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ROUGH IDEAL CONVERGENCE OF DOUBLE SEQUENCES IN PROBABILISTIC NORMED SPACE

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Abstract. In this work, we study the notions of rough I_2-Cauchy and rough I_2^*-Cauchy for double sequences in the setting of random 2-normed space. The notions of rough I_2-convergence, rough I_2^*-convergence, rough I_2-limit points, and rough I_2-cluster points of probabilistic normed double sequences are also developed and investigated. By proving that rough $I2*$-convergence implies rough $I2$-convergence in probabilistic normed space, we discuss the connection between rough I_2- and rough I_2^*-convergence. Additionally, we have demonstrated by an example that rough I_2-convergence in probabilistic normed space does not imply rough I_2^*-convergence. The significance of condition (AP2) in summability analysis utilizing ideals is supported by our findings.

1. Introduction

Menger [16] connected each pair of points in a set to a distribution function in order to generalize the metric axioms. This idea, which was originally described as a statistical metric space and is now known as a probabilistic metric space, has been significantly improved by Schweizer and Sklar [22]. To replace nonnegative real values with distribution functions was one idea. One of the most important families of probabilistic metric spaces is the family of probabilistic normed spaces. Probabilistic normed spaces are real linear spaces where the norm of each vector is a probability distribution function rather than a number. The first reference of these places was made by Šerstnev in 1963, according to [21]. A new definition of PN spaces was offered by Alsina et al. in [2] that incorporates Šerstnev's and inexorably leads to the identification of the Menger spaces as the primary class of PN spaces.

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Phu [24] was the first to propose the concept of rough convergence in finite-dimensional normed spaces. In [24] demonstrated the boundedness, closedness, and convexity of the set LIM_x^r and developed the idea of a rough Cauchy sequence. Additionally, he looked into how rough convergence related to other types of convergence as well as how LIM_x^r was affected by the degree of roughness r. In a similar study [25], he established the rough continuity of linear operators and demonstrated that, with the premise that $\dim Y > 0$ and $r > 0$ and that X and Y are normed spaces, every linear operator $f : X \to Y$ is r-continuous at every point $x \in X$. He expanded the findings from [24] to infinite-dimensional normed spaces in [26]. Aytar [3] researched rough statistical convergence, developed a set of rough statistical limit points for a sequence, and obtained two statistical convergence criteria linked to this set in order to demonstrate that this set is closed and convex. The r-limit set of the sequence is equal to the intersection of these sets, and the r-core of the sequence is equal to the union of these sets, according to Aytar's [4] research. The concepts of rough I-convergence and the set of rough I-limit points of a sequence were recently presented by Dündar and Çakan [7, 8], who also looked into the concepts of rough convergence and the set of rough limit points of a double sequence. In [20], The author broadens the concept of rough convergence by incorporating the notion of ideals, thereby automatically enlarging the initial concepts of rough statistical convergence and rough convergence. In different situation, both statistical convergence and generalized statistical convergence can be employed to explore, extend, and apply rough convergence further; refer to [1, 5, 9, 12, 17, 18] for more details.

We propose the idea of rough I_2-and I_2^*-convergence in probabilistic normed spaces in this study utilizing the concepts of rough convergence and probabilistic normed spaces. We define the set of approximate limit points of a sequence and provide two convergence criteria for the set. We then demonstrate that this set is convex and closed. The relationships between the set of cluster points and the set of approximate limit points of a sequence are next examined.

2. Preliminaries

We will go over some basic terminologies and notations that form the basis of the current work in this section.

Definition 2.1. [22] A triangular norm (t-norm) is a continuous mapping $* : [0,1]^2 \to [0,1]$ such that $([0,1], *)$ is an abelian monoid with unit one and $x * y \le u * v$ if $x \le u$ and $y \le v$ for all $x, y, u, v \in [0,1]$.

Definition 2.2. [10] A function $\phi : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is non-decreasing and left-continuous with $\inf_{u \in \mathbb{R}} \phi(u) = 0$ and $\sup_{u \in \mathbb{R}} \phi(u) = 1$.

By D, we denote the set of all distribution functions.

Definition 2.3. [10] Let X be a real linear space and $\mu : X \to D$. A probabilistic norm or μ-norm is a t-norm satisfying the following conditions:

(i) $\mu_x(0) = 0$,

(ii) $\mu_x(t) = 1$ for all $t > 0$ if and only if $x = 0$,

(iii) $\mu_{cx}(t) = \mu_x\left(\frac{t}{|c|}\right)$ for all $c \in \mathbb{R} \setminus \{0\}$ and all $t > 0$,

(iv) $\mu_{x+y}(s+t) \geq \mu_x(t) * \mu_y(s)$ for all $x, y \in X$ and $s, t \in \mathbb{R}^+$

where μ_x means $\mu(x)$ and $\mu_x(t)$ is the value of μ_x at $t \in \mathbb{R}$. The $(X, \mu, *)$ is called a probabilistic normed space (for short, PNS).

Definition 2.4. [13] Let $(X, \mu, *)$ be an PNS. A sequence $\{x_m\}$ is said to be convergent to $\xi \in X$ with respect to the norm μ, that is, $x_m \xrightarrow{\mu} \xi$ if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists a positive integer m_0 such that $\mu_{x_m - \xi}(\epsilon) > 1 - \lambda$ whenever $m \geq m_0$. In this case, we write μ-$\lim x_m = \xi$.

Definition 2.5. [11] Let E be a subset of $\mathbb{N}$, the set of natural number. The asymptotic density of E denoted by $\delta(E)$, is defined as

$$\delta(E) = \lim_{m \to \infty} \frac{1}{m} \left| \{ s \leq m : s \in E \} \right|.$$

Definition 2.6. [13] Let $(X, \mu, *)$ be an PNS. A sequence $x = \{x_m\}$ is said to be statistically convergent to $\xi \in X$ with respect to the norm μ provided that, for every $\epsilon > 0$ and $\lambda \in (0, 1)$, $\delta(E(\lambda)) = 0$, where

$$E(\lambda) = \{ m \leq n : \mu_{x_m - \xi}(\epsilon) \leq 1 - \lambda \}.$$

or, equivalently,

$$\lim_{n \to \infty} \frac{1}{n} \left| \{ m \leq n : \mu_{x_m - \xi}(\epsilon) \leq 1 - \lambda \} \right| = 0.$$

In this case we write st_μ-$\lim x = \xi$.

Definition 2.7. [23] A non-void class $I \subseteq 2^{\mathbb{N}}$ is called an *ideal* if I is additive (i.e., $A, B \in I \Rightarrow A \cup B \in I$) and hereditary (i.e., $A \in I \;\&\; B \subseteq A \Rightarrow B \in I$). An ideal I is said to be non-trivial if $I \neq 2^{\mathbb{N}}$. A non-trivial ideal I is said to be admissible if I contains every finite subset of $\mathbb{N}$. For any ideal there is a filter $F(I)$ corresponding with I, given by

$$F(I) = \{ K \subseteq \mathbb{N} : \mathbb{N} \setminus K \in I \}.$$

Definition 2.8. [14] An admissible ideal $I \subset 2^{\mathbb{N}}$ is said to possesses (AP) if for every sequence $\{Q_m\}_{m \in \mathbb{N}}$ of pairwise disjoint sets from I there are sets $G_m \subset \mathbb{N}, m \in \mathbb{N}$, such that the symmetric difference $Q_m \triangle G_m$ is a finite set for every m and $\bigcup_{m \in \mathbb{N}} G_m \in I$.

Definition 2.9. [19] Let I be a non trivial ideal of $\mathbb{N} \times \mathbb{N}$ and $(X, \mu, *)$ be a probabilistic normed space. A double sequence $x = \{x_{jk}\}$ of elements of X is said to be I_2-convergent to $\xi \in X$ with respect to norm μ (or I_2^μ-convergent to ξ) if for each $\epsilon > 0$ and $\lambda \in (0, 1)$,

$$\left\{ (j, k) \in \mathbb{N} \times \mathbb{N} : \mu_{x_{jk} - \xi}(\epsilon) \leq 1 - \lambda \right\} \in I_2. \tag{2.1}$$

In this case we write I_2^μ-$\lim x = \xi$.

Definition 2.10. [6] A non trivial ideal I_2 of $\mathbb{N} \times \mathbb{N}$ is said to be strongly admissible if $\{i\} \times \mathbb{N}, \mathbb{N} \times \{i\}$ belong to I_2 for each $i \in \mathbb{N}$. It is clear that a strongly admissible ideal is an admissible ideal.

Let

$$I_0 = \{ A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N}) \, (i, j \geq m(A) \Rightarrow (i, j) \notin A) \}.$$

Then I_0 is a nontrivial strongly admissible ideal and clearly an ideal I is strongly admissible if and only if $I_0 \subset I$.

Definition 2.11. [6] We say that an admissible ideal $I \subset 2^{\mathbb{N}\times\mathbb{N}}$ possesses (AP2) if for every countable family of mutually disjoint sets $\{E_1, E_2, \cdots\}$ belonging to I, there exists a countable family of sets $\{Q_1, Q_2, \cdots\}$ such that $E_j \triangle Q_j \in I_0$ i.e. $E_j \triangle Q_j$ is included in the finite union of rows and columns in $\mathbb{N}\times\mathbb{N}$ for each $j \in \mathbb{N}$ and $Q = \bigcup_{j=1}^{\infty} Q_j \in I$ (so $Q_j \in I$ for each $j \in \mathbb{N}$).

3. Rough I-and I^*-convergence in PNS

We explore the idea of optimal double sequence convergence in PNS in this section. The idea of I_2^*-convergence of double sequences in PNS is also introduced, and it is demonstrated that I_2^*-convergence entails I_2-convergence but not the other way around.

Definition 3.1. Let $(X, \mu, *)$ be an PNS and r be a non-negative real number. A double sequence $x = \{x_{jk}\}$ is said to be rough convergent (r-convergent) to $\xi \in X$ with respect to the norm μ, that is, $x_{jk} \xrightarrow{r\mu} \xi$ if for every $\epsilon > 0$ and $\lambda \in (0,1)$, there exists a positive integer m_0 such that $\mu_{x_{jk}-\xi}(r+\epsilon) > 1-\lambda$ whenever $j, k \geq m_0$. In this case, we write $r\mu\text{-}\lim x_{jk} = \xi$.

Definition 3.2. Let $(X, \mu, *)$ be an PNS and r be a non-negative real number. A double sequence $x = \{x_{jk}\}$ is said to be rough I_2-convergent (rI_2-convergent) to $\xi \in X$ with respect to the norm μ, that is, $x_{jk} \xrightarrow{rI_2^{\mu}} \xi$, provided that, if for every $\epsilon > 0$ and $\lambda \in (0,1)$,

$$\left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu_{x_{jk}-\xi}(\epsilon + r) \leq 1-\lambda\right\} \in I_2. \tag{3.1}$$

Here r is called the roughness degree. If we take $r = 0$, we obtain the notion of I_2^{μ}-convergence.

Theorem 3.3. *Let $(X, \mu, *)$ be an PNS and r be a non-negative real number. Then, the following assertions are equivalent for a double sequence $x = \{x_{jk}\}$ in X.*

(i) $rI_2^{\mu}\text{-}\lim x = \xi$.

(ii) $\left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu_{x_{jk}-\xi}(\epsilon + r) \leq 1-\lambda\right\} \in I_2$ *for every* $\epsilon > 0$ *and* $\lambda \in (0,1)$.

(iii) $\left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu_{x_{jk}-\xi}(\epsilon + r) > 1-\lambda\right\} \in F(I_2)$ *for every* $\epsilon > 0$ *and* $\lambda \in (0,1)$.

Proof. The proof is straightforward, so we omit it. □

The motivation behind introducing this notion is same as in [24] and [3]. For instance assume that the sequence $y = \{y_{jk}\}$ is I_2^{μ}-convergent and can not be measured or calculated exactly and one has to do with an approximated (or I_2^{μ}-approximated) sequence $x = \{x_{jk}\}$ satisfying $\mu_{x-y}(r) \geq 1-\lambda$ for all j and k (or for almost all j,k, that is, $\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu_{x-y}(r) \geq 1-\lambda\} \in I_2$. Then I_2-convergence of the sequence x is not assured, but as the inclusion

$$\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu_{x-\xi}(r+\epsilon) \leq 1-\lambda\} \subseteq \{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu_{x-\xi}(\epsilon) \leq 1-\lambda\}$$

holds so the sequence $x = \{x_{jk}\}$ is rI_2^{μ}-convergent.

In general the rough I_2-limit of a sequence may not be unique for the roughness degree $r > 0$. We define

$$I_2^{\mu} - LIM_x^r = \left\{\xi \in X : x \xrightarrow{rI_2^{\mu}} \xi\right\}.$$

The double sequence x is said to be rI_2^μ-convergent provided that I_2^μ-$LIM_x^r \neq \emptyset$. In [24] (see also [3]) it was observed that for a sequence of real numbers, $LIM_x^r = [\limsup x - r, \liminf x + r]$. Similarly here we can have $I_2^\mu - LIM_x^r = [\limsup x - r, \liminf x + r]$.

Theorem 3.4. *Let $(X, \mu, *)$ be an PNS and r be a non-negative real number.*
(i) If $r\mu$-$\lim x_{jk} = \xi$, then rI_2^μ-$\lim x_{jk} = \xi$.
(ii) If rI_2^μ-$\lim x_{jk} = \xi_1$ and rI_2^μ-$\lim y_{jk} = \xi_2$, then rI_2^μ-$\lim(x_{jk} + y_{jk}) = \xi_1 + \xi_2$.
(iii) If rI_2^μ-$\lim x_{jk} = \xi$, then rI_2^μ-$\lim cx_{jk} = c\xi$ for any $c \in \mathbb{R}$.

Proof. (i) Suppose that $r\mu$-$\lim x_{jk} = \xi$. Then for each $\epsilon > 0$ and $\lambda \in (0, 1)$ there exists a positive integer m_0 such that $\mu_{x_{jk}-\xi}(\epsilon + r) > 1 - \lambda$ for each $j, k > m_0$. Since the set

$$Q(\lambda) = \left\{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu_{x_{jk}-\xi}(\epsilon + r) \leq 1 - \lambda\right\}$$

is contained in the set $\{1, 2, \cdots, m_0 - 1\}$ and the ideal I_2 is admissible so $Q(\lambda) \in I_2$. Hence rI_2^μ-$\lim x_{jk} = \xi$.
(ii) Suppose that rI_2^μ-$\lim x_{jk} = \xi_1$ and rI_2^μ-$\lim y_{jk} = \xi_2$. For a given $\lambda \in (0, 1)$ and $\epsilon > 0$ choose $\eta > 0$ such that $(1 - \eta) * (1 - \eta) > 1 - \lambda$. Define the following sets:

$$\begin{aligned} Q_1(\eta, \epsilon) &= \left\{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu_{x_{jk}-\xi_1}(\epsilon + r) \leq 1 - \eta\right\} \\ Q_2(\eta, \epsilon) &= \left\{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu_{y_{jk}-\xi_2}(\epsilon + r) \leq 1 - \eta\right\} \end{aligned}$$

since rI_2^μ-$\lim x_{jk} = \xi_1$, we have $Q_1(\eta, \epsilon) \in I_2$. Similarly, $Q_2(\eta, \epsilon) \in I_2$ because rI_2^μ-$\lim y_{jk} = \xi_2$. Now let $Q(\eta, \epsilon) = Q_1(\eta, \epsilon) \cup Q_2(\eta, \epsilon)$. This implies that $Q(\eta, \epsilon) \in I_2$ and hence $Q^c(\eta, \epsilon) = \mathbb{N} \times \mathbb{N} \setminus Q(\eta, \epsilon)$ is nonempty set in $F(I_2)$. Now, we want to show $Q^c(\eta, \epsilon) \subset \{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu_{(x_{jk}+y_{jk})-(\xi_1+\xi_2)}(2\epsilon + 2r) > 1 - \lambda\}$. If $(j, k) \in Q^c(\eta, \epsilon)$, then we have $\mu_{x_{jk}-\xi_1}(\epsilon + r) > 1 - \eta$ and $\mu_{y_{jk}-\xi_2}(\epsilon + r) > 1 - \eta$. Consequently,

$$\begin{aligned} \mu_{(x_{jk}+y_{jk})-(\xi_1+\xi_2)}(2\epsilon + 2r) &\geq \mu_{x_{jk}-\xi_1}(\epsilon + r) * \mu_{y_{jk}-\xi_2}(\epsilon + r) \\ &> (1 - \eta) * (1 - \eta) > 1 - \lambda. \end{aligned}$$

This shows that

$$Q^c(\eta, \epsilon) \subset \{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu_{(x_{jk}+y_{jk})-(\xi_1+\xi_2)}(2\epsilon + 2r) > 1 - \lambda\}.$$

Since $Q^c(\eta, \epsilon) \in F(I_2)$ we have rI_2^μ-$\lim(x_{jk} + y_{jk}) = \xi_1 + \xi_2$.
(iii) It is trivial for $c = 0$. Now let $c \neq 0$. Then for a given $\epsilon > 0$ and $\lambda \in (0, 1)$,

$$E(\lambda) = \left\{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu_{x_{jk}-\xi}(\epsilon + r) > 1 - \lambda\right\} \in F(I_2) \tag{3.2}$$

It is sufficient to prove that for each $\epsilon > 0$ and $\lambda \in (0, 1)$

$$E(\lambda) \subset \left\{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu_{cx_{jk}-c\xi}(\epsilon + r) > 1 - \lambda\right\}.$$

Let $(j, k) \in E(\lambda)$. Then we have $\mu_{x_{jk}-\xi}(\epsilon + r) > 1 - \lambda$. Now

$$\begin{aligned} \mu_{cx_{jk}-c\xi}(\epsilon + r) &= \mu_{x_{jk}-\xi}\left(\frac{\epsilon + r}{|c|}\right) \\ &\geq \mu_{x_{jk}-\xi}(\epsilon + r) * \mu_0\left(\frac{\epsilon + r}{|c|} - (\epsilon + r)\right) \\ &= \mu_{x_{jk}-\xi}(\epsilon + r) * 1 \\ &= (1 - \lambda) * 1 > 1 - \lambda. \end{aligned}$$

Hence

$$E(\lambda) \subset \left\{(j, k) \in \mathbb{N} \times \mathbb{N} : \mu_{cx_{jk}-c\xi}(\epsilon + r) > 1 - \lambda\right\}$$

and it follows from (3.2) that rI_2^{μ}-$\lim cx_{jk} = c\xi$ for any $c \in \mathbb{R}$. □

Definition 3.5. Let $(X, \mu, *)$ be an PNS and r be a non-negative real number. We say that a sequence $x = \{x_{jk}\}$ of elements in X is rI_2^*-convergent to $\xi \in X$ with respect to the norm μ if there exists a subset $H = \{(j_s, k_s) : j_1 < j_2 < \cdots ; k_1 < k_2, \cdots\}$ of $\mathbb{N} \times \mathbb{N}$ such that $H \in F(I_2)$ (i.e., $\mathbb{N} \times \mathbb{N} \setminus H \in I_2$) and $r\mu$-$\lim_s x_{j_s k_s} = \xi$. In this case, we write $rI_2^{*,\mu}$-$\lim x = \xi$ and ξ is called the $rI_2^{*,\psi}$-limit of the double sequence $x = \{x_{jk}\}$.

Theorem 3.6. *Let $(X, \mu, *)$ be an PNS and r be a non-negative real number and I_2 be an admissible ideal. If $x = \{x_{jk}\}$ is a double sequence of elements in X and $rI_2^{*,\mu}$-$\lim x = \xi$, then rI_2^{μ}-$\lim x = \xi$.*

Proof. Suppose that $rI_2^{*,\mu}$-$\lim x = \xi$. Then

$$H = \{(j_s, k_s) : j_1 < j_2 < \cdots ; k_1 < k_2, \cdots\}$$

of $\mathbb{N} \times \mathbb{N}$ such that $H \in F(I_2)$ (i.e., $\mathbb{N} \times \mathbb{N} \setminus H \in I_2$) and $r\mu$-$\lim_s x_{j_s k_s} = \xi$. But then for any $\epsilon > 0$, $\lambda \in (0, 1)$ there exists a positive integer N such that $\mu_{j_s k_s - \xi}(\epsilon + r) > 1 - \lambda$ for all $s > N$. Since $\{(j_s, k_s) \in H : \psi_{x_{j_s k_s} - \xi}(r + \epsilon) \leq 1 - \lambda\}$ is contained in $\{j_1 < j_2 < \cdots < j_{N-1}; k_1 < k_2, \cdots < j_{N-1}\}$ and the ideal I_2 is admissible, we have

$$\{(j_s, k_s) \in H : \psi_{x_{j_s k_s - \xi}}(\epsilon + r) \leq 1 - \lambda\} \in I_2.$$

Therefore

$$\begin{aligned} &\{(j, k) \in \mathbb{N} \times \mathbb{N} : \psi_{x_{jk} - \xi}(r + \epsilon)\} \subset \\ &H \cup \{j_1 < j_2 < \cdots < j_{N-1}; k_1 < k_2, \cdots < j_{N-1}\} \in I_2 \end{aligned}$$

for any $\epsilon > 0$, $\lambda \in (0, 1)$ and so rI_2^{ψ}-$\lim x = \xi$. □

The following example shows that the converse of Theorem 3.6 need not be true.

Example 3.7. Take $X = \mathbb{R}$ being equipped with the Euclidean norm $\|x\| = |x|$ and $a * b = ab$ for $a, b \in [0, 1]$. For all $x \in X$, $t > 0$ and nonzero $z \in X$, consider

$$\mu_x(t) = \begin{cases} \frac{t}{t + \|x\|}, & \text{if } t > 0; \\ 0, & \text{if } t \leq 0. \end{cases}$$

Then $(X, \mu, *)$ is a PNS.
Let $\mathbb{N} \times \mathbb{N} = \bigcup_{i,j} \triangle_{ij}$ be a decomposition of $\mathbb{N} \times \mathbb{N}$ such that for any $(m, n) \in \mathbb{N} \times \mathbb{N}$ each $\triangle_{ij}$ contains infinitely many (i, j)'s, where $i \geq m, j \geq n$ and $\triangle_{mn} \cap \triangle_{ij} = \emptyset$ for $(i, j) \neq (m, n)$. Let I_2 be the class of all subsets of $\mathbb{N} \times \mathbb{N}$ which intersect at most a finite number of $\triangle_{ij}$. Then I_2 is an admissible ideal. Now, we define a double sequence $x_{mn} = \frac{1}{ij}$ if $(m, n) \in \triangle_{ij}$. Then

$$\mu_{x_{mn}}(t) = \frac{t}{t + \|x_{mn}\|} \to 1$$

as $m, n \to \infty$. Hence rI_2^{μ}-$\lim_{m,n} x_{mn} = 0$.
Now, suppose that $rI_2^{*,\psi}$-$\lim_{m,n} x_{mn} = 0$. Then there exists a subset $M = \{(m_j, n_j) : m_1 < m_2 < \cdots ; n_1 < n_2 < \cdots\}$ of $\mathbb{N} \times \mathbb{N}$ such that $M \in F(I_2)$ and $r\mu$-$\lim_j x_{m_j n_j} = 0$.

Since $M \in F(I_2)$, there is a set $G \in I_2$ such that $M = \mathbb{N} \times \mathbb{N} \setminus G$. Now, from the definition of I_2, there exist, say $p, q \in \mathbb{N}$ such that

$$G \subset \left(\bigcup_{m=1}^{p} \left(\bigcup_{n}^{\infty} \triangle_{mn} \right) \right) \cup \left(\bigcup_{n=1}^{q} \left(\bigcup_{m}^{\infty} \triangle_{mn} \right) \right)$$

But then $\triangle_{p+1,q+1} \subset M$, and therefore

$$x_{m_j n_j} = \frac{1}{(p+1)(q+1)}$$

for infinitely many (m_j, n_j)'s from M which contradicts $r\mu\text{-}\lim_j x_{m_j n_j} = 0$. Therefore, the assumption $rI_2^{*,\mu}\text{-}\lim_{m,n} x_{mn} = 0$ leads to the contradiction.

The following theorem shows that the converse holds if the ideal I_2 possesses property (AP).

Theorem 3.8. *Let $(X, \mu, *)$ be an PNS and r be a non-negative real number and the admissible ideal I_2 satisfy the condition (AP). If $\{x_{mn}\}$ is a double sequence in X such that $x_{mn} \xrightarrow{rI_2^{\mu}} \xi$, then $x_{mn} \xrightarrow{rI_2^{*,\mu}} \xi$.*

Proof. Suppose I_2 possesses property (AP) and $rI_2^{\mu}\text{-}\lim x = \xi$. Then for each $\epsilon > 0$, $\lambda \in (0, 1)$,

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : \mu_{x_{mn}-\xi}(\epsilon + r) \leq 1 - \lambda\} \in I_2.$$

We define the set E_q for $q \in \mathbb{N}$ and $\epsilon > 0$ as

$$E_q = \left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : 1 - \frac{1}{q} \leq \mu_{x_{mn}-\xi}(r + \epsilon) < 1 - \frac{1}{q+1} \right\}.$$

Obviously, $\{E_1, E_2, \cdots\}$ is countable and belongs to I_2, and $E_i \cap E_j = \emptyset$ for $i \neq j$. By property (AP), there is a countable family of sets $\{Q_1, Q_2, \cdots\} \in I_2$ such that the symmetric difference $E_i \triangle Q_i$ is a finite set for each $i \in \mathbb{N}$ and $Q = \bigcup_{i=1}^{\infty} Q_i \in I_2$. From the definition of associate filter $F(I_2)$ there is a set $M \in F(I_2)$ such that $M = \mathbb{N} \times \mathbb{N} \setminus Q$. To prove the theorem it is sufficient to show that the subsequence $\{x_{mn}\}_{(m,n) \in M}$ is r-convergent to ξ with respect to the probabilistic norm μ. Let $\eta > 0$ and $\epsilon > 0$. Choose $s \in \mathbb{N}$ such that $\frac{1}{s} < \eta$. Then

$$\begin{aligned} &\{(m, n) \in \mathbb{N} \times \mathbb{N} : \mu_{x_{mn}-\xi}(r + \epsilon) \leq 1 - \eta\} \\ &\subset \left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \mu_{x_{mn}-\xi}(r + \epsilon) \leq 1 - \frac{1}{s} \right\} \subset \bigcup_{i=1}^{s+1} E_i. \end{aligned}$$

Since $E_i \triangle Q_i, i = 1, \cdots, s + 1$ are finite, there exists $(m_0, n_0) \in \mathbb{N} \times \mathbb{N}$ such that

$$\bigcup_{i=1}^{s+1} Q_i \cap \{(m, n) : m \geq m_0, n \geq n_0\} = \bigcup_{i=1}^{s+1} E_i \cap \{(m, n) : m \geq m_0, n \geq n_0, \}.$$

If $m \geq m_0, n \geq n_0$ and $(m, n) \in M$ then $(m, n) \notin \bigcup_{i=1}^{s+1} Q_i$ and so $(m, n) \notin \bigcup_{i=1}^{s+1} E_i$. Hence for every $m \geq m_0, n \geq n_0$ and $(m, n) \in M$ we have

$$\mu_{x_{mn}-\xi}(\epsilon + r) > 1 - \eta.$$

Since $\eta > 0$ was arbitrary, we have $rI_2^{*,\mu}\text{-}\lim x = \xi$. □

Theorem 3.9. *Let $(X, \mu, *)$ be an PNS and r be a non-negative real number. Then the following conditions are equivalent:*

(a) $rI_2^{*,\mu}$-$\lim x = \xi$.

(b) *There exist two sequences $y = \{y_{mn}\}$ and $z = \{z_{mn}\}$ in X such that $x = y + z$,$r\mu$-$\lim y = \xi$ and the set $\{(m, n) : z_{mn} \neq \theta\} \in I_2$, where θ denotes the zero element of X.*

Proof. . Suppose that the condition (a) holds. Then there exists a subset $K = \{(m_j, n_j) : m_1 < m_2 < \cdots ; n_1 < n_2 < \cdots, \cdots\}$ of $\mathbb{N} \times \mathbb{N}$ such that

$$K \in F(I_2) \text{ and } r\mu - \lim_j x_{m_j n_j} = \xi. \tag{3.3}$$

We define the sequences $y = \{y_{mn}\}$ and $z = \{z_{mn}\}$ as

$$y_{mn} = \begin{cases} x_{mn}, & \text{if } (m, n) \in K; \\ \xi, & \text{if } (m, n) \notin K. \end{cases}$$

and $z_{mn} = x_{mn} - y_{mn}$ for all $(m, n) \in \mathbb{N} \times \mathbb{N}$. For given $\epsilon > 0$, $\lambda \in (0, 1)$ and $(m, n) \in K$, we have

$$\mu_{y_{mn} - \xi}(r + \epsilon) > 1 - \lambda.$$

Using (3.3) we have $r\mu$- $\lim y = \xi$. Since

$$\{(m, n) : z_{mn} \neq \theta\} \subset K^c,$$

we have $\{(m, n) : z_{mn} \neq \theta\} \in I_2$.
Let the condition (ii) hold. Then $K = \{(m, n) : z_{mn} = \theta\} \in F(I_2)$ is an infinite set. Obviously, $K \in F(I_2)$ is an infinite set. Let $K = \{(m_j, n_j) : m_1 < m_2 < \cdots ; n_1 < n_2 < \cdots\}$. Since $x_{m_j n_j} = y_{m_j n_j}$ and $r\mu$-$\lim_j y_{m_j n_j} = \xi$, $r\mu$-$\lim_j x_{m_j n_j} = \xi$. Hence $rI_2^{*,\mu}$-$\lim_{m,n} x_{mn} = \xi$. So, the proof is complete. □

Theorem 3.10. *Let $(X, \mu, *)$ be an PNS and r be a non-negative real number. Then the following assertions hold.*

(a) *If X has no accumulation point, then rI_2 and rI_2^*-convergence coincide for each strongly admissible ideal I_2.*

(b) *If X has an accumulation point ξ, then there exists a strongly admissible ideal I_2 and a double sequence $y = \{y_{mn}\}$ for which rI_2^{μ}-$\lim y = \xi$ but $rI_2^{*,\mu}$-$\lim y$ does not exist.*

Proof. (a) Let $x = \{x_{mn}\}$ be a double sequence in X. Suppose that $rI_2^{*,\mu}$-$\lim x = \xi$. So there exists a set $M \in F(I_2)$ (i.e., $\mathbb{N} \times \mathbb{N} \setminus M = H \in I_2$) such that

$$r\mu - \lim_{\substack{m,n \to \infty \\ (m,n) \in M}} x = \xi. \tag{3.4}$$

Let $\epsilon > 0$, $\lambda \in (0, 1)$. Then it follows by (3.4) that there exists $N_0 \in \mathbb{N}$ such that $\mu_{x_{mn} - \xi}(r + \epsilon) > 1 - \lambda$ for all $m, n > N_0$. Then

$$\begin{aligned} A(\lambda) &= \{(m, n) \in \mathbb{N} \times \mathbb{N} : \mu_{x_{mn} - \xi}(r + \epsilon) \leq 1 - \lambda\} \\ &\subset H \cup (M \cap M_1), \end{aligned}$$

where $M_1 = (\{1, 2, \cdots N_0 - 1\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, \cdots r_0 - 1\})$.
Now since $H \cup (M \cap M_1) \in I_2$, consequently, we have $A(\lambda) \in I_2$ and so rI_2^{μ}-$\lim x = \xi$.

Next, we show that if rI_2^{μ}-$\lim x = \xi$, then $rI_2^{*,\mu}$-$\lim x = \xi$. To do this, since X has no accumulation point, then there exists $\delta \in (0,1)$ such that for all $\epsilon > 0$

$$Q(\xi, \delta) = \{x \in X : \mu_{x-\xi}(r+\epsilon) > 1-\delta\} = \{\xi\}.$$

Since rI_2^{μ}-$\lim x = \xi$, so $\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu_{x_{mn}-\xi}(r+\epsilon) \leq 1-\delta\} \in I_2$. This gives

$$\{(m,n) : \mu_{x_{mn}-\xi}(r+\epsilon) > 1-\delta\} = \{(m,n) : x_{mnk} = \xi\} \in F(I_2).$$

Consequently, $rI_2^{*,\mu}$-$\lim x = \xi$.
(b) Since ξ is an accumulation point of X, so there exists a sequence $\{z_j\}_{j\in\mathbb{N}}$ of distinct points all different from ξ in X which is convergent to ξ such that, the sequence $\mu_{z_j-\xi}$ is increasing to 1. Let $\{E_j\}_{j\in\mathbb{N}}$ be a decomposition of $\mathbb{N}$ onto infinite sets and put $\triangle_j = \{(m,n) : \max\{m,n\} \in E_j\}$. Then $\{\triangle_j\}_{j\in\mathbb{N}}$ is a decomposition of $\mathbb{N} \times \mathbb{N}$ and the ideal $I_2 = \{A \subset \mathbb{N} \times \mathbb{N} : A \text{ is included in a finite union of } \triangle_j's\}$ is a strongly admissible ideal. Put $x_{mn} = z_j$ if and only if $(m,n) \in \triangle_j$. Put $\epsilon_n = \mu_{z_n-\xi}$ for $n \in \mathbb{N}$. Let $\eta \in (0,1)$, $\epsilon > 0$. Choose $\gamma \in \mathbb{N}$ such that $\epsilon_\gamma > 1-\eta$. Then $A(\eta) = \{(m,n) : \mu_{x_{mn}-\xi}(r+\epsilon) \leq 1-\eta\} \subset \bigcup_{j=1}^{\gamma} \triangle_j$. Hence $A(\eta) \in I_2$ and rI_2^{μ}-$\lim x = \xi$.
Now suppose that $rI_3^{*,\mu}$-$\lim x = \xi$. Then there exists $H \in I_2$ such that for $M = \mathbb{N}\times\mathbb{N}\setminus H$ we have $r\mu$-$\lim_{\substack{m,n\to\infty \\ (m,n)\in M}} x_{mn} = \xi$. By definition of the ideal I_2, there exists $l \in \mathbb{N}$ such that $H \subset \bigcup_{j=1}^{l} \triangle_j$. But then $\triangle_{l+1} \subset \mathbb{N} \setminus H = M$. Then from the construction of $\triangle_{l+1}$ it follows that for any $N_0 \in \mathbb{N}$, $\mu_{x_{mn}-\xi}(r+\epsilon) = 1-\epsilon_{l+1} < 1$ for $\epsilon > 0$ hold for infinitely many (m,n)'s with $(m,n) \in M$ and $m,n, \geq N_0$. This contradicts that $r\mu$-$\lim_{\substack{m,n\to\infty \\ (m,n)\in M}} x_{mn} = \xi$. Also the assumption $rI_2^{*,\mu}$-$\lim_{m,n\to\infty} x_{mn} = p$ for $p \neq \xi$ leads to the contradiction. □

Theorem 3.11. *Let $(X, \mu, *)$ be an PNS and r be a non-negative real number. If $(X, \mu, *)$ has at least one accumulation point and for any arbitrary double sequence $x = \{x_{mn}\}$ of elements of X and for each $\xi \in X$, rI_2^{μ}-$\lim x = \xi$ implies $rI_2^{*,\mu}$-$\lim x = L$, then I_2 has the property (AP2).*

Proof. Suppose $\xi \in X$ is an accumulation point of X. There exists a sequence $\{z_k\}_{k\in\mathbb{N}}$ of distinct elements of X such that $z_k \neq \xi$ for any k, $\xi = \lim_{k\to\infty} z_k = \xi$ and the sequence $\{\mu_{z_k-\xi}\}_{k\in\mathbb{N}}$ is an increasing sequence converging to 1. Put $\epsilon_k = \psi_{z_k-\xi}$ for $k \in \mathbb{N}$. Let $\{A_j\}_{j\in\mathbb{N}}$ be a disjoint family of nonempty sets from I_2. Define a sequence $x = \{x_{mn}\}$ in the following way:

$$x_{mn} = \begin{cases} z_j, & \text{if } (m,n) \in A_j; \\ \xi, & \text{if } (m,n) \notin A_j. \end{cases}$$

for any j. Let $\epsilon > 0, \eta \in (0,1)$. Choose $k \in \mathbb{N}$ such that $\epsilon_k < \eta$. Then $A(\eta) = \{(m,n) : \mu_{x_{mnk}-\xi}(r+\epsilon) \leq 1-\eta\} \subset \bigcup_{j=1}^{k} A_j$. Hence $A(\eta) \in I_2$ and so rI_2^{μ}-$\lim x = \xi$. By virtue of our assumption, we then have $I_2^{*,\mu}$-$\lim x = \xi$. So there exists a set $B \in I_2$ such that $M = \mathbb{N} \times \mathbb{N} \setminus B \in F(I_2)$ and

$$r\mu\text{-}\lim_{\substack{m,n\to\infty \\ (m,n)\in M}} x_{mn} = \xi \tag{3.5}$$

Put $B_j = A_j \cap B$ for $j \in \mathbb{N}$. Then $B_j \in I_2$ for each $j \in \mathbb{N}$. Moreover $\bigcup_{j=1}^{\infty} B_j = B \cap \bigcup_{j=1}^{\infty} A_j \subset B$ and so $\bigcup_{j=1}^{\infty} B_j \in I_2$. Fix $j \in \mathbb{N}$. If $A_j \cap M$ is not included in the finite union of rows and columns in $\mathbb{N}\times\mathbb{N}$, then M must contain an infinite sequence of elements $\{(m_k, n_k)\}$ where both $m_k, n_k \to \infty$ and $x_{m_k n_k} = z_j \neq \xi$ for all $k \in \mathbb{N}$ which contradicts (3.5). Hence $A_j \cap M$ must be contained in the finite union of rows and columns in $\mathbb{N}\times\mathbb{N}$. Hence $A_j \triangle B_j = A_j \setminus B_j = A_j \setminus B = A_j \cap M$ is also included in the finite union of rows and columns. This proves that the ideal I_2 has the property (AP2). □

4. Rough I_2-and I_2^*-Cauchy of Double sequences

In this section we study the concepts of rI_2^{μ}-Cauchy and $I_2^{*,\mu}$-Cauchy double sequences in $(X, \mu, *)$. Moreover, we will study the relations between them.

Definition 4.1. Let $(X, \mu, *)$ be a PNS and r be a non-negative real number and $I_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = \{x_{mnk}\}$ of elements in X is said to be

(a) a rough I_2^{μ}-Cauchy sequence in X if for every $\lambda \in (0,1)$, $\epsilon > 0$, there exist $s = s(\lambda), t = t(\lambda)$ such that
$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu_{x_{mn}-x_{st}}(r+\epsilon) \leq 1-\lambda\} \in I_2.$$

(b) a rough $I_2^{*,\mu}$-Cauchy sequence in X if for every $\lambda \in (0,1)$, $\epsilon > 0$ there exists
$$K = \{(m_j, n_j) : m_1 < m_2 < \cdots ; n_1 < n_2 < \cdots\} \subset \mathbb{N}\times\mathbb{N}$$
such that $K \in F(I_2)$ and $\{x_{m_j n_j}\}$ is $r\mu$-Cauchy sequence in X.

The next theorem gives a relation between rI_2^{μ}- and $rI_2^{*,\mu}$-double Cauchy sequences

Theorem 4.2. *Let $(X, \mu, *)$ be an PNS and r be a non-negative real number and $I_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. If $\{x_{mn}\}$ is an $rI_2^{*,\mu}$-double Cauchy sequence, then $\{x_{mn}\}$ is an rI_2^{μ}-double Cauchy sequence.*

Proof. Suppose that $\{x_{mn}\}$ is an $rI_2^{*,\mu}$-double Cauchy sequence, for any $\epsilon > 0, \lambda \in (0,1)$, there exist
$$K = \{(m_j, n_j) : m_1 < m_2 < \cdots ; n_1 < n_2 < \cdots\} \in F(I_2)$$
and a number $N_0 \in \mathbb{N}$ such that
$$\mu_{x_{m_j n_j}-x_{m_s n_s}}(r+\epsilon) > 1-\lambda$$
for every $j, s \geq N_0$. Now, fix $p = m_{N_0+1}, q = n_{N_0+1}$. Then for every $\lambda \in (0,1)$, $\epsilon > 0$, we have
$$\mu_{x_{m_j n_j}-x_{pq}}(r+\epsilon) > 1-\lambda$$
for every $j \geq N_0$. Let $H = \mathbb{N}\times\mathbb{N} \setminus K$. It is obvious that $H \in F(I_2)$ and
$$\begin{aligned} A(\lambda) &= \Big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu_{x_{m_j n_j}-x_{pq}}(r+\epsilon) \leq 1-\lambda\Big\} \\ &\subset H \cup \{m_1 < m_2 < \cdots ; n_1 < n_2 < \cdots\} \in I_2. \end{aligned}$$
Therefore, for every $\lambda \in (0,1)$, $\epsilon > 0$, we can find $(p,q) \in \mathbb{N}\times\mathbb{N}$ such that $A(\lambda) \in I_2$, i.e., $\{x_{mn}\}$ is an rI_2^{μ}-double Cauchy sequence. □

Lemma 4.3. [15] *Let $\{P_i\}_{i\in\mathbb{N}}$ be a countable collection of subsets of $\mathbb{N}\times\mathbb{N}$ such that $\{P_i\}_{i\in\mathbb{N}} \in F(I_2)$ for each i, where $F(I_2)$ is a filter associate with a strongly admissible ideal I_2 with the property (AP2). Then there exists a set $P \subset \mathbb{N}\times\mathbb{N}$ such that $P \in F(I_2)$ and the set $P \setminus P_i$ is finite for all i.*

Theorem 4.4. *Let $(X, \mu, *)$ be an PNS and r be a non-negative real number and $I_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal with property (AP2). Then the concepts rI_2^{μ}-double Cauchy sequence and $I_2^{*,\mu}$-double Cauchy sequence coincide.*

Proof. If $\{x_{mn}\}$ is $rI_2^{*,\mu}$-double Cauchy sequence, then it is rI_2^{μ}-double Cauchy sequence by Theorem 4.2 (even if I_2 does not have the (AP2) property).
So, we have to prove the converse. Let $\{x_{mn}\}$ be an rI_2^{μ}-double Cauchy sequence. Then by definition, there exists an $m_0 = m_0(\lambda), n_0 = n_0(\lambda)$ such that

$$A(\lambda) = \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu_{x_{mn}-x_{m_0n_0}}(r+\epsilon) \leq 1-\lambda\right\} \in I_2$$

for every $\lambda \in (0,1)$ and $\epsilon > 0$.
Let $P_i = \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu_{x_{mn}-x_{s_it_i},z}(r+\epsilon) > 1-\frac{1}{i}\right\}$, $i = 1, 2, \cdots$, where $s_i = m_0\left(\frac{1}{i}\right), t_i = n_0\left(\frac{1}{i}\right)$. It is clear that $P_i \in F(I_2)$ for $i = 1, 2, \cdots$. Since I_2 has the property (AP2), then by Lemma 4.3 there exists a set $P \subset \mathbb{N}\times\mathbb{N}$ such that $P \in F(I_2)$, and $P \setminus P_i$ is finite for all i. Now we prove that

$$\lim_{\substack{m,n,s,t\to\infty \\ (m,n),(s,t)\in P}} \mu_{x_{mn}-x_{st}}(r+\epsilon) > 1-\lambda.$$

To prove this, let $\epsilon > 0$, $\lambda \in (0,1)$ and $w \in \mathbb{N}$ such that $\left(1-\frac{1}{w}\right) * \left(1-\frac{1}{w}\right) > 1-\lambda$. If $(m,n),(s,t) \in P$, then $P \setminus P_w$ is finite set, so there exists $q = q(w)$ such that $(m,n),(s,t) \in P$ for all $m,n,s,t > q(w)$. Therefore, $\mu_{x_{mn}-x_{s_wt_w}}(r+\epsilon/2) > 1-\frac{1}{w}$ and $\mu_{x_{st}-x_{s_wt_w}}(\epsilon/2) > 1-\frac{1}{w}$ for all $m,n,s,t > q(w)$. Hence it follows that

$$\mu_{x_{mn}-x_{st}}(r+\epsilon) \geq \mu_{x_{mn}-x_{s_wt_w},z}(r+\epsilon/2) * \mu_{x_{st}-x_{s_wt_w}}(\epsilon/2) > \left(1-\frac{1}{w}\right) * \left(1-\frac{1}{w}\right) > 1-\lambda$$

for all $m,n,s,t > q(w)$.
Thus, for any $\epsilon > 0$, $\lambda \in (0,1)$ there exists $q = q(w)$ such that $m,n,s,t > q(w)$ and $m,n,s,t \in P \in F(I_2)$,

$$\mu_{x_{mn}-x_{st}}(r+\epsilon) > 1-\lambda$$

. This shows that x_{mn} is an $rI_2^{*,\mu}$-double Cauchy sequence in X. □

5. Rough limit points and Rough cluster points in PNS

The notions of a rough ideal cluster point and a rough ideal limit point of a sequence in PNS are studied in this section, and some of these concepts' characteristics are examined. We also get a typical ideal convergence criterion linked to a rough ideal cluster point of a sequence in PNS.

Definition 5.1. Let $(X, \mu, *)$ be PNS and r be a non-negative real number. A point $\xi \in X$ is called an I_2^{μ}-limit point of a double sequence $x = \{x_{mn}\}$ if there exists a set $K = \{(m_j, n_j) : j \in \mathbb{N}\} \in F(I_2)$ such that $\lim_{j\to\infty} x_{m_jn_j} = \xi$. The set of all rough I_2^{μ}-limit points of the sequence x will be denoted by $I_2^{\mu}\text{-}LIM_x^r$.

Theorem 5.2. *Let $(X, \mu, *)$ be an PNS and r be a non-negative real number. For a double sequence $x = \{x_{mn}\}$ we have $diam(I_2^{\mu} - LIM_x^r) \leq 2r$. In general $diam(I_2^{\mu} - LIM_x^r)$ has no smaller bound.*

Proof. Let $\epsilon > 0, \lambda \in (0,1)$. Choose $\eta \in (0,1)$ so that $(1-\eta)*(1-\eta) > 1-\lambda$. Assume that $diam(I_2^\mu - LIM_x^r) > 2r$. Then there exists $y, w \in I_2^\mu - LIM_x^r$ such that $\mu_{w-y}(2r) < 1-\lambda$. Put

$$\begin{aligned} A_1 &= \{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu_{x_{mn}-y}(r+\epsilon) \leq 1-\eta\} \text{ and} \\ A_2 &= \{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu_{x_{mn}-w}(r+\epsilon) \leq 1-\eta\} \end{aligned}$$

Then $A_1, A_2 \in I_2$ and hence $M = \mathbb{N}\times\mathbb{N} \setminus (A_1 \cup A_2) \in F(I_2)$ and so $M \neq \emptyset$. Let $(m,n) \in M$. Now

$$\begin{aligned} \mu_{y-w}(2r+2\epsilon) &\geq \mu_{x_{mn}-y}(r+\epsilon) * \mu_{x_{mn}-w}(r+\epsilon) \\ &> (1-\eta)*(1-\eta) > 1-\lambda \end{aligned}$$

which is a contradiction. Thus $diam(I_2^\mu - LIM_x^r) \leq 2r$.
To prove the converse part, consider a sequence $x = \{x_{mn}\}$ such that $rI_2^\mu\text{-}\lim x = \zeta$. Let $\epsilon > 0, \lambda \in (0,1)$ be given. Choose $\eta \in (0,1)$ so that $(1-\eta)*(1-\eta) > 1-\lambda$. Then , $A = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu_{x_{mn}-\zeta,}(\epsilon) \leq 1-\eta\} \in I_2$. Now for each $\xi \in \overline{B}^r(\zeta) = \{\xi \in X : \mu_{\xi-\zeta,z}(r) \leq 1-\eta\}$ we have

$$\begin{aligned} \mu_{x_{mn}-\zeta}(r+\epsilon) &\geq \mu_{x_{mn}-\zeta}(\epsilon) * \mu_{\zeta-\xi}(r) \\ &> (1-\eta)*(1-\eta) > 1-\lambda, \end{aligned}$$

whenever $(m,n) \notin A$. Which shows that $\xi \in I_2^\mu - LIM_x^r$ and consequently we can write $I_2^\mu - LIM_x^r = \overline{B}_\lambda^r(\zeta)$. This shows that in general upper bound $2r$ of the diameter of the set $I_2^\mu - LIM_x^r = \overline{B}_\lambda^r(\zeta)$ can not be decreased anymore. □

Definition 5.3. Let $(X, \mu, *)$ be an PNS and r be a non-negative real number. A double sequence $x = \{x_{mn}\}$ is said to be I_2^ψ-bounded if there exists a real number $M > 0$ such that

$$A = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu_{x_{mn}}(M) \leq 1-\lambda\} \in I_2$$

for every $\lambda \in (0,1)$.

In [24] it was established that there exists a non-negative real number r such that LIM_x^r for a bounded sequence x. As LIM_x^r implies $I - LIM_x^r$, this result is also true for $I_2^\psi - LIM_x^r$ for any admissible ideal I_2. The converse implication is not generally true. For this we have the following result.

Theorem 5.4. *Let $(X, \mu, *)$ be an PNS. A double sequence $x = \{x_{mn}\}$ is I_2^μ-bounded if and only if there is a non-negative real number r such that I_2^μ-$LIM_x^r \neq \emptyset$.*

Proof. Let $x = \{x_{mn}\}$ be an I_2^μ-bounded sequence. Then, for every $\epsilon > 0, \lambda \in (0,1)$ and some $r > 0$, there exists a real number $M > 0$ such that

$$A = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu_{x_{mn}}(M) \leq 1-\lambda\} \in I_2.$$

For $(m,n) \in A^c$ we have $\mu_{x_{mn}}(M) > 1-\lambda$. Also

$$\begin{aligned} \mu_{x_{mn}}(r+M) &\geq \mu_0(r) * \mu_{x_{mn}}(M) \\ &> 1*(1-\lambda) = 1-\lambda. \end{aligned}$$

Hence, $0 \in I_2^\mu - LIM_x^r$ and so I_2^μ-$LIM_x^r \neq \emptyset$.

For the converse, Let I_2^μ-$LIM_x^r \neq \emptyset$ for some $r > 0$. Then there exists $\zeta \in X$ such that $\zeta \in I_2^\mu - LIM_x^r$. For every $\epsilon > 0, \lambda \in (0,1)$ we have

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu_{x_{mn}-\zeta}(r+\epsilon) \leq 1-\lambda\} \in I_2.$$

Therefore, almost all elements x are contained in some ball with center ζ which implies that sequence $x = \{x_{mn}\}$ is I_2^μ-bounded sequence in X. □

The rI_2^μ-limit set of a sequence is then given various topological and geometrical features.

Theorem 5.5. *Let $(X, \mu, *)$ be an PNS and r be a non-negative real number. The r-I_3^ψ-limit set I_2^μ-LIM_x^r of a double sequence $x = \{x_{mn}\}$ is a closed set.*

Proof. If I_2^μ-$LIM_x^r = \emptyset$, then nothing to prove. Assume that I_2^μ-$LIM_x^r \neq \emptyset$ for some $r > 0$ and consider $y = \{y_{mn}\}$ be a convergent sequence in I_2^μ-LIM_x^r with respect to the norm μ to $y_0 \in X$. For $\lambda \in (0,1)$ choose $\eta \in (0,1)$ such that $(1-\eta) * (1-\eta) > 1-\lambda$. Then for every $\epsilon > 0$ and $\eta \in (0,1)$ and each $z \in X$ there exist $m_1, n_1 \in \mathbb{N}$ such that $\mu_{y_{mn}-y_0}(\epsilon) > 1-\eta$ for all $m \geq m_1$ and $n \geq n_1$. Let $(m_0, n_0) \in \mathbb{N} \times \mathbb{N}$ such that $y_{m_0 n_0} \in \{y_{mn}\} \subseteq I_2^\mu$-$LIM_x^r$. Therefore, the set $K = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu_{x_{mn}-y_{m_0 n_0}}(r+\epsilon) \leq 1-\eta\} \in I_2$. Clearly, $E = \mathbb{N} \times \mathbb{N} \setminus K \in F(I_2)$ and hence $E \neq \emptyset$. Let $(p,q,l) \in E$ and choose $m_0 > m_1, n_0 > n_1$. Consequently,

$$\begin{aligned} \mu_{x_{pq}-y_0}(r+2\epsilon) &\geq \mu_{x_{pq}-y_{m_0 n_0}}(r+\epsilon) * \mu_{y_{m_0 n_0}-y_0}(\epsilon) \\ &> (1-\eta) * (1-\eta) > 1-\lambda. \end{aligned}$$

Hence

$$\begin{aligned} E &= \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu_{x_{mn}-y_{m_0 n_0}}(r+\epsilon) > 1-\eta\right\} \\ &\subseteq \{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu_{x_{mn}-y_0}(r+2\epsilon) > 1-\eta\}. \end{aligned}$$

where $\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu_{x_{mn}-y_{m_0 n_0}}(r+\epsilon) > 1-\eta\right\} \in F(I_2)$. Hence

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu_{x_{mn}-y_0}(r+2\epsilon) > 1-\eta\} \in F(I_2)$$

and so $\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu_{x_{mn}-y_0}(r+2\epsilon) \leq 1-\eta\} \in I_2$. Therefore, the proof is complete. □

Theorem 5.6. *Let $(X, \mu, *)$ be an PNS and r be a non-negative real number. The rI_2^μ-limit set I_2^μ-LIM_x^r of a double sequence $x = \{x_{mn}\}$ is convex.*

Proof. Let $\epsilon > 0$ and $\lambda \in (0,1)$ be given. Choose choose $\eta \in (0,1)$ such that $(1-\eta) * (1-\eta) > 1-\lambda$. Let $y_0, y_1 \in I_2^\mu$-LIM_x^r. So, we put

$$\begin{aligned} Q_1 &= \{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu_{x_{mn}-y_0}(r+\epsilon) \leq 1-\eta\} \text{ and} \\ Q_2 &= \{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu_{x_{mn}-y_1}(r+\epsilon) \leq 1-\eta\}. \end{aligned}$$

Then $Q_1 \cup Q_2 \in I_2$ which implies that $E = \mathbb{N} \times \mathbb{N} \setminus (Q_1 \cup Q_2) \in F(I_2)$ and so $E \neq \emptyset$. Now for all $(m,n) \in E$ and each $\alpha \in [0,1]$ we have

$$\begin{aligned} \mu_{x_{mn}-(\alpha y_0+(1-\alpha)y_1)}(r+\epsilon) &= \mu_{x_{mn}-(\alpha y_0+(1-\alpha)y_1)}(\alpha(r+\epsilon)+(1-\alpha)(r+\epsilon) \\ &\geq \mu_{\alpha x_{mn}-\alpha y_0}(\alpha(r+\epsilon)) * \mu_{(1-\alpha)x_{mn}-(1-\alpha)y_0}((1-\alpha)(r+\epsilon)) \\ &= \mu_{x_{mn}-y_0}\left(\alpha\frac{r+\epsilon}{\alpha}\right) * \mu_{x_{mn}-y_0}\left((1-\alpha)\frac{r+\epsilon}{1-\alpha}\right) \\ &> (1-\eta) * (1-\eta) > 1-\lambda. \end{aligned}$$

Hence,

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu_{x_{mn}-(\alpha y_0+(1-\alpha)y_1)}(r+\epsilon)\le 1-\lambda\right\}\in I_2$$

which implies that $\alpha y_0+(1-\alpha)y_1\in I_2^{\mu}\text{-}LIM_x^r$. Consequently, $I_2^{\mu}\text{-}LIM_x^r$ of a double sequence $x=\{x_{mn}\}$ is convex. □

Theorem 5.7. *Let $(X,\mu,*)$ be an PNS and r be a non-negative real number. Then a double sequence $x=\{x_{mn}\}$ is rI_2^{μ}-convergent to ζ if and only if there exists a sequence $y=\{y_{mn}\}$ such that $I_2^{\mu}\text{-}\lim y=\xi$ and $\mu_{x_{mn}-y_{mn}}(r)>1-\lambda$ for all $(m,n)\in\mathbb{N}\times\mathbb{N}$ and every $\lambda\in(0,1)$.*

Proof. Let $\epsilon>0,\lambda\in(0,1)$ and each $z\in X$. Suppose that $y_{mn}\to r^{\mu}\zeta$ and $\mu_{x_{mn}-y_{mn}}(r)>1-\lambda$ for all $m,n\in\mathbb{N}$. For given $\lambda\in(0,1)$ choose $\eta\in(0,1)$ such that $(1-\eta)*(1-\eta)>1-\lambda$. Define

$$\begin{aligned}A&=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu_{y_{mn}-\zeta}(\epsilon)\le 1-\eta\right\}\text{ and}\\ B&=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu_{x_{mn}-y_{mn}}(r)\le 1-\eta\right\}.\end{aligned}$$

Then $A\cup B\in I_2$ and hence $M=\mathbb{N}\times\mathbb{N}\setminus(A\cup B)\in F(I_2)$. For $(m,n)\in M$, we have

$$\begin{aligned}\mu_{x_{mn}-\zeta}(r+\epsilon)&\ge\mu_{y_{mn}-\zeta}(\epsilon)*\mu_{x_{mn}-y_{mn}}(r)\\&>(1-\eta)*(1-\eta)>1-\lambda.\end{aligned}$$

Then $\mu_{x_{mn}-\zeta}(r+\epsilon)>1-\lambda$ for all $(m,n)\in M$. This implies that

$$\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu_{x_{mn}-\zeta}(r+\epsilon)\le 1-\lambda\}\subseteq A\cup B$$

and so

$$\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu_{x_{mn}-\zeta}(r+\epsilon)\le 1-\lambda\}\in I_2$$

and this implies that $x_{mn}\xrightarrow{rI_2^{\mu}}\zeta$.
Conversely, suppose that the given condition holds. For any $\epsilon>0$ and $\lambda\in(0,1)$. Choose $\eta\in(0,1)$ such that $(1-\eta)*(1-\eta)>1-\lambda$. Then the set

$$A=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu_{y_{mn}-\zeta}(\epsilon)\le 1-\eta\}\in I_2.$$

Observe that

$$\begin{aligned}\mu_{x_{mn}-\zeta}(r+\epsilon)&\ge\mu_{x_{mn}-y_{mn}}(r)*\mu_{y_{mn}-\zeta}(\epsilon)\\&>(1-\eta)*(1-\eta)>1-\lambda\end{aligned}$$

for all $(m,n)\in M=\mathbb{N}\times\mathbb{N}\setminus A$. This shows that

$$\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu_{x_{mn}-\zeta}(r+\epsilon)\le 1-\lambda\}\subseteq A\in I_2.$$

Consequently, $x_{mn}\xrightarrow{rI_2^{\mu}}\zeta$. □

Definition 5.8. Let $(X,\psi,*)$ be a PNS and r be a non-negative real number. $\xi\in X$ is called a rough I_2^{ψ}-cluster point of a double sequence $x=\{x_{mn}\}$ if for any $\epsilon>0$ and $\lambda\in(0,1)$, the set $\{(m,n):\psi_{x_{mn}-\xi}(r+\epsilon)>1-\lambda\}\notin I_2$. The set of all I_3^{ψ}-cluster points of x will be denoted by $I_2^{\mu}-C_x^r$.

Theorem 5.9. *Let $(X,\mu,*)$ be an PNS and r be a non-negative real number. For an arbitrary $\nu\in I_2^{\mu}-C_x^r$, where $x=\{x_{mn}\}$ is a double sequence in X, we have $\mu_{\nu-\xi}(r)\ge 1-\lambda$ for all $\xi\in I_3^{\mu}\text{-}LIM_x^r$ and every $\lambda\in(0,1)$.*

Proof. For $\lambda \in (0,1)$ choose $\eta \in (0,1)$ such that $(1-\eta)*(1-\eta) > 1-\lambda$. Let $\nu \in I_2^\mu - C_x^r$. Then, for every $\epsilon > 0, \eta \in (0,1)$ we have

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu_{x_{mn}-\nu}(\epsilon) > 1-\eta\} \notin I_2. \tag{5.1}$$

Now we will show that if for $\xi \in X$ we have $\mu_{\xi-\nu}(r) > 1-\eta$ then $\xi \in I_2^\mu - C_x^r$. Let $(i,j) \in \{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu_{x_{mn}-\nu}(\epsilon) > 1-\eta\}$ then $\mu_{x_{ij}-\nu}(\epsilon) > 1-\eta$. Now

$$\begin{aligned} \mu_{x_{ij}-\xi}(r+\epsilon) &\geq \mu_{x_{ij}-\nu}(\epsilon) * \mu_{\nu-\xi}(r) \\ &> (1-\eta)*(1-\eta) > 1-\lambda. \end{aligned}$$

We have $\mu_{x_{ij}-\xi}(r+\epsilon) > 1-\lambda$. Thus

$$(i,j) \in \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu_{x_{ij}-\xi}(r+\epsilon) > 1-\lambda\right\}.$$

Now the next inclusion holds.

$$\begin{aligned} &\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu_{x_{mn}-\nu}(\epsilon) > 1-\eta\} \\ &\subseteq \{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu_{x_{mn}-\xi}(r+\epsilon) > 1-\lambda\}. \end{aligned}$$

Using equation (5.1) we get

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu_{x_{mn}-\xi}(r+\epsilon) > 1-\lambda\} \notin I_2$$

and hence $\xi \in I_2^\mu - C_x^r$. □

Theorem 5.10. *Let $(X,\mu,*)$ be an PNS and r be a non-negative real number and $\lambda \in (0,1)$. A double sequence $x = \{x_{mn}\}$ is rI_2^μ-convergent to ξ if and only if I_2^μ-$LIM_x^r = \overline{B}_\lambda^r(\xi)$.*

Proof. In Theorem 5.2 we have already proved the necessity part. For the sufficiency, let I_2^μ-$LIM_x^r = \overline{B}_\lambda^r(\xi) \neq \emptyset$. Thus the sequence $x = \{x_{mn}\}$ is I_2^μ-bounded. Suppose that x has another I_2^μ-cluster point ζ different from ξ. The point $\bar{\xi} = \xi + \frac{r}{\|\xi-\zeta\|}(\xi-\zeta)$ satisfies $\mu_{\bar{\xi}-\xi}(r) < 1-\lambda$ for every $\lambda \in (0,1)$. Since $\zeta \in I_3^\mu - C_x^r$, it follows by Theorem 5.9 that $\zeta \notin I_2^\mu$-LIM_x^r. But this is impossible as $\mu_{\bar{\xi}-\xi}(r) > 1-\lambda$ and I_2^μ-$LIM_x^r = \overline{B}_\lambda^r(\xi)$. Since ξ is the unique I_2^μ-cluster point of x, it follows that x is rI_2^μ-convergent. □

Theorem 5.11. *Let $(X,\mu,*)$ be an PNS and r be a non-negative real number. Let $x = \{x_{mn}\}$ be a double sequence in X. Then for every $\lambda \in (0,1)$, we have*

(i) If $c \in I_2^\mu - C_x$, then $I_2^\mu - LIM_x^r \subseteq \overline{B}_\lambda^r(c)$.

(ii) $I_2^\mu - LIM_x^r = \bigcap_{c \in I_2^\mu - C_x} \overline{B}_\lambda^r(c) = \{\zeta \in X : I_2^\mu - C_x \subseteq \overline{B}_\lambda^r(c)\}$.

Proof. (i) Let $\epsilon > 0$. For given $\lambda \in (0,1)$ choose $\eta \in (0,1)$ such that $(1-\eta)*(1-\eta) > 1-\lambda$. Consider $\zeta \in I_2^\mu - C_x^r$ and $c \in I_2^\mu - C_x$. For every $\epsilon > 0, \eta \in (0,1)$, define the sets

$$A = \{(m,n) \in \mathbb{N} : \mu_{x_{mn}-\zeta}(r+\epsilon) > 1-\eta\} \notin I_2,$$

and

$$B = \{(m,n) \in \mathbb{N} : \mu_{x_{mn}-c}(\epsilon) > 1-\eta\} \notin I_2.$$

Now if $(m,n) \in A \cap B$ we have

$$\begin{aligned} \mu_{\zeta-c}(r+2\epsilon) &\geq \mu_{x_{mn}-c}(r+\epsilon) * \mu_{x_{mn}-\zeta}(\epsilon) \\ &> (1-\eta)*(1-\eta) > 1-\lambda. \end{aligned}$$

Consequently, $\zeta \in \overline{B}_\lambda^r(c)$ and so $I_2^\mu - LIM_x^r \subseteq \overline{B}_\lambda^r(c)$.
(ii) By previous part we have $I_2^\mu - LIM_x^r \subseteq \bigcap_{c \in I_2^\mu - C_x} \overline{B}_\lambda^r(c)$.

Assume that $m \in \bigcap_{c\in C_\mu x} \overline{B}^r_\lambda(c)$. Then $\mu_{m-c}(r) > 1-\lambda$ for each $z \in X$ and for all $c \in I_2^\mu - C_x$. This implies that $I_2^\mu - C_x \subseteq \overline{B}^r_\lambda(c)$, i.e.,

$$\bigcap_{c\in I_2^\mu - C_x} \overline{B}^r_\lambda(c) \subseteq \left\{\zeta \in X : I_2^\mu - C_x \subseteq \overline{B}^r_\lambda(c)\right\}.$$

Further, let $m \notin I_2^\mu - LIM_x^r$ then for $\epsilon > 0$ we have

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu_{x_{mn}-m}(r+\epsilon) \leq 1-\lambda\} \notin I_2,$$

which implies that the existence of a I_2^μ-cluster point c of the sequence $x = \{x_{mn}\}$ with $\mu_{x_{mn}-y}(r+\epsilon) \leq 1-\lambda$. Hence $I_2^\mu - C_x \nsubseteq \overline{B}^r_\lambda(c)$ and $m \notin \{\zeta \in X : I_2^\mu - C_x \subseteq \overline{B}^r_\lambda(c)\}$. This implies that $\{\zeta \in X : I_2^\mu - C_x \subseteq \overline{B}^r_\lambda(c)\} \subseteq I_2^\mu - LIM_x^r$ and we obtain $\bigcap_{c\in I_2^\mu - C_x} \overline{B}^r_\lambda(c) \subseteq I_2^\mu - LIM_x^r$. Consequently,

$$I_2^\mu - LIM_x^r = \bigcap_{c\in I_2^\mu - C_x} \overline{B}^r_\lambda(c) = \left\{\zeta \in X : I_2^\mu - C_x \subseteq \overline{B}^r_\lambda(c)\right\}.$$

□

Theorem 5.12. *Let $(X,\mu,*)$ be an PNS and r be a non-negative real number. Let $x = \{x_{mn}\}$ be an I_2^μ-bounded sequence. If $r = diam(I_2^\mu - C_x^r)$, then we have $I_2^\mu - C_x^r \subseteq I_2^\mu - LIM_x^r$.*

Proof. Let $c \notin I_2^\mu - LIM_x^r$. Then there exists an $\epsilon' > 0$ such that the set

$$K = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \mu_{x_{mn}-c}(r+\epsilon') \leq 1-\lambda\} \notin I_2.$$

Since the sequence is I_2^μ-bounded, there exists an I_2^μ-cluster point c' such that $\mu_{c-c'}(r+\epsilon'/2) < 1-\lambda$. Consequently $c \notin I_2^\mu - C_x^r$ and the result follows. □

6. Conclusion and Future Work

In conclusion, our study delved into the concepts of rough $\mathcal{I}_2$ -Cauchy and rough $\mathcal{I}_2^*$-Cauchy for double sequences within the framework of random 2-normed spaces. We extended our exploration to include the development and examination of rough $\mathcal{I}_2$ -convergence, rough $\mathcal{I}_2^*$-convergence, rough $\mathcal{I}_2$-limit points, and rough $\mathcal{I}_2$-cluster points for probabilistic normed double sequences. A key result of our investigation was the establishment of the relationship between rough $\mathcal{I}_2^*$-convergence and rough $\mathcal{I}_2$-convergence in probabilistic normed spaces. Specifically, we demonstrated that rough $\mathcal{I}_2^*$-convergence implies rough $\mathcal{I}_2$-convergence. This insight contributes to a deeper understanding of the connections between these two convergence concepts. Furthermore, we presented an illustrative example showcasing that rough $\mathcal{I}_2$-convergence in probabilistic normed space does not necessarily imply rough $\mathcal{I}_2^*$-convergence. This exemplifies the nuanced nature of these convergence notions and highlights the importance of distinguishing between them in certain scenarios. Our findings underscore the significance of condition (AP2) in summability analysis when employing ideals. The role of this condition in shaping the convergence behavior within the context of probabilistic normed spaces has been elucidated through our study.

As for future work, potential directions include investigating the interplay between rough $\mathcal{I}_2$-and $\mathcal{I}_2^*$-convergence in more specialized settings, exploring applications of these concepts in specific mathematical or statistical contexts, and extending our analysis to encompass other types of normed spaces or mathematical structures. Additionally, further exploration of the implications of condition (AP2) in various mathematical analyses and the development of new conditions for enhanced convergence understanding could be valuable avenues for future research.

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