$$\begin{aligned}
\|\sum_{j=0}^{n} c_j \chi_{(a_j+\alpha,b_j+\alpha)} - \sum_{j=0}^{n} c_j \chi_{(a_j+\alpha_{2k},b_j+\alpha_{2k})}\| &\le \sum_{j=0}^{n} |c_j| \ \|\chi_{(a_j+\alpha,b_j+\alpha)} - \chi_{(a_j+\alpha_{2k},b_j+\alpha_{2k})}\| \\
&= C\sqrt{|\alpha_{2k} - \alpha|} \\
&\le 2^{-n-1}.
\end{aligned}$$

Thus, we get $\|T_\alpha f - \sum\limits_{j=0}^{n} c_j \ \chi_{(a_j+\alpha_{2k},b_j+\alpha_{2k})}\| < 2^{-n}$, where the α_{RSF} name of $\sum\limits_{j=0}^{n} c_j \ \chi_{(a_j+\alpha_{2k},b_j+\alpha_{2k})} \in RSF$ can be easily computed. $\square$

As $\|T_a\| = 1$, for all $a \in \mathbb{R}$, we obtain the following corollary.

Corollary 3.2. The map $\mathbb{R} \to B(L^2(\mathbb{R}), L^2(\mathbb{R}))$ defined as $\alpha \to T_\alpha$ is computable.

In [12], it was shown that the Fourier transform $\mathcal{F} : L_p \to L_q$ is $(\delta_{L_p}, \delta_{L_q})$ computable for $1 < p \le 2$ and $q = \frac{p}{p-1}$, where $\mathcal{F}f$ or $\hat{f}$ is defined by

$$\hat{f}(x) = \mathcal{F}f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(\xi)\ e^{-\iota x\xi}\ d\xi,$$

where $x \in \mathbb{R}, f \in L^2(\mathbb{R})$. Since $\mathcal{F}$ is unitary on $L^2(\mathbb{R})$, the inversion formula holds trivially on $L^2(\mathbb{R})$,

$$f = (\hat{f})^{\vee} = (\check{f})^{\wedge} \text{ for all } f \in L^2(\mathbb{R}),$$

where $\check{f} = \mathcal{F}^{-1}f$ is the inverse Fourier transform of f given by $\check{f}(x) = \hat{f}(-x)$. Here, the equality of functions in $L^2(\mathbb{R})$ holds pointwise only in the almost everywhere sense. It can be easily verified that the inverse Fourier transform $\mathcal{F}^{-1} : L_2 \to L_2$ defined by $f \mapsto \check{f}$ is $(\delta_{L_2}, \delta_{L_2})$ computable.

We now consider the Modulation operator on $L_2(\mathbb{R})$ defined by $(M_b f)(x) = e^{2\pi ibx} f(x),\ b, x \in \mathbb{R}$. It is well known that the Fourier transform interchanges translation with modulation. For all $f \in L^2(\mathbb{R})$, we have

$$(T_b f)^{\wedge}(\xi) = M_{-b}\hat{f}(\xi) \tag{3.1}$$

pointwise almost everywhere. We prove the computability of the modulation operator in the following:

Theorem 3.3. *The map* $\mathbb{R} \mapsto C(L^2(\mathbb{R}), L^2(\mathbb{R}))$ *given by* $b \to M_b$ *is* $(\delta_{\mathbb{R}}, [\delta_{L^2(\mathbb{R})} \to \delta_{L^2(\mathbb{R})}])$ *computable where* $M_b : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *is the modulation operator given by* $(M_b f)(x) = e^{2\pi ibx} f(x),\ x \in \mathbb{R}$.

Proof. The relation $\mathcal{F}(T_b\check{f}) = M_{-b}f$, for all $f \in L^2(\mathbb{R})$ can be obtained from (3.1) by replacing f by $\check{f}$ and using the fact that $f = (\check{f})^{\wedge}$, for all $f \in L^2(\mathbb{R})$. The map $(b, f) \to T_{-b}\check{f}$ is $([\delta_{\mathbb{R}}, \delta_{L^2(\mathbb{R})}], \delta_{L^2(\mathbb{R})})$ computable by Theorem 3.1. Since the Fourier transform $\mathcal{F} : L^2 \to L^2$ is $(\delta_{L^2(\mathbb{R})}, \delta_{L^2(\mathbb{R})})$ computable, we obtain the $([\delta_{\mathbb{R}}, \delta_{L^2(\mathbb{R})}], \delta_{L^2(\mathbb{R})})$ computability of the map $(b, f) \mapsto M_b f$. By Type conversion, the result follows. □

As Modulation is a unitary operator, we obtain the following :

Corollary 3.4. The map $\mathbb{R} \to B(L^2(\mathbb{R}), L^2(\mathbb{R}))$ given by $b \to M_b$ is computable.

The next result deals with the computability of the Dilation operator $D_c : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by $D_c f(x) = \frac{1}{\sqrt{|c|}} f(\frac{x}{c})$ for $c \neq 0$.

Theorem 3.5. *The map* $\phi :\subseteq \mathbb{R} \to C(L^2(\mathbb{R}), L^2(\mathbb{R}))$ *given by* $c \to D_c$*, with* $dom\phi = \{c \in \mathbb{R} : c > 0\}$ *is* $(\delta_{\mathbb{R}}, [\delta_{L^2(\mathbb{R})} \to \delta_{L^2(\mathbb{R})}])$ *computable, where* $D_c : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *is the Dilation operator given by* $D_c g(x) = \frac{1}{\sqrt{c}} g(\frac{x}{c})$.

Proof. For a given $\delta_{\mathbb{R}}$ name of c and $\delta_{L^2(\mathbb{R})}$ name of g, which contains a coded list of names of elements $\sum_{j=0}^{n} d_j \chi_{(a_j,b_j)} \in$ RSF such that

$$\|g - \sum_{j=0}^{n} d_j \chi_{(a_j,b_j)}\|_{L^2(\mathbb{R})} < 2^{-n-1},$$

for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \|D_c g - \sum_{j=0}^{n} \frac{d_j}{\sqrt{c}} \chi_{(ca_j, cb_j)}\|^2_{L^2(\mathbb{R})} &= \int_{-\infty}^{\infty} |\frac{1}{\sqrt{c}} g(\frac{x}{c}) - \frac{1}{\sqrt{c}} \sum_{j=0}^{n} d_j \chi_{(ca_j, cb_j)}(x)|^2 \, dx \\ &= \int_{-\infty}^{\infty} |g(x) - \sum_{j=0}^{n} d_j \chi_{(a_j, b_j)}(x)|^2 \, dx \\ &= \|g - \sum_{j=0}^{n} d_j \chi_{(a_j, b_j)}\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

Thus, we get $\|D_c g - \sum\limits_{j=0}^{n} \frac{d_j}{\sqrt{c}} \chi_{(ca_j, cb_j)}\| < 2^{-n-1}$. Now, let $R = \sum\limits_{j=0}^{n} |d_j| \sqrt{\frac{a_j + b_j}{c}}$ and for given $\delta_{\mathbb{R}}$ name of c, given $n \in \mathbb{N}$, we can get $c_n > c$ such that $|c_n - c| < 2^{-n}$,. Choose $k \in \mathbb{N}$ such that $2^{-k} < 2^{-n-1}/R$ and $(c_{2k} a_j, c_{2k} b_j) \cap (ca_j, cb_j) \neq \phi$ for all $j = 0, 1, ..., n$. Since $\|\frac{1}{\sqrt{c}} \chi_{(ca_j, cb_j)} - \frac{1}{\sqrt{c_{2k}}} \chi_{(c_{2k} a_j, c_{2k} b_j)}\|^2 < \frac{(a_j + b_j)}{c} |c_{2k} - c|$, we have

$$\begin{aligned} \|\sum_{j=0}^{n} \frac{d_j}{\sqrt{c}} \chi_{(ca_j, cb_j)} - \sum_{j=0}^{n} \frac{d_j}{\sqrt{c_{2k}}} \chi_{(c_{2k} a_j, c_{2k} b_j)}\| &\leq \sum_{j=0}^{n} |d_j| \; \|\frac{1}{\sqrt{c}} \chi_{(ca_j, cb_j)} - \frac{1}{\sqrt{c_{2k}}} \chi_{(c_{2k} a_j, c_{2k} b_j)}\| \\ &\leq R \sqrt{|c_{2k} - c|} < 2^{-n-1}. \end{aligned}$$

Thus, we get $\|D_c g - \sum\limits_{j=0}^{n} \frac{d_j}{\sqrt{c_{2k}}} \; \chi_{(c_{2k} a_j, c_{2k} b_j)}\| < 2^{-n}$, where the α_{RSF} name of $\sum\limits_{j=0}^{n} \frac{d_j}{\sqrt{c_{2k}}} \chi_{(c_{2k} a_j, c_{2k} b_j)}$ can be easily computed. □

The following result shows the computability of an operator defined in terms of the modulation and translation operators.

Theorem 3.6. *The map $\phi : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to C(L^2(\mathbb{R}), L^2(\mathbb{R}))$ defined by $(g_1, g_2) \to \phi_{g_1, g_2}$ where $\phi_{g_1, g_2} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is defined by*

$$\phi_{g_1, g_2}(f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle f, M_b T_a g_1 \rangle M_b T_a g_2 \; db \; da$$

is $([\delta_{L^2(\mathbb{R})}, \delta_{L^2(\mathbb{R})}], [\delta_{L^2(\mathbb{R})} \to \delta_{L^2(\mathbb{R})}])$ computable.

Proof. For $g_1, g_2 \in L^2(\mathbb{R})$ and $f \in L^2(\mathbb{R})$, define the map $\psi_{f, g_1, g_2} : L^2(\mathbb{R}) \to \mathbb{R}$ by $f_1 \mapsto \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle f, M_b T_a g_1 \rangle \langle M_b T_a g_2, f_1 \rangle \; db \; da$. Since

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle f, M_b T_a g_1 \rangle \langle M_b T_a g_2, f_1 \rangle \; db \; da = \langle f, f_1 \rangle \langle g_2, g_1 \rangle$$

by Proposition 8.1.2 [7], we get by Type conversion that the map $(f, g_1, g_2) \to \psi_{f, g_1, g_2}$ is $([\delta_{L^2(\mathbb{R})}, \delta_{L^2(\mathbb{R})}, \delta_{L^2(\mathbb{R})}], [\delta_{L^2(\mathbb{R})} \to \delta_{\mathbb{R}}])$ computable. Also, since $\|\psi_{f, g_1, g_2}\| = \|f\| |\langle g_2, g_1 \rangle|$, the above map is $([\delta_{L^2(\mathbb{R})}, \delta_{L^2(\mathbb{R})}, \delta_{L^2(\mathbb{R})}], \delta'_{L^2(\mathbb{R})})$ computable. By computable Theorem of Frechet-Riesz, Theorem 4.3 [4], we can get $\delta_{L^2(\mathbb{R})}$ name of the unique element $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle f, E_b T_a g_1 \rangle E_b T_a g_2 \; db \; da \in L^2(\mathbb{R})$. Now, by Type Conversion, the result follows. □

4. Computability of Gabor Frames

A Gabor system is a sequence in $L^2(\mathbb{R})$ of the form $G(g,a,b) = \{e^{2\pi ibnx}g(x-ak)\}_{k,n\in\mathbb{Z}}$, where $g \in L^2(\mathbb{R})$ and $a, b > 0$ are fixed. In terms of the translation and modulation operators, the system can be expressed as $G(g,a,b) = \{M_{bn}\, T_{ak}g\}_{k,n\in\mathbb{Z}}$. A Gabor system $G(g,a,b)$ is a Gabor frame for $L^2(\mathbb{R})$, if there exist two constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{k,n\in\mathbb{Z}} |\langle f, M_{bn}T_{ak}g\rangle|^2 \leq B\|f\|^2, \quad \text{for all} \quad f \in L^2(\mathbb{R}).$$

We now formally define a computable Gabor frame as follows:

Definition 4.1. A Gabor frame $G(g,a,b)$ is said to be a computable Gabor frame if the sequence $G(g,a,b)$ is computable as a double sequence.

Since the maps $(a,f) \mapsto T_af$ and $(b,f) \mapsto M_bf$ are $([\delta_{\mathbb{R}}, \delta_{L^2(\mathbb{R})}], \delta_{L^2(\mathbb{R})})$ computable, it can be easily observed that by Type conversion, the map $(a,b,g) \to ((n,k) \to M_{bn}T_{ak}g)$ is $([\delta_{\mathbb{R}}, \delta_{\mathbb{R}}, \delta_{L^2(\mathbb{R})}], ([\delta_{\mathbb{Z}}, \delta_{\mathbb{Z}}] \to \delta_{L^2(\mathbb{R})}))$ computable. Thus, given computable real numbers $a, b > 0$ and computable $g \in L^2(\mathbb{R})$, a Gabor frame $\{M_{bn}T_{ak}g\}_{k,n\in\mathbb{Z}}$ is a computable Gabor frame.

Remark 4.2. If $G(g,a,b)$ be a frame for $L^2(\mathbb{R})$, where $g \in L^2(\mathbb{R})$ is computable and $a, b > 0$ be computable real numbers, then for every computable real number $c > 0$, $G(D_cg, a/c, bc)$ forms a computable frame in $L^2(\mathbb{R})$. Also, since the Fourier transform is a computable operator on $L^2(\mathbb{R})$, the frame $G(\hat{g}, b, a)$ also forms a computable Gabor frame.

Example 4.3. The simplest example of a Gabor frame that forms a computable Gabor frame is $G(\chi_{[0,1]}, 1, 1) = \{e^{2\pi inx}\chi_{[k,k+1](x)}\}_{k,n\in\mathbb{Z}}$.

With functions g that is compactly supported in an interval of length $1/b$, we can easily create computable Gabor frames $G(g,a,b)$ for $L^2(\mathbb{R})$ as proved in the following result.

Theorem 4.4. *If $a, b > 0$ are computable real numbers such that $0 < ab < 1$, then there exists a computable element g in $L^2(\mathbb{R})$, supported in $[0, 1/b]$ such that $G(g,a,b)$ forms a computable frame for $L^2(\mathbb{R})$.*

Proof. Let $g = \chi_{[0,1/2b]} * \chi_{[0,1/2b]}$, the convolution of $\chi_{[0,1/2b]}$ with itself. Then, g is a continuous function such that $g(x) = 0$ outside $[0, 1/b]$ and $g(x) > 0$ on $(0, 1/b)$. Then $G(g,a,b)$ forms a frame for $L^2(\mathbb{R})$ by Theorem 11.4 [10]. Since the convolution map $* : L_1 \times L_p \to L_p$ is $([\delta_{L_1}, \delta_{L_p}], \delta_{L_p})$ computable by Theorem 3.4 [12], we get that g is a computable element of $L^2(\mathbb{R})$. Thus, $G(g,a,b)$ forms a computable frame for $L^2(\mathbb{R})$. □

We now consider the synthesis operator associated with a Gabor frame $G(g,a,b)$.

$$T : l^2(\mathbb{Z}^2) \to L^2(\mathbb{R})$$
$$T((c_{m,n})_{m,n\in\mathbb{Z}}) = \sum_{m,n\in\mathbb{Z}} c_{m,n}M_{bm}T_{an}g,$$

where $l^2(I) = \{(x_k)_{k\in I} \subseteq \mathbb{C} : \sum_{k\in I} |x_k|^2 < \infty\},\ I$ being a countable index set. The space $l^2(I)$ forms a Hilbert space with respect to the inner product $\langle \{x_k\}, \{y_k\}\rangle = \sum_{k\in I} x_k\ \overline{y}_k$. The sequence $\{e_{m,n}\}_{m,n\in\mathbb{Z}}$ forms an orthonormal basis for $l^2(\mathbb{Z}^2)$, where $e_{m,n} = \{\delta_{m,m'}\ \delta_{n,n'}\}_{m',n'\in\mathbb{Z}}$. Then, it can be easily verified that $(l^2(\mathbb{Z}^2), \langle\cdot\rangle, \{e_{m,n}\}_{m,n\in\mathbb{Z}})$ forms a computable Hilbert space if the representation induced by the countable orthonormal basis $\{e_{m,n}\}_{m,n\in\mathbb{Z}}$ is considered.

The following result shows that the synthesis operator associated to a computable Gabor frame is computable.

Theorem 4.5. *If $G(g, a, b) = \{M_{bn}T_{ak}g\}_{k,n\in\mathbb{Z}}$ be a computable Gabor frame for $L^2(\mathbb{R})$, then the synthesis operator $T : l^2(\mathbb{Z}^2) \to L^2(\mathbb{R})$ defined by $(c_{m,n})_{m,n\in\mathbb{Z}} = \sum_{m,n\in\mathbb{Z}} c_{m,n}M_{bn}T_{ak}g$ is a computable operator.*

Proof. As $G(g, a, b)$ is a frame for $L^2(\mathbb{R})$, T is a well defined bounded linear operator from $l^2(\mathbb{Z}^2)$ onto $L^2(\mathbb{R})$. The computability of T follows from the fact that the sequence $T((e_{m,n})_{m,n\in\mathbb{Z}}) = (M_{bm}T_{an}g)_{m,n\in\mathbb{Z}}$ forms a computable sequence in $L^2(\mathbb{R})$. □

The Zak transform is a fundamental tool for analyzing Gabor frames, especially for the case $ab = 1$. The Zak transform $Z : L^2(\mathbb{R}) \to L^2(\mathbb{Q})$ is defined as $f \mapsto Zf$, where $Zf : [0, 1] \times [0, 1] \to \mathbb{C}$ is given by

$$Zf(x, \xi) = \sum_{j\in\mathbb{Z}} f(x - j)\ e^{2\pi i j\xi}.$$

Here, the space $L^2(\mathbb{Q})$ is a Hilbert space, where $\mathbb{Q} = [0, 1]^2$, with norm and inner product given by

$$\|F\|^2_{L^2(\mathbb{Q})} = \int_0^1 \int_0^1 |F(x, \xi)|^2\ dx\ d\xi$$

and

$$\langle F, G\rangle = \int_0^1 \int_0^1 F(x, \xi)\ \overline{G(x, \xi)}\ dx\ d\xi.$$

The sequence $\{E_{n,k}\}_{n,k\in\mathbb{Z}}$, where $E_{n,k}(x, \xi) = e^{2\pi inx}\ e^{-2\pi ik\xi}$ forms an orthonormal basis for $L^2(\mathbb{Q})$. If we consider the representation induced by this orthonormal basis, then it can be easily verified that $(L^2(\mathbb{Q}), \langle\cdot\rangle, \{E_{n,k}\}_{n,k\in\mathbb{Z}})$ forms a computable Hilbert space. The Zak transform is the unique unitary map $Z : L^2(\mathbb{R}) \to L^2(\mathbb{Q})$ that satisfies $Z(M_nT_k\chi_{[0,1]}) = E_{n,k}$ for all $n, k \in \mathbb{Z}$.

Theorem 4.6. *The Zak transform $Z : L^2(\mathbb{R}) \to L^2(\mathbb{Q})$ defined as $Zf(x, \xi) = \sum_{j\in\mathbb{Z}} f(x - j)\ e^{2\pi ij\xi}$ is $(\delta_{L^2(\mathbb{R})}, \delta_{L^2(\mathbb{Q})})$ computable.*

Proof. The computable sequence $(\phi_{n,k})_{n,k\in\mathbb{Z}} = (M_nT_k\chi_{[0,1]})_{n,k\in\mathbb{Z}}$ forms an orthonormal basis of $L^2(\mathbb{R})$. Thus, the space $(L^2(\mathbb{R}), \langle\cdot\rangle, (\phi_{n,k})_{n,k\in\mathbb{Z}})$ is a computable Hilbert space computably equivalent to $(L^2(\mathbb{R}), \langle\cdot\rangle, RSF, \alpha_{RSF})$. Given $\delta_{L^2(\mathbb{R})}$ name of $f \in L^2(\mathbb{R})$ and precision $k \in \mathbb{N}$, we can effectively find $n, p \in \mathbb{N}$ such that $\|f - \phi_{n,p}\| < 2^{-k}$. Then

$$\begin{aligned}\|Zf - E_{n,p}\| &= \|Z(f - M_nT_p\chi_{[0,1]})\| \\ &= \|f - \phi_{n,p}\| \\ &< 2^{-k}.\end{aligned}$$

Hence, the map $Z : L^2(\mathbb{R}) \to L^2(\mathbb{Q})$ is $(\delta_{L^2(\mathbb{R})}, \delta_{L^2(\mathbb{Q})})$ computable. □

For $f, g \in L^2(\mathbb{R})$, $a, b > 0$ and $k \in \mathbb{Z}$, the series $\sum\limits_{n\in\mathbb{Z}} f(x-na)\overline{g(x-na-k/b)}$ converges absolutely for a.e. $x \in \mathbb{R}$. This defines a function $x \mapsto \sum\limits_{n\in\mathbb{Z}} |f(x-na)\overline{g(x-na-k/b)}|$ in $L^1(0, a)$. We prove the computability of such a function. We first prove the following result in this direction.

Theorem 4.7. *The map $\phi : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to L^1(\mathbb{R})$ defined as $(f, g) \to f\overline{g}$ is $([\delta_{L^2(\mathbb{R})}, \delta_{L^2(\mathbb{R})}], \delta_{L^1(\mathbb{R})})$ computable.*

Proof. Let $\delta_{L^2(\mathbb{R})}(p) = f$ and $\delta_{L^2(\mathbb{R})}(q) = g$, where $p = i(u_0)i(u_1)\dots$ and $g = i(v_0)i(v_1)\dots$ such that $\|f - \alpha_{RSF}(u_n)\| < 2^{-n}$ and $\|g - \alpha_{RSF}(v_n)\| < 2^{-n}$, for all $n \in \mathbb{N}$. Set $\tau_n = \alpha_{RSF}(u_n)$ and $\sigma_n = \alpha_{RSF}(v_n)$. Then

$$\begin{aligned}\|f\overline{g} - \tau_n\overline{\sigma_n}\| &\le \|f\|\ \|g - \sigma_n\| + \|f - \tau_n\|\ \|\sigma_n\| \\ &\le (1 + \|\tau_1\|)\|g - \sigma_n\| + \|f - \tau_n\|(1 + \|\sigma_1\|).\end{aligned}$$

Compute $n \in \mathbb{N}$ such that $2^{-n} < \min\{\frac{2^{-(k+1)}}{\|\tau_1\|+1}, \frac{2^{-(k+1)}}{\|\sigma_1\|+1}\}$.Then $\|f\overline{g} - \tau_n\overline{\sigma_n}\| < 2^{-(k+1)} + 2^{-(k+1)} = 2^{-k}$, where clearly α_{RSF} name of $\tau_n\overline{\sigma_n}$ can be computed. □

Theorem 4.8. *The map $L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to L^1(0, a)$ defined as $(f, g) \to \phi$ is $([\delta_{L^2(\mathbb{R})}, \delta_{L^2(\mathbb{R})}], \delta_{L^1(0,a)})$ computable, where $k \in \mathbb{Z}$, $a, b > 0$ are computable real numbers and $\phi : (0, a) \to \mathbb{R}$ is defined as $\phi(x) = \sum\limits_{n\in\mathbb{Z}} |f(x-na)\overline{g(x-na-k/b)}|$.*

Proof. The given map is well defined by Lemma 8.2.2 [7]. Since $\phi(x) = \sum\limits_{n\in\mathbb{Z}} |f\overline{T_{k/b}g(x-na)}|$, by Theorem 3.13, the map $(f, g) \mapsto f\overline{T_{k/b}g}$ is $([\delta_{L^2(\mathbb{R})}, \delta_{L^2(\mathbb{R})}], \delta_{L^1(\mathbb{R})})$ computable. Thus, it is sufficient to show that the map $f \mapsto \phi_f$ is $(\delta_{L^1(\mathbb{R})}, \delta_{L^1(0,a)})$ computable where $\phi_f : (0, a) \to \mathbb{R}$ is given by $\phi_f(x) = \sum\limits_{n\in\mathbb{Z}} |f(x-na)|$. Now, for a $\delta_{L^1(\mathbb{R})}$ name of f and precision $k \in \mathbb{N}$, we can get $\tau_k \in RSF$ such that $\|f - \tau_k\| < 2^{-k-1}$. Define

$\psi_k(x) = \sum\limits_{n\in\mathbb{Z}} |\tau_k(x-na)|$. Then

$$\begin{aligned}\|\phi - \psi_k\|_{L^1(0,a)} &= \int_0^a |\phi(x) - \psi_k(x)| \; dx \\ &= \int_0^a \Big| \sum_{n\in\mathbb{Z}} |f(x-na)| - \sum_{n\in\mathbb{Z}} |\tau_k(x-na)| \Big| \; dx \\ &\le \int_0^a \sum_{n\in\mathbb{Z}} |f(x-na) - \tau_k(x-na)| \; dx \\ &= \int_{-\infty}^{\infty} |f(x) - \tau_k(x)| \; dx \\ &< 2^{-k-1}.\end{aligned}$$

Now, it suffices to consider $\tau_k = \chi_{[c,d]}$ for some $c, d \in \mathbb{R}$. Then $\psi_k(x) = \sum\limits_{n\in\mathbb{Z}} |\chi_{[c+na,d+na]}(x)|$. Since only finitely many intervals of the type $[c+na, d+na]$ can intersect $[0,a]$, compute $n' \in \mathbb{N}$ in finitely many steps such that $[c+ka, d+ka] \cap [0,a] = \phi$, for all $k \notin \{-n', ..., n'\}$. Then

$$\|\psi_k - \sum_{k=-n'}^{n'} \chi_{[c+ka,d+ka]}\| \le \sum_{\substack{k\in\mathbb{Z} \\ k\neq\{-n',...,n'\}}} \int_0^a \chi_{[c+ka,d+ka]}(x) \; dx = 0.$$

Since a is a computable real number, $\delta_{L^2(\mathbb{R})}$ name of $\sum\limits_{k=-n'}^{n'} \chi_{[c+ka,d+ka]}$ can be computed. Hence, the result holds. □

Given a computable $g \in L^2(\mathbb{R})$ and computable real numbers $a, b > 0$, for the Gabor frame $G(g,a,b)$, the following theorem guarantees the existence of a corresponding computable element in $L^1(0,a)$.

Theorem 4.9. *If $g \in L^2(\mathbb{R})$ be computable and $a, b > 0$ be computable real numbers such that $G(g,a,b)$ is a frame for $L^2(\mathbb{R})$ with bounds $A, B > 0$, then there exists a computable element $G_0 \in L^1(0,a)$ such that $Ab \le G_0 \le Bb$ almost everywhere.*

Proof. By Theorem 11.6 [10], we have that if $G(g,a,b)$ is a frame for $L^2(\mathbb{R})$, then $G_0(x) = \sum\limits_{k\in\mathbb{Z}} |g(x-ak)|^2$ satisfies $Ab \le \sum\limits_{k\in\mathbb{Z}} |g(x-ak)|^2 \le Bb$ a.e.. Since $|g|^2$ is computable in $L^1(\mathbb{R})$, by Theorem 3.13, $\phi_g(x) = \sum\limits_{k\in\mathbb{Z}} \big||g|^2(x-ak)\big| = \sum\limits_{k\in\mathbb{Z}} |g(x-ak)|^2$ is computable in $L^1(0,a)$ by Theorem 3.14. Hence the result holds. □

The following result is a computable version of Proposition 9.3.8 [7]. This result gives a procedure to find some duals of a frame $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ having the Gabor structure. We give conditions to generate a computable dual frame.

Theorem 4.10. *Let $g \in L^2(\mathbb{R})$ be computable, $a, b > 0$ be computable real numbers. Let $(E_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$ be a frame for $L^2(\mathbb{R})$ with computable frame operator. Then, for any* *computable* $f \in L^2(\mathbb{R})$, *for which* $(E_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$ *is a Bessel sequence, the function*

$$h = S^{-1}g + f - \sum_{m,n\in\mathbb{Z}} \langle S^{-1}g, E_{mb}T_{na}g\rangle E_{mb}T_{na}f$$

generates a computable dual frame $(E_{mb}T_{na}h)_{m,n\in\mathbb{Z}}$ *of* $(E_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$.

Proof. Since $g \in L^2(\mathbb{R})$ is computable and $S : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is computable, we get $S^{-1}g$ is a computable element in $L^2(\mathbb{R})$. Again, as $(E_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$ is a computable sequence, the map $(m, n) \mapsto \langle S^{-1}g, E_{mb}T_{na}g\rangle$ is $([\delta_{\mathbb{Z}}, \delta_{\mathbb{Z}}], \delta_{\mathbb{R}})$ computable. Also as

$$\sum_{m,n} |\langle S^{-1}g, E_{mb}T_{na}g\rangle|^2 = \langle SS^{-1}g, S^{-1}g\rangle = \langle g, S^{-1}g\rangle.$$

We get that $(\langle S^{-1}g, E_{mb}T_{na}g\rangle)_{m,n\in\mathbb{Z}}$ is a computable sequence in $l^2(\mathbb{Z}^2)$. Since corresponding to the computable Bessel sequence $(E_{mb}T_{na}f)_{m,n\in\mathbb{Z}}$, the synthesis operator is computable, we obtain $\delta_{L^2(\mathbb{R})}$ computability of $\sum_{m,n\in\mathbb{Z}} \langle S^{-1}g, E_{mb}T_{na}g\rangle E_{mb}T_{na}f$. □

5. Conclusion

Computable analysis is the study of mathematical analysis from the standpoint of computability theory in mathematics and computer science. Mantry and Sharma [14] discussed the Computability of Hilbert Schmidt Operator. In this article, the computability of Gabor systems has been discussed for the first time and computable versions of some of the known results related to Gabor systems are obtained. Further, the computability of Zak transform is deliberated. Finally, the existence of the computable dual frames associated with the Gabor Systems is discussed.

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