Poincare Journal of Analysis & Applications
Vol. 11, No. 1 (2024), 15-34

DOI: 10.46753/pjaa.2024.v011i01.002
Online Published on 30. 04. 2024

TRANSCENDENTAL ENTIRE SOLUTIONS OF COMPLEX NONLINEAR PARTIAL DIFFERENTIAL DIFFERENCE EQUATIONS IN $\mathbb{C}^2$

ABHIJIT BANERJEE, JHUMA SARKAR, AND GAUTAM HALDAR†

Date of Receiving : 04. 08. 2023
Date of Revision : 30. 11. 2023
Date of Acceptance : 10. 01. 2024

***Abstract*.** In this paper, we explore the existence and forms of transcendental entire solutions of some second order partial differential-difference equations

$$f(z)^2 + 2\,\alpha\, f(z)\left\{L(z,f) + P_{L(f(z))}\right\} + \left\{L(z,f) + P_{L(f(z))}\right\}^2 = e^{g(z)},$$

$$f(z)^2 + 2\,\alpha\, f(z)\left\{a_1 f(z+c) + P_{L(f(z+c))}\right\} + \left\{a_1 f(z+c) + P_{L(f(z+c))}\right\}^2 = e^{g(z)},$$

and

$$P^2_{L(f(z))} + 2\,\alpha\, P_{L(f(z))} L(z,f) + L(z,f)^2 = e^{g(z)},$$

where $g(z)$ is a non-constant polynomial in $\mathbb{C}^2$, $L(z,f)$ and $P_{L(f(z))}$ are introduced in the Definition 2.1 known as first order linear shift operator and second order partial differential operator respectively. Our results are the extensions of some of the recent results due to Xu *et. al* [31], Li-Xu [19]. Moreover, we exhibit a series of examples in support of our results.

1. Introduction and Preliminaries

The well known classical Fermat-type functional equation is of the following form:

$$f(z)^n + g(z)^n = 1 \tag{1.1}$$

over the field of complex numbers $\mathbb{C}$, where n is a positive integer. Gross [6, 8, 7], Baker [1] and Montel [22], independently characterized the existence of entire and meromorphic solutions of (1.1) when $n \geq 2$. For detailed study, we insist the readers to go through [1, 6, 8, 7, 22]. In 2004, when $m = 2$, Yang and Li [32] studied (1.1) with the replacement

2010 *Mathematics Subject Classification.* 39A10, 30D35, 39B32.
Key words and phrases. Nevanlinna theory, quadratic trinomial, partial-differential operator, Fermat-type equations, finite order.
Communicated by: Hongyan Xu
† *Corresponding author*

of g with f' and proved that the transcendental entire solution of $f(z)^2+f'(z)^2=1$ has the form $f(z)=Ae^{\alpha z}/2+e^{-\alpha z}/2A$, where A,α are non-zero complex constants.

In recent years, the classical Fermat-type functional equation $f(z)^2+g(z)^2=1$ sparked interest among authors, leading to the investigation of various types of trinomial equations derived from classical Fermat-type functional equation [6]. This indicates the importance of this research direction. Trinomial equations, which involve three terms, often arise in different contexts and can exhibit rich and intriguing behavior. These equations are more complex than simple quadratic equations and naturally require novel techniques and approaches for their analysis to discover the interesting properties of the solutions and to extend the related equations.

Before going to the further discussion, we assume that readers are familiar with the basic notations of Nevalinna theory[10, 12, 33] such as $T(r,f)$, $m(r,f)$, $N(r,f)$, $N(r,\frac{1}{f})$, $S(r,f)$ etc in $\mathbb{C}$ as well as in $\mathbb{C}^2$. As far as the authors' knowledge are concerned, Saleeby [24] first considered quadratic functional equation

$$f^2+2\alpha fg+g^2=1, \tag{1.2}$$

and investigated the existence and form of transcendental entire and meromorphic solution in $\mathbb{C}^n$ and proved that the transcendental entire solution of (1.2), $\alpha^2\neq 1,0$ must be of the form

$$f(z)=\frac{1}{\sqrt{2}}\left(\frac{\cos(h(z))}{\sqrt{1+\alpha}}+\frac{\sin(h(z))}{\sqrt{1-\alpha}}\right)\ \text{ and }\ g(z)=\frac{1}{\sqrt{2}}\left(\frac{\cos(h(z))}{\sqrt{1+\alpha}}-\frac{\sin(h(z))}{\sqrt{1-\alpha}}\right),$$

where h is entire $\mathbb{C}^n$. The meromorphic solution of (1.2) must be of the form $f(z)=\dfrac{\alpha_1-\alpha_2\beta^2(z)}{(\alpha_1-\alpha_2)\beta(z)}$ and $g(z)=\dfrac{1-\beta^2(z)}{(\alpha_1-\alpha_2)\beta(z)}$, where $\beta(z)$ is meromorphic in $\mathbb{C}^n$ and $\alpha_1=-\alpha+\sqrt{\alpha^2-1}$, $\alpha_2=-\alpha-\sqrt{\alpha^2-1}$. Using this result and by utilizing difference analogue lemma of the logarithmic derivative lemma (see, [5, 9]), Liu and Yang [20] proved that if $\alpha^2\neq 1,0$, then the finite order transcendental entire solution of the trinomial difference equation

$$f(z)^2+2\,\alpha\,f(z)\,f(z+c)+f(z+c)^2=1 \tag{1.3}$$

must be of order equal to one. In 2021, *Luo et al.* [21] investigated the existence and exact form of transcendental entire solutions of the trinomial shift equation

$$f(z)^2+2\,\alpha\,f(z)\,f(z+c)+f(z+c)^2=e^{g(z)}, \tag{1.4}$$

where $g(z)$ is a polynomial in $\mathbb{C}^2$ and obtained the exact form of transcendental entire solutions as follows.

Theorem A. [21] Let $\alpha\neq 0,1$, $c\neq 0\in\mathbb{C}$ and $g(z)$ be a polynomial. If the difference equation (1.4) admits a finite order transcendental entire solution $f(z)$, then $g(z)$ must be of the form $g(z)=az+b$, where $a,b\in\mathbb{C}$. Furthermore, $f(z)$ must satisfy the following cases:

(i)

$$f(z)=\frac{1}{\sqrt{2}}(A_1\eta+A_2\eta^{-1})e^{\frac{1}{2}(az+b)},$$

where $\eta(\neq 0),a,c\in\mathbb{C}$ satisfying $e^{\frac{1}{2}ac}=\frac{(A_2\eta+A_1\eta^{-1})}{A_1\eta+A_2\eta^{-1}}$

(ii) $f(z) = \frac{1}{\sqrt{2}}(A_1 e^{a_1 z + b_1} + A_2 e^{a_2 z + b_2})$, where $a_j, b_j \in \mathbb{C}$, $(j = 1, 2)$ satisfy $a_1 \neq a_2$, $g(z) = (a_1 + a_2)z + b_1 + b_2 = az + b$, $e^{a_1 c} = \frac{A_2}{A_1}$, $e^{a_2 c} = \frac{A_1}{A_2}$ and $e^{ac} = 1$.

In the same paper, they have also considered the trinomial differential equation

$$f(z+c)^2 + 2\,\alpha\, f(z+c)f'(z) + f'(z)^2 = e^{g(z)}, \tag{1.5}$$

and obtained the exact form of transcendental entire solutions with finite order. For details, we refer the readers to go through [21].

The study of entire and meromorphic solutions for partial differential equations is a crucial aspect of understanding the behavior of various physical phenomena and mathematical models. Thats why partial differential equations occur in various areas of applied mathematics such as the propagation of heat or sound, fluid flow, elasticity, electrostatics, electrodynamics, thermodynamics, etc. In general, it is very much difficult task to find entire and meromorphic solutions for a nonlinear partial differential equation. In 1995, Khavinson [13] proved that any entire solutions of the partial differential equation $\left(\frac{\partial f}{\partial z_1}\right)^2 + \left(\frac{\partial f}{\partial z_2}\right)^2 = 1$ in $\mathbb{C}^2$ must be linear, i.e., $f(z_1, z_2) = c_1 z_1 + c_2 z_2 + c$, where $c_1, c_2, c \in \mathbb{C}$ such that $c_1^2 + c_2^2 = 1$. Thus the partial differential equation, mentioned in the work of Khavinson, is a nonlinear PDE but the result they proved about entire solutions being linear is significant. Later on, Li and his coauthors[18, 16, 17, 15] extended the investigation to variations of the original partial differential equation and made additional interesting discoveries.

After the development of the difference version of logarithmic derivative lemma for meromorphic functions in several complex variables [3, 4, 14], and its subsequent applications to the study of Fermat type partial differential-difference equations garnered significant attention from multiple researchers, resulting in various interesting findings [27, 28, 29, 30, 31, 34]. The investigations carried out by researchers in this area have focused on establishing existence results for solutions to specific types of Fermat type partial differential-difference equations, as well as analyzing the properties of these solutions. Understanding the existence and behaviour of solutions in such equations is crucial to gaining insights into the underlying mathematical models and the phenomena they describe.

In 2021, Xu *et al.* [19], extended Theorem A and the result corresponding to the equation (1.5) in the field of several complex variables. Before stating their result, let us make some following notations below.

Throughout the paper, by z and c, we mean $z = (z_1, z_2)$, $c = (c_1, c_2) \in \mathbb{C}^2$. We also use $z + c = (z_1 + c_1,\ z_2 + c_2)$ and set

$$A_1 = \frac{1}{2\sqrt{1+\alpha}} - \frac{i}{2\sqrt{1-\alpha}},\ A_2 = \frac{1}{2\sqrt{1+\alpha}} + \frac{i}{2\sqrt{1-\alpha}}. \tag{1.6}$$

Now, we state a result of Xu *et al.* [19] as follows.

Theorem B. ([19], Theorem 9) Let $\alpha^2 \neq 0, 1$, $c_1 \neq c_2$ and $g(z)$ be a non-constant polynomial in $\mathbb{C}^2$ not in the form of $\phi(z_2 - z_1)$. If $f(z)$ be a finite order transcendental entire solution of the partial differential-difference equation

$$f(z+c)^2 + 2\,\alpha\, f(z+c)\left(\frac{\partial f(z)}{\partial z_1} + \frac{\partial f(z)}{\partial z_2}\right) + \left(\frac{\partial f(z)}{\partial z_1} + \frac{\partial f(z)}{\partial z_2}\right)^2 = e^{g(z)},$$

then $f(z)$, $g(z)$ must satisfy one of the following cases:
(i) $g(z_1,\ z_2)=a_1z_1+a_2z_2+b$ and

$$f(z_1,\ z_2)=\frac{\sqrt{2}}{a_1+a_2}\left(A_2\xi+A_1\xi^{-1}\right)e^{\frac{1}{2}(a_1z_1+a_2z_2+b)},$$

where $\xi(\neq 0)\in\mathbb{C}$ and $a_1,\ a_2,\ c_1,\ c_2,\ \xi,\ A_1,\ A_2$ satisfying

$$e^{\frac{1}{2}(a_1c_1+a_2c_2)}=\frac{A_1\xi+A_2\xi^{-1}}{A_2\xi+A_1\xi^{-1}}(a_1+a_2);$$

(ii) $g(z_1,z_2)=L_1(z_1,z_2)+L_2(z_1,z_2)+b_1+b_2$ and

$$f(z_1,z_2)=\frac{1}{\sqrt{2}}\left(\frac{A_2}{a_{11}+a_{12}}e^{L_1(z_1,\ z_2)+b_1}+\frac{A_1}{a_{21}+a_{22}}e^{L_2(z_1,\ z_2)+b_2}\right),$$

where $L_1(z_1,\ z_2)=a_{11}z_1+a_{12}z_2$, $L_2(z_1,\ z_2)=a_{21}z_1+a_{22}z_2$ and $a_{ij},\ b_j\in\mathbb{C}$, $(i,j=1,2)$ satisfy $a_{11}+a_{12}\neq 0$, $a_{21}+a_{22}\neq 0$, $L_1(z_1,\ z_2)\neq L_2(z_1,\ z_2)$ and

$$e^{a_{11}c_1+a_{12}c_2}=\frac{A_1}{A_2}(a_{11}+a_{12}),\quad e^{a_{21}c_1+a_{22}c_2}=\frac{A_2}{A_1}(a_{21}+a_{22}).$$

2. Motivation

To proceed further we are now going to define first order linear c-shift operator and second order partial differential operator respectively as follows:

Definition 2.1. Let c be a non-zero constant in $\mathbb{C}^2$. We define
(i) $L(z,f)=a_0f(z)+a_1f(z+c),\ \ a_0,\ a_1\in\mathbb{C}\setminus\{0\}$ and
(ii) $P_{L(f(z))}=\sum_{l+k=1}^{2}b_{lk}\dfrac{\partial^{(l+k)}f(z)}{\partial z_1^l\partial z_2^k}$, where $b_{10},\ b_{01},\ b_{11},\ b_{20},\ b_{02}\in\mathbb{C}$.

From the discussion it's evident that the literature in this area is growing, and researchers are actively contributing to expanding the understanding of partial differential-difference equations with several variables. Generalizing results and exploring analogous results in different contexts can lead to valuable insights and a deeper understanding of the underlying mathematical structures. Generalizing the operators of Theorem B in terms of quadratic trinomial equations, we aim to extend the applicability of the theorem to a broader class of mathematical models so that they can capture additional complexities in the behaviour of solutions. This is the motivation of writing this paper. In fact, we will investigate Theorem B's analogous result in terms of the following three quadratic trinomial equations the third of which automatically includes Theorem B:

$$f(z)^2+2\ \alpha\ f(z)\left\{L(z,f)+P_{L(f(z))}\right\}+\left\{L(z,f)+P_{L(f(z))}\right\}^2=e^{g(z)},\tag{2.1}$$

$$f(z)^2+2\ \alpha\ f(z)\ \left\{a_1f(z+c)+P_{L(f(z+c))}\right\}+\left\{a_1f(z+c)+P_{L(f(z+c))}\right\}^2=e^{g(z)},\tag{2.2}$$

$$P_{L(f(z))}^2+2\ \alpha\ P_{L(f(z))}\ L(z,f)+L(z,f)^2=e^{g(z)},\tag{2.3}$$

where $g(z)$ is a polynomial in $\mathbb{C}^2$, $L(z,f)$ and $P_{L(f(z))}$ are defined in Definition 2.1.

Such findings can shed light on the connections between different mathematical objects and provide a unifying perspective. Throughout our research, we strive to encounter various techniques and methods for analysing these quadratic trinomial equations, as well as exploring the properties of their solutions. The development of analytical tools and approaches to handle these equations will be effective for the execution of our investigation.

3. Main Theorems and Examples

For convenience, throughout the paper we use the following notations.

$$\begin{cases} M' = \frac{1}{2}b_{10}\dfrac{\partial}{\partial z_1} + \dfrac{1}{2}b_{01}\dfrac{\partial}{\partial z_2} + b_{11}\left(\dfrac{1}{2}\dfrac{\partial^2}{\partial z_1 \partial z_2} + \dfrac{1}{4}\dfrac{\partial}{\partial z_1}\dfrac{\partial}{\partial z_2}\right) \\ \qquad +b_{20}\left(\dfrac{1}{2}\dfrac{\partial^2}{\partial z_1^2} + \dfrac{1}{4}\left(\dfrac{\partial}{\partial z_1}\right)^2\right) + b_{02}\left(\dfrac{1}{2}\dfrac{\partial^2}{\partial z_2^2} + \dfrac{1}{4}\left(\dfrac{\partial}{\partial z_2}\right)^2\right), \\ B = b_{20}c_2^2 + b_{02}c_1^2 - b_{11}c_1c_2, \\ E_1 = -b_{10}c_2 + b_{01}c_1 + \frac{1}{2}b_{11}\left(d_1c_1 - d_2c_2\right) - b_{20}d_1c_2 + b_{02}d_2c_1, \\ E_2 = -b_{10}c_2 + b_{01}c_1 + b_{11}(d_{11}c_1 - d_{12}c_2) - 2b_{20}d_{11}c_2 + 2b_{02}d_{12}c_1, \\ \tilde{E}_2 = -b_{10}c_2 + b_{01}c_1 + b_{11}(d_{21}c_1 - d_{22}c_2) - 2b_{20}d_{21}c_2 + 2b_{02}d_{22}c_1, \end{cases} \tag{3.1}$$

where d_1, d_2, d_{11}, d_{12}, d_{21}, d_{22} are constants in $\mathbb{C}$.

Now we list our main results as follows.

Theorem 3.1. Let $g(z)$ be a non-constant polynomial in $\mathbb{C}^2$, $\alpha \in \mathbb{C}$ such that $\alpha^2 \neq 0, 1$, $(c_1, c_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$ and $A_1, A_2, B, E_1, E_2, \tilde{E}_2$ are defined as in (3.1). Let $f(z)$ be a transcendental entire solution of (2.1), then $f(z)$ and $g(z)$ must satisfy one of the following forms:

(i) When $B \neq 0$ or $B = 0, E_1 \neq 0$, then $g(z) = L(z) + d$,

$$f(z) = \frac{1}{\sqrt{2}}\left(A_2\xi + A_1\xi^{-1}\right)e^{\frac{L(z)+d}{2}},$$

where $L(z) = d_1z_1 + d_2z_2$, $\xi(\neq 0)$, $a_1(\neq 0)$, d_1, d_2, $b_{ij} \in \mathbb{C}$ satisfying

$$\left[\frac{(A_1\xi + A_2\xi^{-1})}{a_1(A_2\xi + A_1\xi^{-1})} - \frac{a_0}{a_1} - \frac{1}{2a_1}\left\{b_{10}d_1 + b_{01}d_2 + \frac{1}{2}b_{11}d_1d_2 + \frac{1}{2}b_{20}d_1^2 + \frac{1}{2}b_{02}d_2^2\right\}\right]$$
$$= e^{\frac{1}{2}L(c)}.$$

(ii) When $B \neq 0$ or $B = 0$, $E_2 \neq 0$, $\tilde{E}_2 \neq 0$, then $g(z) = L_1(z) + L_2(z) + d_1 + d_2$ and

$$f(z) = \frac{1}{\sqrt{2}}\left(A_2e^{L_1(z)+d_1} + A_1e^{L_2(z)+d_2}\right),$$

where $L_1(z) = d_{11}z_1 + d_{12}z_2$, $L_2(z) = d_{21}z_1 + d_{22}z_2$, $a_1(\neq 0)$, $d_{ij} \in \mathbb{C}$, $i, j = 1, 2$ satisfying

$$e^{L_1(c)} = \frac{A_1}{a_1A_2} - \frac{a_0}{a_1} - \frac{1}{a_1}\left(b_{10}d_{11} + b_{01}d_{12} + b_{11}d_{11}d_{12} + b_{20}d_{11}^2 + b_{02}d_{12}^2\right),$$

$$e^{L_2(c)} = \frac{A_2}{a_1A_1} - \frac{a_0}{a_1} - \frac{1}{a_1}\left(b_{10}d_{21} + b_{01}d_{22} + b_{11}d_{21}d_{22} + b_{20}d_{21}^2 + b_{02}d_{22}^2\right)$$

and $L_1(z) \neq L_2(z)$.

The following examples justify Theorem 3.1.

Example 3.1. Let $\alpha = \frac{1}{2}$, $\xi = 1$, $c = (\pi i, -\pi i)$, $d_1 = 1$, $d_2 = 1$, $d = 0$, $b_{10} = b_{01} = b_{11} = b_{20} = b_{02} = 1$, $a_1 = \frac{1}{4}$, $a_0 = -1$. Then $f(z) = \frac{1}{\sqrt{3}} e^{\frac{z_1+z_2}{2}}$ is a solution of (2.1) with $g(z) = z_1 + z_2$.

Example 3.2. Let $\alpha = \frac{1}{2}$, $b_{10} = 1$, $b_{01} = -1$, $b_{11} = 1$, $b_{20} = 1$, $b_{02} = -1$, $c_1 = -\frac{\pi i}{3}$, $c_2 = \frac{\pi i}{3}$, $a_1 = 1$, $a_0 = -1$, $d_{11} = -3$, $d_{12} = 1$, $d_{21} = 1$, $d_{22} = 1$. Then $f(z) = \frac{1}{\sqrt{3}} \left(e^{-3z_1+z_2+\frac{\pi i}{3}} + e^{z_1+z_2-\frac{\pi i}{3}} \right)$ is a solution of (2.1) with $g(z) = -2z_1 + 2z_2$.

Corollary 3.1. Let $\alpha^2 \neq 0, 1$, $c = (c_1, c_2) \neq (0, 0)$. If degree of the polynomial $g(z)$ is more than one, then (2.1) can not have any finite order transcendental entire solution.

Remark 3.1. The next two examples, respectively show that under the condition $B = 0$, $E_1 = 0$ and under the condition $B = 0$, $E_2 = 0$, $\tilde{E}_2 = 0$, then (2.1) possesses different solutions than those provided in Theorem 3.1.

Example 3.3. Let $\alpha = \frac{1}{2}$, $a_0 = -1$, $a_1 = 1$, $b_{10} = 1$, $b_{01} = 1$, $b_{11} = 2$, $b_{20} = 1$, $b_{02} = 1$, $c_1 = 1$, $c_2 = 1$, d_1, d_2 be constants satisfying

$$e^{\frac{d_1+d_2}{2}} = 2 - \frac{1}{2}(d_1 + d_2 + d_1 d_2 + \frac{1}{2} d_1^2 + \frac{1}{2} d_2^2).$$

Then

$$f(z) = \frac{1}{\sqrt{3}} e^{\frac{d_1 z_1 + d_2 z_2 + G(-z_1+z_2)}{2}},$$

where $G(-z_1 + z_2)$ is a polynomial of degree greater than one with $g(z) = d_1 z_1 + d_2 z_2 + G(-z_1 + z_2)$, is a solution of equation (2.1) under *Theorem 3.1.*

Example 3.4. Let $c_1 = 1$, $c_2 = 1$, $b_{10} = 1$, $b_{01} = 1$, $b_{20} = b_{02} = 1$, $b_{11} = 2$, $a_0 = 1$, $a_1 = 1$. Also let d_{11}, d_{12}, d_{21}, d_{22} be constants satisfying

$$e^{d_{11}+d_{12}} = e^{\frac{-2\pi i}{3}} - 1 - \left\{ d_{11} + d_{12} + 2d_{11}d_{12} + d_{11}^2 + d_{12}^2 \right\}.$$

$$e^{d_{21}+d_{22}} = e^{\frac{2\pi i}{3}} - 1 - \left\{ d_{21} + d_{22} + 2d_{21}d_{22} + d_{21}^2 + d_{22}^2 \right\}.$$

Then

$$f(z) = \frac{1}{\sqrt{3}} \left(e^{d_{11}z_1 + d_{12}z_2 + \frac{\pi i}{3} + F_1(-z_1+z_2)} + e^{d_{21}z_1 + d_{22}z_2 - \frac{\pi i}{3} + F_1(-z_1+z_2)} \right),$$

where $F_1(-z_1 + z_2)$ is a polynomial of degree greater than one, $g(z) = (d_{11} + d_{21}) z_1 + (d_{21} + d_{22}) z_2 + 2F_1(-z_1 + +z_2)$, is a solution of equation (2.1) under *Theorem 3.1.*

Theorem 3.2. Let $g(z)$ be a non-constant polynomial in $\mathbb{C}^2$, $\alpha^2 \neq 0, 1$, $(c_1, c_2) \neq (0, 0) \in \mathbb{C}^2$ and A_1, A_2, B, E_1, E_2, $\tilde{E}_2$ are defined as in (3.1). Let $f(z)$ be a finite order transcendental entire solution of (2.2), then $g(z)$ and $f(z)$ must satisfy one of the following forms:

(i) When $B \neq 0$ or $B = 0, E_1 \neq 0$, then $g(z) = L(z) + d$ and

$$f(z) = \frac{1}{\sqrt{2}} \left(A_2 \xi + A_1 \xi^{-1} \right) e^{\frac{L(z)+d}{2}},$$

where $L(z)=d_1z_1+d_2z_2$, $\xi(\neq 0)\in\mathbb{C}$ and $a_1,\ A_1,\ A_2,\ \xi,\ d_1,\ d_2,\ b_{ij}\ (i,j=1,2)$ are constants satisfying

$$e^{-\frac{L(c)}{2}}=\frac{a_1(A_2\xi+A_1\xi^{-1})}{(A_1\xi+A_2\xi^{-1})}+\frac{1}{2}\left\{b_{10}d_1+b_{01}d_2+\frac{1}{2}b_{11}d_1d_2+\frac{1}{2}b_{20}d_1^2+\frac{1}{2}b_{02}d_2^2\right\}.$$

(ii) When $B\neq 0$ or $B=0$, $E_2\neq 0$, $\tilde{E}_2\neq 0$, then $g(z)=L_1(z)+L_2(z)+d_1+d_2$ with $L_1(z)\neq L_2(z)$ and

$$f(z)=\frac{1}{\sqrt{2}}\left(A_2e^{L_1(z)+d_1}+A_1e^{L_2(z)+d_2}\right),$$

where $L_1(z)=d_{11}z_1+d_{12}z_2$, $L_2(z)=d_{21}z_1+d_{22}z_2$, $a_1,\ d_{ij}\in\mathbb{C}$, $i,j=1,2$ satisfying

$$e^{-L_1(c)}=\frac{a_1A_2}{A_1}+\frac{A_2}{A_1}\left\{b_{10}d_{11}+b_{01}d_{12}+b_{11}d_{11}d_{12}+b_{20}d_{11}^2+b_{02}d_{12}^2\right\}\quad\text{and}$$

$$e^{-L_2(c)}=\frac{a_1A_1}{A_2}+\frac{A_1}{A_2}\left\{b_{10}d_{21}+b_{01}d_{22}+b_{11}d_{21}d_{22}+b_{20}d_{21}^2+b_{02}d_{22}^2\right\}.$$

The following examples justify Theorem 3.2.

Example 3.5. Let $\alpha=1/2$, $c_1=\pi i$, $c_2=3\pi i$, $a_1=3/4$, $b_{10}=b_{01}=b_{11}=b_{20}=b_{02}=1$, $d_1=1$, $d_2=-1$, $d=0$. Then $f(z)=\frac{1}{\sqrt{3}}e^{\frac{z_1-z_2}{2}}$ is a solution of (2.2) with $g(z)=z_1-z_2$.

Example 3.6. Let $\alpha=1/2$, $a_1=1$, $c_1=-\pi i/3$, $c_2=-\pi i/3$, $d_{11}=4$, $d_{12}=1$, $d_{21}=-1$, $d_{22}=2$, b_{10}, b_{01}, b_{11}, b_{20}, b_{02} be constants satisfying

$$4b_{10}+b_{01}+4b_{11}+16b_{20}+b_{02}=-2,\ -b_{10}+2b_{01}-2b_{11}+b_{20}+4b_{02}=-2,$$
$$-b_{11}+b_{20}^2+b_{02}^2\neq 0.$$

Then

$$f(z)=\frac{1}{\sqrt{3}}\left(e^{4z_1+z_2+d_1+\frac{\pi i}{3}}+e^{-z_1+2z_2+d_2-\frac{\pi i}{3}}\right),$$

is a solution of (2.2) with $g(z)=5z_1+3z_2+d_1+d_2$.

Corollary 3.2. Let $\alpha^2\neq 0,1$ and $c\neq(0,0)$ is a constant in $\mathbb{C}^2$. If degree of $g(z)$ in Theorem 3.2 is more than one, then (2.2) can not have any finite order transcendental entire solution.

The following example shows that we can get a solution of (2.2) even under the condition $E_2=0$ and $\tilde{E}_2=0$ and the order is not restricted to one.

Example 3.7. Let $\alpha=1/2$, $b_{10}=1$, $b_{01}=1$, $b_{11}=2$, $b_{20}=1$, $b_{02}=1$, $c_1=c_2=1$, $a_1=1$ and

$$e^{-(d_{11}+d_{12})}=e^{\frac{2\pi i}{3}}+e^{\frac{2\pi i}{3}}\left(d_{11}+d_{12}+2d_{11}d_{12}+d_{11}^2+d_{12}^2\right),$$

$$e^{-(d_{21}+d_{22})}=e^{-\frac{2\pi i}{3}}+e^{-\frac{2\pi i}{3}}\left(d_{21}+d_{22}+2d_{21}d_{22}+d_{21}^2+d_{22}^2\right).$$

Then

$$f(z)=\frac{1}{\sqrt{3}}\left(e^{d_{11}z_1+d_{12}z_2+d_1+F_1(-z_1+z_2)+\frac{\pi i}{3}}+e^{d_{21}z_1+d_{22}z_2+d_2+F_1(-z_1+z_2)-\frac{\pi i}{3}}\right),$$

where $F_1(-z_1+z_2)$ is a polynomial of degree properly greater than one and $g(z)=(d_{11}+d_{21})\,z_1+(d_{12}+d_{22})\,z_2+F_1(-z_1+z_2)$.

The next example shows that under the condition **(i)** if $E_1 = 0$, we get a different solution of Theorem 3.2.

Example 3.8. Let $\alpha = 1/2$, $b_{10} = 1$, $b_{01} = 1$, $b_{11} = 2$, $b_{20} = 1$, $b_{01} = 1$ and d_1, d_2, $c_1 = 1$, $c_2 = 1$ satisfying

$$e^{-\frac{d_1+d_2}{2}} = 1 + \frac{1}{2}\left(d_1 + d_2 + d_1d_2 + \frac{1}{2}d_1^2 + \frac{1}{2}d_2^2\right).$$

Then

$$f(z) = \frac{1}{\sqrt{3}} e^{\frac{d_1z_1+d_2z_2+F(-z_1+z_2)}{2}},$$

where $F(-z_1 + z_2)$ is a polynomial of degree greater than one, is a solution of equation (2.2) under Theorem 3.2 with $g(z) = d_1z_1 + d_2z_2 + F(-z_1 + z_2)$.

Theorem 3.3. Let $g(z)$ be a non-constant polynomial in $\mathbb{C}^2$ and $M'g \neq 0$, $\alpha^2 \neq 0, 1$, $(c_1, c_2) \neq (0, 0) \in \mathbb{C}^2$ and A_1, A_2, B, E_1, E_2, $\tilde{E}_2$ are defined as in (3.1). Let $f(z)$ be a finite order transcendental entire solution of (2.3), then $f(z)$ and $g(z)$ must satisfy one of the following forms:

(i) When $B \neq 0$ or $B = 0, E_1 \neq 0$, then $g(z) = L(z) + d$ and

$$f(z) = \frac{\sqrt{2}\left(A_2\xi + A_1\xi^{-1}\right)}{\left(b_{10}d_1 + b_{01}d_2 + \frac{1}{2}b_{11}d_1d_2 + \frac{1}{2}b_{20}d_1^2 + \frac{1}{2}b_{02}d_2^2\right)} e^{\frac{L(z)+d}{2}},$$

where $L(z) = d_1z_1 + d_2z_2$, $\xi(\neq 0)$, a_0, a_1, ξ, d_1, d_2, $b_{ij} \in \mathbb{C}$, $i, j = 1, 2$ satisfying

$$\begin{aligned}&\frac{\left(A_1\xi + A_2\xi^{-1}\right)}{2}\left\{b_{10}d_1 + b_{01}d_2 + \frac{1}{2}b_{11}d_1d_2 + \frac{1}{2}b_{20}d_1^2 + \frac{1}{2}b_{02}d_2^2\right\}\\ &= a_0(A_2\xi + A_1\xi^{-1}) + a_1(A_2\xi + A_1\xi^{-1})e^{\frac{L(c)}{2}}.\end{aligned}$$

(ii) When $B \neq 0$ or $B = 0$, $E_2 \neq 0$, $\tilde{E}_2 \neq 0$, then $g(z) = L_1(z) + L_2(z) + d_1 + d_2$ and

$$\begin{aligned}f(z) = \frac{1}{\sqrt{2}}&\left(\frac{A_2}{b_{10}d_{11} + b_{01}d_{12} + b_{11}d_{11}d_{12} + b_{20}d_{11}^2 + b_{02}d_{12}^2} e^{L_1(z)+d_1}\right.\\ &\left.+\frac{A_1}{b_{10}d_{21} + b_{01}d_{22} + b_{11}d_{21}d_{22} + b_{20}d_{21}^2 + b_{02}d_{22}^2} e^{L_2(z)+d_2}\right),\end{aligned}$$

where $L_1(z) = d_{11}z_1 + d_{12}z_2$, $L_2(z) = d_{21}z_1 + d_{22}z_2$ with $L_1(z) \neq L_2(z)$ and $a_0, a_1(\neq 0), d_{ij} \in \mathbb{C}$, $i, j = 1, 2$ satisfying

$$e^{L_1(c)} + \frac{a_0}{a_1} = \frac{A_1}{a_1A_2}\left(b_{10}d_{11} + b_{01}d_{12} + b_{11}d_{11}d_{12} + b_{20}d_{11}^2 + b_{02}d_{12}^2\right) \quad \text{and}$$

$$e^{L_2(c)} + \frac{a_0}{a_1} = \frac{A_2}{a_1A_1}\left(b_{10}d_{21} + b_{01}d_{22} + b_{11}d_{21}d_{22} + b_{20}d_{21}^2 + b_{02}d_{22}^2\right).$$

Remark 3.2. If we put $a_0 = 0$, $a_1 = 1$, $b_{10} = b_{01} = 1$, $b_{11} = b_{20} = b_{02} = 0$ in Theorem 3.3, we get Theorem B. Thus, Theorem 3.3 generalizes Theorem B.

The following examples show that results of Theorem 3.3 hold.

Example 3.9. Let $\alpha = 1/2$, $a_1 = 1$, $a_0 = -2$, $b_{10} = 1$, $b_{01} = -1$, $b_{11} = 4$, $b_{20} = -4$, $b_{02} = -4$, $\xi = 1$, $c_1 = 1$, $c_2 = -1$, $d_1 = 1$, $d_2 = 1$, $d = 1$. Then $f(z) = -\frac{1}{\sqrt{3}}e^{\frac{z_1+z_2+1}{2}}$ is a solution of (2.3) with $g(z) = z_1 + z_2 + 1$.

Example 3.10. Let $a_0 = -1$, $a_1 = 1$, $\alpha = \frac{1}{2}$, $c_1 = \frac{\pi i}{3}$, $c_2 = \frac{\pi i}{3}$, $d_{11} = 3$, $d_{12} = 2$, $d_{21} = 2$, $d_{22} = -1$, b_{10}, b_{01}, b_{11}, b_{20}, b_{02} are constants such that

$$3b_{10} + 2b_{01} + 6b_{11} + 9b_{20} + b_{02} = 1,\ 2b_{10} - b_{01} - 2b_{11} + 4b_{20} + b_{02} = 1,$$
$$-b_{11} + b_{20} + b_{02} \neq 0.$$

Then

$$f(z) = \frac{1}{\sqrt{3}}\left(e^{3z_1+2z_2+d_1+\frac{\pi i}{3}} + e^{2z_1-z_2+d_2-\frac{\pi i}{3}}\right),$$

is a solution of (2.3) with $g(z) = 5z_1 + z_2 + d_1 + d_2$.

Corollary 3.3. Let $c = (c_1, c_2) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ and $\alpha \in \mathbb{C}$ such that $\alpha^2 \neq 0, 1$. If the degree of $g(z)$ in *Theorem 3.3* is greater than one, then the equation (2.3) can not have any transcendental entire solution of finite order.

We see that under the condition **(i)** if $E_1 = 0$, equation (2.3) admits a solution in which degree of $g(z)$ is greater than one.

Example 3.11. Let $\alpha = 1/2$, $c_1 = c_2 = 1$, $b_{10} = 1$, $b_{01} = 1$, $b_{11} = 2$, $b_{20} = 1$, $b_{02} = 1$, $a_1 = 1$, $a_0 = -1$, d_1, d_2 be constants satisfying

$$e^{\frac{d_1+d_2}{2}} = 1 + \frac{1}{2}\left(d_1 + d_2 + d_1 d_2 + \frac{1}{2}d_1^2 + \frac{1}{2}d_2^2\right).$$

Then

$$f(z) = \frac{2}{\sqrt{3}\left(d_1 + d_2 + d_1 d_2 + \frac{1}{2}d_1^2 + \frac{1}{2}d_2^2\right)}e^{\frac{d_1 z_1 + d_2 z_2 + H(-z_1+z_2)}{2}},$$

$H(-z_1 + z_2)$ is a polynomial of degree greater than one, is a solution of (2.3) with $g(z) = d_1 z_1 + d_2 z_2 + H(-z_1 + z_2)$.

Then next example shows that under the condition **(ii)**, if $E_2 = 0$, $\tilde{E}_2 = 0$, equation (2.3) possesses a solution which is different from Theorem 3.3.

Example 3.12. Let $\alpha = -1/2$, $c_1 = c_2 = 1$, $b_{10} = 1$, $b_{01} = 1$, $b_{11} = 2$, $b_{20} = 1$, $b_{02} = 1$, $a_0 = -1$, $a_1 = 1$ and

$$e^{d_{11}+d_{12}} = 1 + e^{-\frac{2\pi i}{6}}\left\{d_{11} + d_{12} + 2d_{11}d_{12} + d_{11}^2 + d_{12}^2\right\}.$$

$$e^{d_{21}+d_{22}} = 1 + e^{\frac{2\pi i}{6}}\left\{d_{21} + d_{22} + 2d_{21}d_{22} + d_{21}^2 + d_{22}^2\right\}.$$

Then,

$$f(z) = \frac{1}{\sqrt{3}}\left(\frac{1}{d_{11} + d_{12} + 2d_{11}d_{12} + d_{11}^2 + d_{12}^2}e^{d_{11}z_1+d_{12}z_2+d_1+\frac{\pi i}{6}+F_1(-z_1+z_2)}\right.$$
$$\left.+\frac{1}{d_{21} + d_{22} + 2d_{21}d_{22} + d_{21}^2 + d_{22}^2}e^{d_{21}z_1+d_{22}z_2+d_2-\frac{\pi i}{6}+F_1(-z_1+z_2)}\right);$$

where $F_1(-z_1 + z_2)$ is a polynomial of degree greater than one is a solution of (2.3) with $g(z) = (d_{11} + d_{21})z_1 + (d_{12} + d_{22})z_2 + d_1 + d_2 + 2F_1(-z_1 + z_2)$.

4. Lemmas

The following lemmas will be useful in proving our results.

Lemma 4.1. [2] Let f be a non-constant meromorphic function in $\mathbb{C}^n$. Then for any $I \in (Z^+)^n$, $T(r, \partial^I f) = o(T(r,f))$ for all r except possibly a set of finite Lebesgue measure and where $I = (i_{1,\dots,i_n}) \in (Z^+)^n$ denotes a multiple index with $|I| = i_1 + i_2 + \cdots + i_n$, $Z^+ = 0, 1, 2, \dots$, and $\partial^I f = \frac{\partial^I f}{\partial^{i_n}\xi_n \dots \partial^{i_1}\xi_1}$.

Lemma 4.2. [25] For any entire function F in $\mathbb{C}^n$, $F(0) \neq 0$ and put $\rho(n_F) = \rho < \infty$. Then there exists a canonical function f_F and a function $g_F \in \mathbb{C}^n$ such that $F(z) = f_{F(z)} e^{g_{F(z)}}$. For special case $n = 1$, f_F is the canonical product of Weierstrass. Here $\rho(n_f)$ denotes the order of the counting function of zeros of F.

Lemma 4.3. [11] Let $f_j (\not\equiv 0)$, $j = 1, 2, 3$ be meromorphic functions in $\mathbb{C}^n$ such that f_1 is not constant, $f_1 + f_2 + f_3 = 1$ and

$$\sum_{j=1}^{3} \left\{ N_2\left(r, \frac{1}{f_j}\right) + 2\overline{N}(r, f_j) \right\} < \lambda T(r, f_1) + o(log^+ T(r, f_1)),$$

for all r outside possibly a set with finite logarithmic measure, where $\lambda < 1$ is a positive number, then either $f_2 \equiv 1$ or $f_3 \equiv 1$.

Lemma 4.4. [11] Suppose that $a_0(z)$, $a_1(z)$, $\dots$, $a_n(z)$ $(n \geq 1)$ are meromorphic functions on $\mathbb{C}^n$ and $g_0(z)$, $g_1(z)$, $\dots$, $g_n(z)$ are entire functions in $\mathbb{C}^n$ such that $g_j(z) - g_k(z)$ are not constants for $0 \leq j < k \leq n$. If

$$\sum_{j=0}^{n} a_j(z) e^{g_j(z)} = 0 \text{ and } T(r, a_j) = o(T(r)),\ j = 0, 1, 2, \dots n,$$

where $T(r) = \min_{0 \leq j < k \leq n} \ T(r, e^{g_k - g_j})$, then $a_j(z) \equiv 0, j = 0, 1, 2, \dots, n$.

5. Proofs of the Theorems

Proof of Theorem 3.1. Let f be a finite order transcendental entire solution of (2.1) in $\mathbb{C}^2$. Let us take

$$u = \frac{1}{\sqrt{2}} \left\{ \left(P_{L(f(z))} + L(z, f)\right) + f(z) \right\},$$

$$v = \frac{1}{\sqrt{2}} \left\{ \left(P_{L(f(z))} + L(z, f)\right) - f(z) \right\}.$$

From the above two equations, we get

$$L(z, f) + P_{L(f(z))} = \frac{1}{\sqrt{2}}(u + v) \text{ and } f(z) = \frac{1}{\sqrt{2}}(u - v). \tag{5.1}$$

Then, (2.1) turns into

$$(1 + \alpha)u^2 + (1 - \alpha)v^2 = e^{g(z)}.$$

This leads to

$$\left(\frac{\sqrt{1+\alpha}\ u}{e^{\frac{g(z)}{2}}}\right)^2 + \left(\frac{\sqrt{1-\alpha}\ v}{e^{\frac{g(z)}{2}}}\right)^2 = 1,$$

which implies

$$\left(\frac{\sqrt{1+\alpha}\,u}{e^{\frac{g(z)}{2}}}+i\frac{\sqrt{1-\alpha}\,v}{e^{\frac{g(z)}{2}}}\right)\left(\frac{\sqrt{1+\alpha}\,u}{e^{\frac{g(z)}{2}}}-i\frac{\sqrt{1-\alpha}\,v}{e^{\frac{g(z)}{2}}}\right)=1.$$

Since f is a finite order transcendental entire function and $g(z)$ is a polynomial, in view of Lemma 4.2, there exists a polynomial $p(z)$ in $\mathbb{C}^2$ such that

$$\frac{\sqrt{1+\alpha}\,u}{e^{\frac{g(z)}{2}}}+i\frac{\sqrt{1-\alpha}\,v}{e^{\frac{g(z)}{2}}}=e^{p(z)}, \tag{5.2}$$

$$\frac{\sqrt{1+\alpha}\,u}{e^{\frac{g(z)}{2}}}-i\frac{\sqrt{1-\alpha}\,v}{e^{\frac{g(z)}{2}}}=e^{-p(z)}. \tag{5.3}$$

$$\text{Let}\quad \sigma_1(z)=\frac{g(z)}{2}+p(z)\ \text{ and }\ \sigma_2(z)=\frac{g(z)}{2}-p(z). \tag{5.4}$$

From (5.2) and (5.3), we get

$$u=\frac{e^{\sigma_1(z)}+e^{\sigma_2(z)}}{2\sqrt{1+\alpha}}\ \text{ and }\ v=\frac{e^{\sigma_1(z)}-e^{\sigma_2(z)}}{2i\sqrt{1-\alpha}}. \tag{5.5}$$

So, from (5.1) and (5.5), we have

$$L(z,f)+P_{L(f(z))}=\frac{1}{\sqrt{2}}\left(A_1e^{\sigma_1(z)}+A_2e^{\sigma_2(z)}\right) \tag{5.6}$$

$$\text{and}\qquad f(z)=\frac{1}{\sqrt{2}}\left(A_2e^{\sigma_1(z)}+A_1e^{\sigma_2(z)}\right), \tag{5.7}$$

where A_1, A_2 are defined as in (3.1).

Now, in view of (5.6) and (5.7), we obtain

$$\begin{aligned}&\left(\frac{A_1-A_2M_1(z)-a_0A_2}{a_1A_2}\right)e^{\sigma_1(z)-\sigma_1(z+c)}-\frac{A_1}{A_2}e^{\sigma_2(z+c)-\sigma_1(z+c)}\\&+\left(\frac{A_2-A_1M_2(z)-a_0A_1}{a_1A_2}\right)e^{\sigma_2(z)-\sigma_1(z+c)}=1,\end{aligned} \tag{5.8}$$

where

$$\begin{aligned}M_j(z)\ &=\ b_{10}\frac{\partial\sigma_j(z)}{\partial z_1}+b_{01}\frac{\partial\sigma_j(z)}{\partial z_2}+b_{11}\left(\frac{\partial^2\sigma_j(z)}{\partial z_1\partial z_2}+\frac{\partial\sigma_j(z)}{\partial z_1}\frac{\partial\sigma_j(z)}{\partial z_2}\right)\\&+\ b_{20}\left(\frac{\partial^2\sigma_j(z)}{\partial z_1^2}+\left(\frac{\partial\sigma_j(z)}{\partial z_1}\right)^2\right)+b_{02}\left(\frac{\partial^2\sigma_j(z)}{\partial z_2^2}+\left(\frac{\partial\sigma_j(z)}{\partial z_2}\right)^2\right),\ j=1,2\end{aligned}$$

Now we consider the following two possible cases.

Case 1: Let $\sigma_2(z+c)-\sigma_1(z+c)$ is a constant. This implies $p(z)$ is a constant. Let $e^{p(z)}=\xi$. Then, in view of (5.7) and (5.8) we get

$$f(z)=\frac{1}{\sqrt{2}}\left(A_2\xi+A_1\xi^{-1}\right)e^{\frac{g(z)}{2}} \tag{5.9}$$

$$L(z,f)+P_{L(f(z))}=\frac{1}{\sqrt{2}}\left(A_1\xi+A_2\xi^{-1}\right)e^{\frac{g(z)}{2}}. \tag{5.10}$$

Equations (5.9) and (5.10) lead to

$$\begin{aligned}&\left[(A_1\xi+A_2\xi^{-1})-a_0(A_2\xi+A_1\xi^{-1})-(A_2\xi+A_1\xi^{-1})M'g\right]\\&=a_1(A_2\xi+A_1\xi^{-1})e^{\frac{g(z+c)-g(z)}{2}}.\end{aligned}\tag{5.11}$$

Suppose $g(z+c)-g(z)$ is not a constant. Since $e^{\frac{g(z+c)-g(z)}{2}}$ has no zeros and no poles, from (5.11), we see that the only possible case is both $A_1\xi+A_2\xi^{-1}=0$ and $A_2\xi+A_1\xi^{-1}=0$. Otherwise, one side is polynomial and the other side is transcendental. But, this lead to $A_1^2=A_2^2$ and which implies $\alpha^2=0,1$, a contradiction. Hence, $g(z+c)-g(z)$ is a constant. This implies that $g(z)=L(z)+H(s)+d$, where $L(z)=d_1z_1+d_2z_2$, $H(s)$ is a polynomial in $s:=e_2z_1+e_1z_2$ such that

$$e_2c_1+e_1c_2=0.\tag{5.12}$$

From (5.11), we see that

$$\begin{aligned}&(A_1\xi+A_2\xi^{-1})-a_0(A_2\xi+A_1\xi^{-1})-(A_2\xi+A_1\xi^{-1})N\\&=a_1(A_2\xi+A_1\xi^{-1})e^{\frac{L(c)}{2}},\end{aligned}\tag{5.13}$$

where

$$\begin{aligned}N\;=\;&\left\{\frac{1}{2}b_{10}(d_1+H'(s)e_2)+\frac{1}{2}b_{01}(d_2+H'(s)e_1)+b_{11}\left(\frac{1}{2}H''(s)e_1e_2\right.\right.\\&\left.+\frac{1}{4}(d_1+H'(s)e_2)(d_2+H'(s)e_1)\right)+b_{20}\left(\frac{1}{2}H''(s)e_2^2+\frac{1}{4}\left(d_1+H'(s)e_2\right)^2\right)\\&\left.+b_{02}\left(\frac{1}{2}H''(s)e_1^2+\frac{1}{4}\left(d_2+H'(s)e_1\right)^2\right)\right\}.\end{aligned}$$

For the sake of convenience, let us denote the coefficients of $H'(s)$, $H'(s)^2$, $H'(s)^2$ and the constant term of (5.13) by A, B', B'' and C, respectively. That is to say $A=\frac{1}{2}b_{10}e_2+\frac{1}{2}b_{01}e_1+\frac{1}{4}b_{11}(d_1e_1+d_2e_2)+\frac{1}{2}b_{20}d_1e_2+\frac{1}{2}b_{02}d_2e_1$, $B'=\frac{1}{4}b_{11}e_1e_2+\frac{1}{4}b_{20}e_2^2+\frac{1}{4}b_{02}e_1^2$, $B''=2B'$, $C=\frac{1}{2}b_{10}d_1+\frac{1}{2}b_{01}d_2+\frac{1}{4}b_{11}d_1d_2+\frac{1}{4}b_{20}d_1^2+\frac{1}{4}b_{02}d_2^2$.

Now, we discuss the four possible cases as follows. **(i)** $A\neq 0$, $B'\neq 0$. **(ii)** $A=0$, $B'\neq 0$. **(iii)** $A\neq 0$, $B'=0$. **(iv)** $A=0$, $B'=0$.

Since one side of (5.13) is constant and other side is polynomial comparing degree of $H(s)$ in both sides of (5.13) under the cases (**i**)-(**iii**), it is easy to see that the degree of $H(s)\leq 1$. Under the case (**iv**), $H(s)$ is an arbitrary polynomial.

Now, in view of(5.12) and the fact that $B\neq 0$, we have (**i**) or (**ii**). Whereas using (5.12) and the fact that $B=0$ and $E_1\neq 0$, we get (**iii**). So let $H(s)=l(e_2z_1+e_1z_2)+d$; l, d are constants in $\mathbb{C}$. Therefore $g(z)=L(z)+H(s)+d$.

Since $g(z)$ becomes linear, we may take $H(s)\equiv 0$. Hence (5.11) we get

$$\begin{aligned}&\left[\frac{(A_1\xi+A_2\xi^{-1})}{a_1(A_2\xi+A_1\xi^{-1})}-\frac{a_0}{a_1}-\frac{1}{2a_1}\left\{b_{10}d_1+b_{01}d_2+\frac{1}{2}b_{11}d_1d_2+\frac{1}{2}b_{20}d_1^2+\frac{1}{2}b_{02}d_2^2\right\}\right]\\&=e^{\frac{L(c)}{2}}.\end{aligned}$$

Hence, we get from (5.9)

$$f(z)=\frac{1}{\sqrt{2}}\left(A_2\xi+A_1\xi^{-1}\right)e^{\frac{L(z)+d}{2}}.$$

Case 2: Let $\sigma_2(z+c)-\sigma_1(z+c)$ is not a constant.

In (5.8), let us denote $Q_1=\left(\frac{A_1-A_2M_1(z)-a_0A_2}{a_1A_2}\right)$ and $Q_2=\left(\frac{A_2-A_1M_2(z)-a_0A_1}{a_1A_2}\right)$.

Subcase 2.1: Let $Q_1\equiv 0$ and $Q_2\equiv 0$. Then from (5.8) we get

$$-\frac{A_1}{A_2}e^{\sigma_2(z+c)-\sigma_1(z+c)}=1,$$

which shows that $\sigma_2(z+c)-\sigma_1(z+c)$ is a constant, contradicts our assumption.

Subcase 2.2: Let $Q_1\equiv 0$ and $Q_2\not\equiv 0$. Then from (5.8) we get

$$Q_2e^{\sigma_2(z)-\sigma_1(z+c)}-\frac{A_1}{A_2}e^{\sigma_2(z+c)-\sigma_1(z+c)}=1, \tag{5.14}$$

which shows that $\sigma_2(z)-\sigma_1(z+c)$ is not a constant. Now (5.14) can be rewritten as

$$Q_2e^{\sigma_2(z)}-\frac{A_1}{A_2}e^{\sigma_2(z+c)}-e^{\sigma_1(z+c)}=0. \tag{5.15}$$

Now we claim that $\sigma_2(z+c)-\sigma_2(z)$ is non-constant. On the contrary Let $\sigma_2(z+c)-\sigma_2(z)=k$, then $\sigma_2(z+c)=k+\sigma_2(z)$, then from (5.14) we see that

$$\left(Q_2e^{-k}-\frac{A_1}{A_2}\right)e^{\sigma_2(z+c)-\sigma_1(z+c)}=1,$$

which is a contradiction. Now applying Lemma 4.4 in (5.15) we get $Q_2\equiv 0$, again a contradiction.

Subcase 2.3: Let $Q_1\not\equiv 0$ and $Q_2\equiv 0$. Then by similar arguments as used in Subcase 2.2 we get a contradiction.

Subcase 2.4: Let $Q_1\not\equiv 0$ and $Q_2\not\equiv 0$. Then using Lemma 4.3 in (5.8) we get

$$Q_1e^{\sigma_1(z)-\sigma_1(z+c)}\equiv 1,$$

or,

$$Q_2e^{\sigma_2(z)-\sigma_1(z+c)}\equiv 1.$$

Subcase 2.4.1: Now if $Q_1e^{\sigma_1(z)-\sigma_1(z+c)}\equiv 1$, it shows that $\sigma_1(z)-\sigma_1(z+c)$ is constant otherwise one side is polynomial and one side is transcendental. Let $\sigma_1(z)=L_1(z)+H_1(s)+d_1$ where $L_1(z)=d_{11}z_1+d_{12}z_2$, $H_1(s)$ is a polynomial in s, $s=e_{12}z_1+e_{11}z_2$ such that

$$e_{12}c_1+e_{11}c_2=0. \tag{5.16}$$

and d_1 is a constant in $\mathbb{C}$. Thus we get

$$\frac{A_1}{a_1A_2}-\frac{a_0}{a_1}-\frac{1}{a_1}N'=e^{L_1(c)}, \tag{5.17}$$

where

$$\begin{aligned} N' \;=\; & \{b_{10}(d_{11}+H_1'(s)e_{12})+b_{01}(d_{12}+H_1'(s)e_{11}) \\ & +b_{11}\left(H_1''(s)e_{11}e_{12}+(d_{11}+H_1'(s)e_{12})(d_{12}+H_1'(s)e_{11})\right) \\ & +b_{20}\left(H_1''(s)e_{12}^2+(d_{11}+H_1'(s)e_{12})^2\right)+b_{02}\left(H_1''(s)e_{11}^2+(d_{12}+H_1'(s)e_{11})^2\right)\Big\}. \end{aligned}$$

Now, we proceed with the similar arguments as done in Case 1. Here using $B\neq 0$ and (5.16) or using $B=0$ and $E_2\neq 0$ with (5.16), we get $H_1(s)\equiv 0$.

Then, from (5.17), we obtain

$$e^{L_1(c)} = \frac{A_1}{a_1A_2} - \frac{a_0}{a_1} - \frac{1}{a_1}\left(b_{10}d_{11} + b_{01}d_{12} + b_{11}d_{11}d_{12} + b_{20}d_{11}^2 + b_{02}d_{12}^2\right).$$

In view of (5.8), we get

$$\left[\frac{A_2}{a_1A_1} - \frac{a_0}{a_1} - \frac{1}{a_1}M_2(z)\right]e^{\sigma_2(z)-\sigma_2(z+c)} \equiv 1. \tag{5.18}$$

In view of (5.18), it follows that $\sigma_2(z) - \sigma_2(z+c)$ is a constant. Otherwise one side is polynomial and one side is transcendental. Let $\sigma_2(z) = L_2(z) + H_2(\tilde{s}) + d_2$ where $L_2(z) = d_{21}z_1 + d_{22}z_2$, $H_2(\tilde{s})$ is a polynomial in $\tilde{s}$, $\tilde{s} = e_{22}z_1 + e_{21}z_2$ such that $e_{22}c_1 + e_{21}c_2 = 0$.

From (5.18), we get

$$\frac{A_2}{a_1A_1} - \frac{a_0}{a_1} - \frac{1}{a_1}N'' = e^{L_2(c)}, \tag{5.19}$$

where

$$\begin{aligned} N'' \;=\; & \{b_{10}(d_{21} + H_2'(\tilde{s})e_{22}) + b_{01}(d_{22} + H_2'(\tilde{s})e_{21}) \\ & + b_{11}\left(H_2''(\tilde{s})e_{21}e_{22} + (d_{21} + H_2'(\tilde{s})e_{22})(d_{22} + H_2'(\tilde{s})e_{21})\right) \\ & + b_{20}\left(H_2''(\tilde{s})e_{22}^2 + (d_{21} + H_2'(\tilde{s})e_{22})^2\right) + b_{02}\left(H_2''(\tilde{s})e_{21}^2 + (d_{22} + H_2'(\tilde{s})e_{21})^2\right)\Big\}. \end{aligned}$$

Now, we proceed with the similar arguments as done in Case 1, using $B \neq 0$ and $e_{22}c_1 + e_{21}c_2 = 0$ or using $B = 0$ and $\tilde{E_2} \neq 0$ with $e_{22}c_1 + e_{21}c_2 = 0$, we get $H_2(\tilde{s}) \equiv 0$. Hence, $\sigma_2(z) = L_2(z) + d_2$ and from (5.19) we get

$$e^{L_2(c)} = \frac{A_2}{a_1A_1} - \frac{a_0}{a_1} - \frac{1}{a_1}\left(b_{10}d_{21} + b_{01}d_{22} + b_{11}d_{21}d_{22} + b_{20}d_{21}^2 + b_{02}d_{12}^2\right).$$

From (5.7) we get

$$f(z) = \frac{1}{\sqrt{2}}\left(A_2e^{L_1(z)+d_1} + A_1e^{L_2(z)+d_2}\right).$$

Since $\sigma_2(z+c) - \sigma_1(z+c)$ is not constant we clearly have $L_1(z) \neq L_2(z)$.
Subcase 2.4.2: Let

$$Q_2e^{\sigma_2(z)-\sigma_1(z+c)} \equiv 1.$$

Then we have $\sigma_2(z) - \sigma_1(z+c)$ is a constant. Now in view of (5.8) we get $\sigma_1(z) - \sigma_2(z+c)$ is a constant, say μ_2. Then

$$\mu_2 - \mu_1 = p(z) + p(z+c),$$

this shows that $p(z)$ is a constant, which contradicts the fact that $p(z)$ is not a constant. □

Proof of Theorem 3.2. Let $f(z)$ is a finite order transcendental entire solution of (2.2). By using similar arguments as done in the proof of Theorem 3.1 we get

$$a_1f(z+c) + P_{L(f(z+c))} = \frac{1}{\sqrt{2}}\left(A_1e^{\sigma_1(z)} + A_2e^{\sigma_2(z)}\right), \tag{5.20}$$

$$f(z)=\frac{1}{\sqrt{2}}\left(A_2e^{\sigma_1(z)}+A_1e^{\sigma_2(z)}\right). \tag{5.21}$$

Now, in view of (5.20) and (5.21), we get

$$\frac{[a_1A_2+A_2M_1(z+c)]}{A_1}e^{\sigma_1(z+c)-\sigma_1(z)}+\frac{[a_1A_1+A_1M_2(z+c)]}{A_1}e^{\sigma_2(z+c)-\sigma_1(z)}$$
$$-\frac{A_2}{A_1}e^{\sigma_2(z)-\sigma_1(z)}=1, \tag{5.22}$$

where $M_1(z+c)$, $M_2(z+c)$ are defined as in Theorem 3.1.

Now we consider the following cases.

Case 1: Let $\sigma_2(z)-\sigma_1(z)$ is a constant, which means $p(z)$ is a constant. Let $e^{p(z)}=\xi$. Then (5.20) and (5.21) gives

$$a_1f(z+c)+P_{L(f(z+c))}=\frac{1}{\sqrt{2}}\left(A_1\xi+A_2\xi^{-1}\right)e^{\frac{g(z)}{2}}, \tag{5.23}$$

$$f(z)=\frac{1}{\sqrt{2}}\left(A_2\xi+A_1\xi^{-1}\right)e^{\frac{g(z)}{2}}. \tag{5.24}$$

In view of (5.23) and (5.24), we get

$$\begin{aligned}&a_1\left(A_2\xi+A_1\xi^{-1}\right)+\left(A_2\xi+A_1\xi^{-1}\right)M'\left(g(z+c)\right)\\&=\left(A_1\xi+A_2\xi^{-1}\right)e^{\frac{g(z)-g(z+c)}{2}}.\end{aligned} \tag{5.25}$$

Let $g(z)-g(z+c)$ be not a constant, we see that from (5.25) both $A_1\xi+A_2\xi^{-1}=0$ and $A_2\xi+A_1\xi^{-1}=0$, which yields $A_1^2=A_2^2$ and this contradicts the assumption $\alpha^2\neq 0,1$. Hence, $g(z)-g(z+c)$ must be constant, otherwise one side is polynomial and one side is a constant. Let $g(z)=L(z)+H(s)+d$, where $L(z)=d_1z_1+d_2z_2$ and $H(s)$ is a polynomial in s, $s=e_2z_1+e_1z_2$ such that

$$e_2c_1+e_1c_2=0.$$

$$\frac{a_1(A_2\xi+A_1\xi^{-1})}{(A_1\xi+A_2\xi^{-1})}+N=e^{\frac{-L(c)}{2}}, \tag{5.26}$$

where N is defined as in case 1 of Theorem 3.1. Now using similar arguments like the proof of Theorem 3.1, we get $H(s)\equiv 0$.

Thus, we have $g(z)=L(z)+d$ and from (5.26), we get

$$e^{\frac{-L(c)}{2}}=\frac{a_1(A_2\xi+A_1\xi^{-1})}{(A_1\xi+A_2\xi^{-1})}+\frac{1}{2}\left\{b_{10}d_1+b_{01}d_2+\frac{1}{2}b_{11}d_1d_2+\frac{1}{2}b_{20}d_1^2+\frac{1}{2}b_{02}d_2^2\right\}.$$

From (5.21), we get $f(z)=\frac{1}{\sqrt{2}}\left(A_2\xi+A_1\xi^{-1}\right)e^{\frac{L(z)+d}{2}}$.

Case 2: Let $\sigma_2(z)-\sigma_1(z)$ is not a constant. Let us denote by $\tilde{Q_1}=\frac{[a_1A_2+A_2M_1(z+c)]}{A_1}$ and $\tilde{Q_2}=\frac{[a_1A_1+A_1M_2(z+c)]}{A_1}$ in (5.22).

Subcase 2.1: Let both $\tilde{Q_1}\equiv 0$ and $\tilde{Q_2}\equiv 0$. Then from (5.22) we get $\sigma_2(z)-\sigma_1(z)$ is a constant, which contradicts our assumption.

Subcase 2.2: Let $\tilde{Q_1}\equiv 0$ and $\tilde{Q_2}\not\equiv 0$ or $\tilde{Q_1}\not\equiv 0$ and $\tilde{Q_2}\equiv 0$. Then proceeding in the similar way as done in Subcase 2.2 of Theorem 3.1 we get a contradiction.

Subcase 2.3: Let $\tilde{Q_1} \not\equiv 0$ and $\tilde{Q_2} \not\equiv 0$. Applying Lemma 4.3 in (5.22), we get

$$\tilde{Q_1}e^{\sigma_1(z+c)-\sigma_1(z)} \equiv 1, \quad \text{or} \quad \tilde{Q_2}e^{\sigma_2(z+c)-\sigma_1(z)} \equiv 1.$$

Subcase 2.3.1: If $\tilde{Q_1}e^{\sigma_1(z+c)-\sigma_1(z)} \equiv 1$ we see that $\sigma_1(z+c)-\sigma_1(z)$ is a constant. Let $\sigma_1(z) = L_1(z) + H_1(s) + d_1$, where $L_1(z) = d_{11}z_1 + d_{12}z_2$, $H_1(s)$ is a polynomial in s, $s = e_{12}z_1 + e_{11}z_2$ such that

$$e_{12}c_1 + e_{11}c_2 = 0.$$

Then, we get

$$\frac{a_1A_2}{A_1} + \frac{A_2}{A_1}N' = e^{-L_1(c)}, \tag{5.27}$$

where N' is defined as in Theorem 3.1. By proceeing in the similar way as used in Theorem 3.1, we get $H_1(s) \equiv 0$.

Thus, we have $\sigma_1(z) = L_1(z) + d_1$. From (5.27) we get

$$e^{-L_1(c)} = \frac{a_1A_2}{A_1} + \frac{A_2}{A_1}\left\{b_{10}d_{11} + b_{01}d_{12} + b_{11}d_{11}d_{12} + b_{20}d_{11}^2 + b_{02}d_{12}^2\right\}.$$

In view of (5.22) we get

$$\left[\frac{a_1A_1}{A_2} + \frac{A_1M_2(z+c)}{A_2}\right]e^{\sigma_2(z+c)-\sigma_2(z)} \equiv 1. \tag{5.28}$$

From (5.28) we see that $\sigma_2(z+c)-\sigma_2(z)$ is a constant. Let $\sigma_2(z) = L_2(z)+H_2(\tilde{s})+d_2$, where $L_2(z) = d_{21}z_1 + d_{22}z_2 + d_2$, $H_2(\tilde{s})$ is a polynomial in $\tilde{s}$, $s = e_{22}z_1 + e_{21}z_2$ such that

$$e_{22}c_1 + e_{21}c_2 = 0.$$

Then (5.28) gives

$$\frac{a_1A_1}{A_2} + \frac{A_1}{A_2}N'' = e^{-L_2(c)}, \tag{5.29}$$

where N'' is defined as in Theorem 3.1. Using the similar arguments like Theorem 3.1 we have $H_2(\tilde{s}) = 0$. Thus we have $\sigma_2(z) = L_2(z) + d_2$. From (5.29) we have

$$\frac{a_1A_1}{A_2} + \frac{A_1}{A_2}\left\{b_{10}d_{21} + b_{01}d_{22} + b_{11}d_{21}d_{22} + b_{20}d_{21}^2 + b_{02}d_{22}^2\right\} = e^{-L_2(c)}.$$

From (5.21) we get $f(z) = \frac{1}{\sqrt{2}}\left(A_2e^{L_1(z)+d_1} + A_1e^{L_2(z)+d_2}\right)$, and clearly $L_1(z) \neq L_2(z)$ because $\sigma_1(z) - \sigma_2(z)$ is not a constant.

Subcase 2.3.2: Let $\tilde{Q_2}e^{\sigma_2(z+c)-\sigma_1(z)} \equiv 1$. Then by the similar arguments as done in the proof of Theorem 3.1 we can get a contradiction. □

Proof of Theorem 3.3. Let $f(z)$ be a finite order transcendental entire solution of (2.3). Then by similar arguments as used in the proof Theorem 3.1 we get

$$P_{L(f(z))} = \frac{1}{\sqrt{2}}\left(A_2e^{\sigma_1(z)} + A_1e^{\sigma_2(z)}\right), \tag{5.30}$$

$$L(z,f) = \frac{1}{\sqrt{2}}\left(A_1e^{\sigma_1(z)} + A_2e^{\sigma_2(z)}\right). \tag{5.31}$$

Using (5.30) and (5.31) we get

$$\left(\frac{A_1M_1(z)-a_0A_2}{a_1A_2}\right)e^{\sigma_1(z)-\sigma_1(z+c)}+\left(\frac{A_2M_2(z)-a_0A_1}{a_1A_2}\right)e^{\sigma_2(z)-\sigma_1(z+c)}$$
$$-\frac{A_1}{A_2}e^{\sigma_2(z+c)-\sigma_1(z+c)}=1, \tag{5.32}$$

where $M_1(z)$, $M_2(z)$ are as defined in Theorem 3.1.

Here we cosider the following cases:

Case 1: Let $\sigma_2(z+c)-\sigma_1(z+c)$ is a constant, i.e., $p(z)$ is a constant. Let $e^{p(z)}=\xi$. Then (5.30) and (5.31) becomes

$$P_{L(f(z))}=\frac{1}{\sqrt{2}}\left(A_2\xi+A_1\xi^{-1}\right)e^{\frac{g(z)}{2}}, \tag{5.33}$$

$$L(z,f)=\frac{1}{\sqrt{2}}\left(A_1\xi+A_2\xi^{-1}\right)e^{\frac{g(z)}{2}}. \tag{5.34}$$

Equations (5.33) and (5.34) gives

$$\left(A_1\xi+A_2\xi^{-1}\right)M'g(z)-a_0\left(A_2\xi+A_1\xi^{-1}\right)=a_1\left(A_2\xi+A_1\xi^{-1}\right)e^{\frac{g(z+c)-g(z)}{2}}. \tag{5.35}$$

We claim that $g(z+c)-g(z)$ is constant. Since $e^{\frac{g(z+c)-g(z)}{2}}$ has no zeros and no poles (5.35) shows that only two cases can arise:

(i) $M'g(z)=0$ and $A_2\xi+A_1\xi^{-1}=0$. But $M'g(z)=0$ contradicts our assumption.

(ii) $A_1\xi+A_2\xi^{-1}=0$ and $A_2\xi+A_1\xi^{-1}=0$. These two identities implies $A_1^2=A_2^2$, contradicts the assumption $\alpha^2\neq 0,1$. Let $g(z)=L(z)+H(s)+d$, $L(z)=d_1z_1+d_2z_2$, $H(s)$ is a polynomial in s where $s=e_2z_1+e_1z_2$ such that

$$e_2c_1+e_1c_2=0.$$

Then from (5.35) we get

$$\left(A_1\xi+A_2\xi^{-1}\right)N=a_0(A_2\xi+A_1\xi^{-1})+a_1(A_2\xi+A_1\xi^{-1})e^{\frac{L(c)}{2}}, \tag{5.36}$$

where N is defined as in Theorem 3.1. For the same reason as described in Theorem 3.1 we have $H(s)=0$. Hence $g(z)=L(z)+d$ and (5.36) gives

$$\frac{\left(A_1\xi+A_2\xi^{-1}\right)}{2}\left\{b_{10}d_1+b_{01}d_2+\frac{1}{2}b_{11}d_2d_1+\frac{1}{2}b_{20}d_1^2+\frac{1}{2}b_{02}d_2^2\right\}$$
$$=a_0(A_2\xi+A_1\xi^{-1})+a_1(A_2\xi+A_1\xi^{-1})e^{\frac{L(c)}{2}}.$$

Then the solution is

$$f(z)=\frac{\sqrt{2}\left(A_2\xi+A_1\xi^{-1}\right)}{(b_{10}d_1+b_{01}d_2+b_{11}d_1d_2+b_{20}d_1^2+b_{02}d_2^2)}e^{\frac{L(z)+d}{2}};$$

such that $P_{L(f(\frac{L(z)}{2}))}\neq 0$.

Case 2: Let $\sigma_2(z+c)-\sigma_1(z+c)$ is not a constant. Suppose $Q_1'=\frac{A_1M_1(z)-a_0A_2}{a_1A_2}$ and $Q_2'=\frac{A_2M_2(z)-a_0A_1}{a_1A_2}$ in (5.32).

Subcase 2.1: Let Both $Q_1'=Q_2'\equiv 0$. Then from (5.32) we get a contradiction that $\sigma_2(z+c)-\sigma_1(z+c)$ is a constant.

Subcase 2.2: Let $Q_1'\not\equiv 0$, $Q_2'\equiv 0$ or $Q_1'\equiv 0$, $Q_2'\not\equiv 0$, then by the same arguments as used in the proof of Theorems 3.1 we get contradiction.

Subcase 2.3: Let $Q_1' \not\equiv 0$ and $Q_2' \not\equiv 0$.

Using Lemma 4.3 in (5.32) we get

$$Q_1' e^{\sigma_1(z)-\sigma_1(z+c)} \equiv 1,$$

or

$$Q_2' e^{\sigma_2(z)-\sigma_1(z+c)} \equiv 1.$$

Subcase 2.3.1: Now if $Q_1' e^{\sigma_1(z)-\sigma_1(z+c)} \equiv 1$, it yield that $\sigma_1(z)-\sigma_1(z+c)$ is a constant. Let $\sigma_1(z) = L_1(z) + H_1(s) + d_1$, where $L_1(z) = d_{11}z_1 + d_{12}z_2$, $H_1(s)$ is a polynomial in s, $s = e_{12}z_1 + e_{11}z_2$ such that

$$e_{12}c_1 + e_{11}c_2 = 0.$$

Then we get

$$\frac{A_1}{a_1 A_2} N' = \frac{a_0}{a_1} + e^{L_1(c)}, \tag{5.37}$$

where N' is defined as in Theorem 3.1. As like Theorem 3.1 we get $H_1(s) \equiv 0$. Then from (5.37) we get

$$e^{L_1(c)} + \frac{a_0}{a_1} = \frac{A_1}{a_1 A_2}\left(b_{10}d_{11} + b_{01}d_{12} + b_{11}d_{11}d_{12} + b_{20}d_{11}^2 + b_{02}d_{12}^2\right).$$

Now from (5.32) we get

$$\left(\frac{A_2 M_2(z) - a_0 A_1}{a_1 A_1}\right) e^{\sigma_2(z)-\sigma_2(z+c)} \equiv 1. \tag{5.38}$$

This shows that $\sigma_2(z) - \sigma_2(z+c)$ is a constant. Let $\sigma_2(z) = L_2(z) + H_2(s) + d_2$, where $L_2(z) = d_{21}z_1 + d_{22}z_2$, $H_2(\tilde{s})$ is a polynomial in $\tilde{s}$, $\tilde{s} = e_{22}z_1 + e_{21}z_2$ such that

$$e_{22}c_1 + e_{21}c_2 = 0.$$

From (5.38) we get

$$\frac{A_2}{a_1 A_1} N'' = \frac{a_0}{a_1} + e^{L_2(c)}, \tag{5.39}$$

where N'' is defined as in Theorem 3.1. Similar to the arguments as used in case 1 we have $H_2(\tilde{s}) = 0$. From (5.39) we have

$$e^{L_2(c)} + \frac{a_0}{a_1} = \frac{A_2}{a_1 A_1}\left(b_{10}d_{21} + b_{01}d_{22} + b_{11}d_{21}d_{22} + b_{20}d_{21}^2 + b_{02}d_{22}^2\right).$$

Hence $\sigma_2(z) = L_2(z) + d_2$. Clearly $L_1(z) \neq L_2(z)$. And from (5.30) we have

$$f(z) = \frac{1}{\sqrt{2}}\left(\frac{A_2}{b_{10}d_{11} + b_{01}d_{12} + b_{11}d_{11}d_{12} + b_{20}d_{11}^2 + b_{02}d_{12}^2} e^{L_1(z)+d_1} \right.$$
$$\left. + \frac{A_1}{b_{10}d_{21} + b_{01}d_{22} + b_{11}d_{21}d_{22} + b_{20}d_{21}^2 + b_{02}d_{22}^2} e^{L_2(z)+d_2}\right).$$

Subcase 2.3.2: Suppose $Q_2' e^{\sigma_2(z)-\sigma_1(z+c)} \equiv 1$, then by similar arguments as done in the proof of Theorem 3.1 we get a contradiction. $\square$

Acknowledgement

The authors would like to thank the referee for his/her useful suggestions which improved the presentation of the paper. The second author is thankful to Council of Scientific and Industrial Research (India, for financial help under File no.-09/0106(13572)/2022-EMR-I).

References

1. I. N. Baker, On a class of meromorphic functions, Proc. Am. Math. Soc., 17 (1966), no. 4, 819–822.
2. C. A. Berenstein, D. C. Chang and Li B. Q., On the shared values of entire functions and their partial differential polynomials in $\mathbb{C}^n$, Forum Math., 8 (1996), 379–396.
3. T. B. Cao and R. J. Korhonen, A new version of the second main theorem for meromorphic mappings intersecting hyperplanes in several complex variables, J. Math. Anal. Appl., 444 (2016), no. 2, 1114–1132.
4. T. B. Cao and L. Xu, Logarithmic difference lemma in several complex variables and partial difference equations, Ann. di Mat., 199 (2020), 767–794.
5. Y. M. Chiang and S. J. Feng, On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane, Ramanujan J., 16 (2008), 105–129.
6. F. Gross, On the equation $f^n + g^n = 1$, Bull. Am. Math. Soc., 72 (1966), 86–88.
7. F. Gross, On the functional equation $f^n + g^n = h^n$, Am. Math. Mon., 73 (1966), no. 10, 1093–1096.
8. F. Gross, On the equation $f^n + g^n = 1$, II. Bull. Am. Math. Soc., 74 (1968), no. 4, 647–648.
9. R. G. Halburd and R. J. Korhonen, Difference analogue of the lemma on the logarithmic derivative with applications to difference equations, J. Math. Anal. Appl., 314 (2006), 477–487.
10. W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.
11. P. C. Hu, P. Li and C.-C. Yang, *Unicity of meromorphic mappings*, Adv. Complex Anal. Appl., Volume 1, Kluwer Academic Publishers, Dordrecht, 2003.
12. P. C. Hu and C. C. Yang, Uniqueness of meromorphic functions on $\mathbb{C}^m$, Complex Var., 30 (1996) 235–270.
13. D. Khavinson, A note on entire solutions of the eiconal equation, Amer. Math. Mon., 102 (1995), 159–161.
14. R. J. Korhonen, A difference Picard theorem for meromorphic functions of several variables, Comput. Methods Funct. Theo., 12 (2012), 343–361.
15. B. Q. Li, On reduction of functional-differential equations, Complex Var., 31 (1996), 311–324.
16. B. Q. Li, On entire solutions of Fermat-type partial differential equations, Int. J. Math., 15 (2004), 473–485.
17. B. Q. Li, Entire solutions of $(u_{z_1})^m + (u_{z_2})^n = e^g$, Nagoya Math. J., 178 (2005), 151–162.
18. B. Q. Li, Entire solutions of eiconal type equations, Arch. Math., 89 (2007), 350–357.
19. H. Li and H. Y. Xu, Solutions for several quadratic trinomial difference equations and partial differential difference equations in $\mathbb{C}^2$, Axioms, 10 (2021), no. 2, Art. ID 126, 19 pp.
20. K. Liu and L. Z. Yang, A note on meromorphic solutions of Fermat-types equations, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.), 62 (2016), no. 2, 317–325.
21. J. Luo, H. Y. Xu and F. Hu, Entire solutions for several general quadratic trinomial differential difference equations, Open Math., 19 (2021), 1018–1028.
22. P. Montel, Leçons sur les familles normales de fonctions analytiques et leurs applications, Collect. Borel, Paris, (1927), 135–136.

23. G. Poyla, On the integral function of an integral function, J. Lond. Math. Soc., 1 (1926), no. 1, 12–15.
24. E. G. Saleeby, On complex analytic solutions of certain trinomial functional and partial differential equations, Aequationes Math., 85 (2013), 553–562.
25. W. Stoll, *Holomorphic functions of finite order in several complex variables*, Expository Lectures from the CBMS Regional Conference held at the University of Wisconsin, Watewater, Wis., June 18–22, 1973, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 21, American Mathematical Society, Providence, RI, 1974.
26. Q. Wang, W. Chen and P. Hu, On entire solutions of two certain Fermat-Type differential-difference equations, Bull. Malays. Math. Sci. Soc., 43 (2020), no. 4, 2951–2961.
27. L. Xu and T.-B. Cao, Solutions of complex Fermat-type partial difference and differential-difference equations, Mediterr. J. Math., 15 (2018), no. 6, Paper No. 227, 14 pp.
28. L. Xu and T.-B. Cao, Correction to: Solutions of complex Fermat-Type partial difference and differential-difference equations, Mediterr. J. Math., 17 (2020), no. 1, Paper No. 8, 4 pp.
29. H. Y. Xu, S. Y. Liu and Q. P. Li, Entire solutions for several systems of nonlinear difference and partial differential-difference equations of Fermat-type, J. Math. Anal. Appl., 483 (2020), 123–641.
30. H. Y. Xu and H. Wang, Notes on the existence of entire solutions for several partial differential-difference equations, Bull. Iran. Math. Soc., 47 (2021), 1477–1489.
31. H. Y. Xu, K. Y. Zhang and X. M. Zheng, Entire and meromorphic solutions for several Fermat-type partial differential difference equations in $\mathbb{C}^2$, Rocky Mountain J. Math., 52 (2022), no. 6, 2169–2187.
32. C. C. Yang and P. Li, On the transcendental solutions of a certain type of non-linear differential equations, Arch. Math., 82 (2004), 442–448.
33. Z. Ye, On Nevanlinna's second main theorem in projective space, Invent. Math., 122 (1995), 475–507.
34. X. M. Zheng and X. Y. Xu, Entire solutions of some Fermat-type functional equations concerning difference and partial differential in $\mathbb{C}^2$, Anal. Math., 48 (2022), 199–226.

(Abhijit Banerjee) Department of Mathematics, University of Kalyani, West Bengal 741235, India.
E-mail address: `abanerjee_kal@yahoo.co.in`

(Jhuma Sarkar) Department of Mathematics, University of Kalyani, West Bengal 741235, India.
E-mail address: `jhumasarkar928@gmail.com`

(Gautam Haldar) Department of Mathematics, Ghani Khan Choudhury Institute of Engineering and Technology, Narayanpur, Malda, Pin 732141, West Bengal, India.
E-mail address: `goutamiit1986@gmail.com`