Poincare Journal of Analysis & Applications
Vol. 11, No. 1 (2024), 53-66
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DOI: 10.46753/pjaa.2024.v011i01.004
Online Published on 20. 06. 2024

A BANACH BUNDLE APPROACH TO DYNAMICAL PROCESSES ON CHANGING NETWORKS

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Date of Receiving	:	14. 03. 2023
Date of Revision	:	08. 09. 2023
Date of Acceptance	:	22. 01. 2024

Abstract. In this paper, we study dynamical processes where the underlying network is allowed to change in time. In order to do so, we use the theory of Banach bundles which allow us to analyze all changes in the dynamics at the same time in a uniform manner.

1. Introduction

When modeling dynamical systems it is common that the dynamics interact between certain subsystems. Nowadays, it is important to couple thousands or even millions of sub-systems linked in a network-like manner. The largest and most well-known network is the world wide web. But there are numerous examples of networks, each with its accompanying dynamics, e.g., electric distribution networks, water supply, or road and railway networks. The difficulty of analyzing such dynamics is then due to the diversity of interactions. Much of the diversity derives from interactions, or coupling, of partial differential equations, that act on sub-systems. There are applications for example in control theory, quantum chaos, spectral theory or inverse problems, just to mention a few.

However, if real-life situations it might happen, that the underlying graph structure changes with time. If we think for example about a road network, it might happen that some roads will be closed due to maintenance or some roads are added to prevent traffic congestion. In the abstract framework of graphs, one would then delete or add certain edges. The dynamical process that are described on the graphs can therefore also change and the question is, whether we are still able to analyze the changed evolution of the system.

2010 *Mathematics Subject Classification.* 47D06, 65J08, 37L05, 82C70, 35R02, 14D21.
Key words and phrases. strongly continuous semigroups, Banach bundles, transport problems, networks.
Communicated by: Masoumeh Aghajani
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A first operator theoretical approach to evolution equations on networks by means of flows on finite graphs was first presented by Kramar and Sikolya [26] and further used in [33, 19, 17, 6, 2, 5]. Later on, this theory has been extended even to infinite (and locally finite) graphs, see for example [15, 16, 17, 3, 4]. We also refer to the monograph by Mugnolo [34] for a general overview of evolution equations on networks, especially if one want to study differential operators on networks in more detail. Recently, also bi-continuous semigroups on networks have attracted some interest [10].

Evolution equations in connection with variable underlying networks occurred already in the framework of non-autonomous evolution equations, see for example [1, 6, 30]. In a partially distinguished research area, namely in the framework of piecewise deterministic Markov processes, switched systems, quantum dissipative systems as well as random evolution equations, one studies evolution equations on families of state spaces that are connected in a graph-like fashion, cf. [7, 14, 24, 35]. In this paper, we concentrate on a certain functional analytic approach to evolution equations on changing networks rather than on explicit evolution equations.

The idea of how the changing networks will be monitored in a mathematical framework is the following: every graph with its corresponding dynamic give rise to a new phase space (which is a Banach space) and another linear operator on that space. We will collect all the information we have by means of so-called Banach bundles which for example have been recently studied by Kreidler and Siewert [27] to obtain a Gelfand-type theorem for dynamical Banach modules.

2. Preliminaries

Here, we want to recall the definition and basic properties of so-called strongly continuous one-parameter semigroups of linear operators, or C_0-semigroups for short. For more details and a broader overview we refer for example to the monographs by Engel and Nagel [20], Goldstein [22] or Pazy [36].

Definition 2.1. A family of bounded linear operators $(T(t))_{t\geq 0}$ on a Banach space X is called *strongly continuous one-parameter semigroup of linear operators*, or *C_0-semigroup*, if the following properties are satisfied:

(i) $T(t+s)=T(t)T(s)$ and $T(0)=\mathrm{I}$ for all $t,s\geq 0$.

(ii) $\lim_{t\searrow 0}\|T(t)f-f\|=0$ for each $f\in X$.

Each C_0-semigroup $(T(t))_{t\geq 0}$ yields an operator $(A,\mathrm{D}(A))$ called the (infinitesimal) generator defined by:

$$Af:=\lim_{t\searrow 0}\frac{T(t)f-f}{t},\quad \mathrm{D}(A):=\left\{f\in X:\ \lim_{t\searrow 0}\frac{T(t)f-f}{t}\text{ exists}\right\}.$$

The question which operator $(A,\mathrm{D}(A))$ yields a generator of a C_0-semigroups is more involved and is answered by the so-called Hille–Yosida theorem, cf. [20, Chapter II, Thm. 3.8]. Operator semigroups serve as solutions of Banach space valued initial problems, so-called abstract Cauchy problems, which are of the form

$$\begin{cases}\dot{u}(t)=Au(t) & ,\quad t\geq 0,\\ u(0)=x\in X,\end{cases}\tag{ACP}$$

where $(A, \mathrm{D}(A))$ is some linear operator on a Banach space X. Let us recall how (classical) solutions of (ACP) are defined and when we call (ACP) well-posed.

Definition 2.2. A function $u : \mathbb{R}_+ \to X$ is called a *(classical) solution* of (ACP) if u is continuously differentiable with respect to t, $u(t) \in \mathrm{D}(A)$ for all $t \geq 0$ and (ACP) holds.

Definition 2.3. The abstract Cauchy problem (ACP) is called *well-posed* if for every $f \in \mathrm{D}(A)$, there exists a unique solution $u(\cdot, f)$ of (ACP), $\mathrm{D}(A)$ is dense in X and if for every sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{D}(A)$ with $\lim_{n\to\infty} f_n = 0$, one has $\lim_{n\to\infty} u(t, f_n) = 0$ uniformly in compact intervals.

Remark 2.4. By [20, Chapter II, Cor. 6.9] the abstract Cauchy problem (ACP) is well-posed in the sense of Definition 2.3 if and only if the operator $(A, \mathrm{D}(A))$ is the generator of a C_0-semigroup. In this case, the solution of (ACP) is given by $u(t) = T(t)f$.

3. Banach bundles and well-posedness

3.1. Banach bundles. Firstly, we introduce the main tool of this paper, the Banach bundles. One distinguishes between continuous (or topological) and measurable Banach bundles, cf. [27]. As a matter of fact, the idea worked out in [23] and [11] are special cases for continuous and measurable Banach bundles, respectively. This shows, that Banach bundles are powerful analytical tools. For convenience, we stick to the continuous Banach bundles.

Definition 3.1. A *(continuous) Banach bundle* over a locally compact Hausdorff space Ω is a pair $(\mathcal{E}, \pi_{\mathcal{E}})$ consisting of a topological space $\mathcal{E}$ and a continuous, open and surjective mapping $\pi_{\mathcal{E}} : \mathcal{E} \to \Omega$ satisfying the following conditions:

(i) There exists a norm $\|\cdot\|_\omega$ such that each fiber $X_\omega := \pi_{\mathcal{E}}^{-1}(\{\omega\})$, $\omega \in \Omega$, is a complete vector space with respect to $\|\cdot\|_\omega$, i.e., $(X_\omega, \|\cdot\|_\omega)$ is a Banach space.

(ii) The mappings

$$+ : \mathcal{E} \times_\Omega \mathcal{E} \to \mathcal{E} : \quad (x, y) \mapsto x +_{X_{\pi_{\mathcal{E}}(y)}} y,$$
$$\cdot : \mathbb{K} \times \mathcal{E} \to \mathcal{E} : \quad (\lambda, x) \mapsto \lambda \cdot_{X_{\pi_{\mathcal{E}}(y)}} x,$$

are continuous, where $\mathcal{E} \times_\Omega \mathcal{E} := \{(x, y) \in \mathcal{E} \times \mathcal{E} : \pi_{\mathcal{E}}(x) = \pi_{\mathcal{E}}(y)\}$ is equipped with the subspace topology. Moreover, with $+_{X_{\pi_{\mathcal{E}}(x)}}$ and $\cdot_{X_{\pi_{\mathcal{E}}(x)}}$ we denote the addition and scalar multiplication on the Banach space $X_{\pi_{\mathcal{E}}(x)}$, respectively. In what follow, we assume $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

(iii) The map

$$\|\cdot\| : \mathcal{E} \to \mathbb{R}_{\geq 0}, \quad x \mapsto \|x\|_{X_{\pi_{\mathcal{E}}(x)}},$$

is continuous.

(iv) For each $\omega \in \Omega$ and each open $W \subseteq \mathcal{E}$ containing the zero $0_\omega \in X_\omega$ there exists $\varepsilon > 0$ and an open neighborhood U of $\omega \in \Omega$ such that

$$\left\{x \in \pi_{\mathcal{E}}^{-1}(U) : \|x\| \leq \varepsilon\right\} \subseteq W.$$

Remark 3.2. Observe that $\mathcal{E} = \coprod_{\omega\in\Omega} X_\omega$ and that condition (iv) is equivalent to the following condition: if $\omega \in \Omega$ and $(x_\alpha)_\alpha$ is a net in $\mathcal{E}$ such that $\|x_\alpha\|_{X_{\pi_{\mathcal{E}}(x_\alpha)}} \to 0$ and $\pi_{\mathcal{E}}(x_\alpha) \to \omega$ in Ω, then $x_\alpha \to 0_\omega$ in $\mathcal{E}$ (we notice, that this equivalent condition is for example used in [12, Sect. 5.1]).

Indeed, if condition (iv) holds true and if $\omega \in \Omega$ and $(x_\alpha)_\alpha$ is a net in $\mathcal{E}$ such that $\|x_\alpha\|_{X_{\pi_{\mathcal{E}}(x_\alpha)}} \to 0$ and $\pi_{\mathcal{E}}(x_\alpha) \to \omega$ in Ω, then we want to show that $x_\alpha \to 0_\omega$ in $\mathcal{E}$. For this purposed take a neighbourhood of 0_ω. By condition (iv) this neighbourhood is of the form $\left\{x \in \pi_{\mathcal{E}}^{-1}(U) : \|x\| \leq \varepsilon\right\}$ for some open neighbourhood U of Ω and $\varepsilon > 0$. Moreover, as $\pi_{\mathcal{E}}(x_\alpha) \to \omega$ there exists an index α_0 such that $x_\alpha \in \pi_{\mathcal{E}}^{-1}(U)$ for $\alpha \succ \alpha_0$. Furthermore, we assumed that $\|x_\alpha\|_{X_{\pi_{\mathcal{E}}(x_\alpha)}} \to 0$. Hence, there exists an index α_1 such that $\|x_\alpha\|_{X_{\pi_{\mathcal{E}}(x_\alpha)}} < \varepsilon$ for $\alpha \succ \alpha_1$. In summary, there exists an index such that x_α in this aforementioned neighbourhood.

For the converse implication we assume for the sake of a contradiction that on the one side holds that if $\omega \in \Omega$ and $(x_\alpha)_\alpha$ is a net in $\mathcal{E}$ such that $\|x_\alpha\|_{X_{\pi_{\mathcal{E}}(x_\alpha)}} \to 0$ and $\pi_{\mathcal{E}}(x_\alpha) \to \omega$ in Ω, then $x_\alpha \to 0_\omega$ in $\mathcal{E}$, but condition (iv) is not true. Then there exist ω_0 a neighbourhood V of 0_{ω_0} such that the sets of the form $\left\{x \in \pi_{\mathcal{E}}^{-1}(U) : \|x\| \leq \varepsilon\right\}$ do not form a neighbourhood basis. This means, that for every open neighborhood U and every $\varepsilon > 0$ there is an $x_{U,\varepsilon}$ that does not lie in V but whose base point lies in U and that has a norm smaller than ε. The set of all open neighbourhoods of ω_0 is directed by reverse inclusion, and $\mathbb{R}_+$ is also directed by reverse order, so the product set of neighborhoods of ω_0 and $\mathbb{R}_+$ is also directed and thus $(x_{U,\varepsilon})$ is a net. However, its base points then converge towards ω_0 and its norms tend to 0. On the other hand, it cannot converge towards 0_ω because it is always outside of V.

A continuous Banach bundle gives naturally rise to a space of certain continuous functions vanishing at infinity.

Definition 3.3. A continuous section for a Banach bundle $(\mathcal{E}, \pi_{\mathcal{E}})$ is a continuous map $s : \Omega \to \mathcal{E}$ such that $\pi_{\mathcal{E}} \circ s = \mathrm{Id}_\Omega$. The set of continuous sections of $(\mathcal{E}, \pi_{\mathcal{E}})$ is denoted by $\Gamma(\mathcal{E})$. The subspace $\Gamma_0(\mathcal{E})$ of all continuous sections vanishing at infinity is defined by

$$\Gamma_0(\mathcal{E}) := \{s \in \Gamma(\mathcal{E}) : \forall \varepsilon > 0\ \exists K \subseteq \Omega \text{ compact } \forall \omega \in \Omega \setminus K : \|s(\omega)\|_\omega < \varepsilon\}.$$

Notice that $\Gamma_0(\mathcal{E})$ naturally becomes a Banach space if equipped with the supremum norm $\|\cdot\|_{\Gamma_0(\mathcal{E})}$ defined by

$$\|s\|_{\Gamma_0(\mathcal{E})} := \sup_{\omega\in\Omega} \|s(\omega)\|_\omega, \quad s \in \Gamma_0(\mathcal{E}),$$

see for example [31, p. 450] or [32, p. 18].

Remark 3.4. The choice of the space $\Gamma_0(\mathcal{E})$ has different reasons. On the one hand, the space of continuous functions over a locally compact Hausdorff space vanishing at infinity is the prototype of commutative C*-algebras, i.e., every commutative C*-algebra is isometric isomorphic to $\mathrm{C}_0(\Omega)$ for some locally compact Hausdorff space Ω. In the language of Banach bundles, one can rephrase this: every commutative C*-algebra is of the form $\Gamma_0(\mathcal{E})$ for some trivial bundle $\mathcal{E}$ with fiber $\mathbb{C}$, i.e., a bundle of the form $\mathcal{E} = \Omega\times\mathbb{C}$ with $\pi_{\mathcal{E}}\colon \Omega\times\mathbb{C} \to \Omega$ the projection onto the first component. In particular, for a trivial bundle $\mathcal{E} = \Omega \times \mathbb{C}$ one has $\Gamma_0(\mathcal{E}) = \mathrm{C}_0(\Omega)$, i.e., [21, Chapter VII, Sect. 8.1 & Chapter VI, Thm. 4.3]. On the other hand, the space $C_0(\Omega)$ seems to be a suitable Banach space to work with in the framework of C_0-semigroups, cf. [20, Chapter I, Sect. 4]. For example, the larger space $\mathrm{C_b}(\Omega)$ of bounded continuous functions is mostly too big for

C_0-semigroups. In fact, in the literature Markov process are generally considered on $C_0(\Omega)$ as $\mathrm{C_b}(\Omega)$ lead to weaker types of operator semigroups, cf. [28, 25, 29, 9].

3.2. Well-posedness on Banach bundles. As $\Gamma_0(\mathcal{E})$ is a Banach space, we are also able to consider abstract Cauchy problems on $\Gamma_0(\mathcal{E})$. So let $(\mathcal{E}, \pi_{\mathcal{E}})$ be a continuous Banach bundle and assume that for each $\omega \in \Omega$ there is a linear operator $(A_\omega, \mathrm{D}(A_\omega))$ on the Banach space $X_\omega := \pi_{\mathcal{E}}^{-1}(\{\omega\})$. Each operator $(A_\omega, \mathrm{D}(A_\omega))$ on its own yields an abstract Cauchy problem on X_ω, i.e., one considers

$$\begin{cases} \dot{u}_\omega(t) = A_\omega u_\omega(t), & t \geq 0, \\ u_\omega(0) = x_\omega \in \mathrm{D}(A_\omega), \end{cases} \tag{ACP_ω}$$

where $u_\omega : \mathbb{R}_{\geq 0} \to X_\omega$ for each $\omega \in \Omega$. We now turn to the Banach bundle view and introduce an operator $(\mathscr{A}, \mathrm{D}(\mathscr{A}))$ on $\Gamma_0(\mathcal{E})$ built up from the operators $(A_\omega, \mathrm{D}(A_\omega))$, $\omega \in \Omega$. In particular, one defines $(\mathscr{A}, \mathrm{D}(\mathscr{A}))$, similarly to [23, Def. 3.1] and [11, Def. 2.1],

$$(\mathscr{A}s)(\omega) := A_\omega s(\omega), \quad \mathrm{D}(\mathscr{A}) := \{s \in \Gamma_0(\mathcal{E}) : \ s(\omega) \in \mathrm{D}(A_\omega), \ (\omega \mapsto A_\omega s(\omega)) \in \Gamma_0(\mathcal{E})\} \tag{3.1}$$

Indeed, this is a well-defined linear operator on $\Gamma_0(\mathcal{E})$. In fact, for $s \in \mathrm{D}(\mathscr{A})$ the expression $A_\omega s(\omega)$, $\omega \in \Omega$, is well-defined as $s(\omega) \in \mathrm{D}(A_\omega)$. Moreover, $(\mathscr{A}, \mathrm{D}(\mathscr{A}))$ is an operator on $\Gamma_0(\mathcal{E})$ as $(\omega \mapsto A_\omega s(\omega)) \in \Gamma_0(\mathcal{E})$. Of course, one can always define this operator $(\mathscr{A}, \mathrm{D}(\mathscr{A}))$ and $\mathrm{D}(\mathscr{A}) \neq \varnothing$ as $\omega \to 0_\omega$ is always element of $\mathrm{D}(\mathscr{A})$. Let us prove that under reasonable assumptions the operator $(\mathscr{A}, \mathrm{D}(\mathscr{A}))$ is a closed operator.

Lemma 3.5. *Let $(A_\omega, \mathrm{D}(A_\omega))$, $\omega \in \Omega$, be closed operators. Then $(\mathscr{A}, \mathrm{D}(\mathscr{A}))$ is also a closed operator.*

Proof. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{D}(\mathscr{A})$ such that $s_n \to s$ and $\mathscr{A}s_n \to u$ in $\Gamma_0(\mathcal{E})$ for $n \to \infty$. In particular, one has that $s_n(\omega) \to s(\omega)$ and $(\mathscr{A}s)(\omega) = A_\omega s(\omega) \to u(\omega)$ for all $\omega \in \Omega$. However, by assumption each operator $(A_\omega, \mathrm{D}(A_\omega))$, $\omega \in \Omega$, is closed, showing that $s(\omega) \in \mathrm{D}(A_\omega)$ and $A_\omega s(\omega) = u(\omega)$ for all $\omega \in \Omega$. Hence $s \in \mathrm{D}(\mathscr{A})$ and $\mathscr{A}s = u$, which shows that $(\mathscr{A}, \mathrm{D}(\mathscr{A}))$ is a closed operator. □

The linear operator $(\mathscr{A}, \mathrm{D}(\mathscr{A}))$ yields again an abstract Cauchy problem on $\Gamma_0(\mathcal{E})$ given by

$$\begin{cases} \dot{v}(t) = \mathscr{A}v(t), & t \geq 0, \\ v(0) = s_0, \end{cases} \tag{$\mathscr{ACP}$}$$

where $v : \mathbb{R}_{\geq 0} \to \Gamma_0(\mathcal{E})$ and $s_0(\omega) := x_\omega$, $\omega \in \Omega$. The requirement one needs for well-posedness will be that $s_0 \in \mathrm{D}(\mathscr{A})$, see also Section 2. The following result relates the well-posedness of $(\mathscr{ACP})$ to the well-posedness of (ACP_ω) in a one-to-one way and generalizes [23, Thm. 3.4]. Recall, that for a general linear operator $(A, \mathrm{D}(A))$ the *spectrum* $\sigma(A)$ is defined by $\sigma(A) = \{\lambda \in \mathbb{C} : \ \lambda - A \text{ not invertible}\}$. The *resolvent set* is the defined by $\rho(A) = \mathbb{C} \setminus \sigma(A)$.

Theorem 3.6. *Let $(\mathcal{E}, \pi_{\mathcal{E}})$ be a Banach bundle over Ω with corresponding Banach spaces X_ω. Moreover, let $(A_\omega, \mathrm{D}(A_\omega))$ be an operator on X_ω for each $\omega \in \Omega$ and $(\mathscr{A}, \mathrm{D}(\mathscr{A}))$ be the operator on $\Gamma_0(\mathcal{E})$ defined by (3.1). Suppose that $\rho(\mathscr{A}) \cap \rho(A_\omega) \neq \varnothing$ for all $\omega \in \Omega$. Then the following assertions are equivalent:*

(a) *The operator* $(\mathscr{A}, \mathrm{D}(\mathscr{A}))$ *generates a* C_0*-semigroup* $(\mathscr{T}(t))_{t\geq 0}$ *on* $\Gamma_0(\mathcal{E})$ *satisfying* $\|\mathscr{T}(t)\| \leq M\mathrm{e}^{\alpha t}$ *for all* $t \geq 0$.
(b) *Each operator* $(A_\omega, \mathrm{D}(A_\omega))$ *generates a* C_0*-semigroup* $(T_\omega(t))_{t\geq 0}$ *on* X_ω *satisfying* $\|T_\omega(t)\| \leq M\mathrm{e}^{\alpha t}$ *for all* $t \geq 0$ *and all* $\omega \in \Omega$. *Furthermore, the function* $\Omega \times \mathbb{R}_+ \to \mathcal{E}$ *defined by* $(\omega, t) \mapsto T_\omega(t)s(\omega)$ *is continuous for each* $s \in \Gamma_0(\mathcal{E})$.

Observe that Theorem 3.6 indeed relates the well-posedness of $(\mathscr{ACP})$ to the well-posedness of (ACP_ω) in a one-to-one way in view of Remark 2.4.

Proof. (i) $\Rightarrow$ (ii): Assume that the operator $(\mathscr{A}, \mathrm{D}(\mathscr{A}))$ generates a C_0-semigroup $(\mathscr{T}(t))_{t\geq 0}$ on $\Gamma_0(\mathcal{E})$ satisfying $\|\mathscr{T}(t)\| \leq M\mathrm{e}^{\alpha t}$ for all $t \geq 0$. We need to define strongly continuous operator semigroups $(T_\omega(t))_{t\geq 0}$ on X_ω for each $\omega \in \Omega$. Let $x \in X_\omega$ be arbitrary. Since we consider continuous Banach bundles over locally compact Hausdorff spaces, it is already ensured that we have enough sections, cf. [21, pp. 640–641] or [31, Cor. 2.10]. In particular, we can find $s \in \Gamma_0(\mathcal{E})$ such that $x = s(\omega)$. Hence we define $T_\omega(t)x := (\mathscr{T}(t)s)(\omega)$, $t \geq 0$. We observe that

$$\|T_\omega(t)x - x\|_\omega = \|\mathscr{T}(t)s(\omega) - s(\omega)\|_\omega = \|(\mathscr{T}(t)s - s)(\omega)\| \leq \|\mathscr{T}(t)s - s\|_{\Gamma_0(\mathcal{E})},$$

showing that $(T_\omega(t))_{t\geq 0}$ is indeed strongly continuous. We now show that $(T_\omega(t))_{t\geq 0}$ is actually generated by $(A_\omega, \mathrm{D}(A_\omega))$. To do so, we use the Laplace transform. In fact, from the assumption that $\rho(\mathscr{A}) \cap \rho(A_\omega) \neq \varnothing$ for each $\omega \in \Omega$ one obtains that $R(\lambda, \mathscr{A})s(\omega) = R(\lambda, A_\omega)s(\omega)$. To see this one can follow and imitate the proofs of [23, Prop. 3.2 & Lemma 3.5]. Hence, for all $\lambda > \alpha$ we obtain that

$$R(\lambda, A_\omega)s(\omega) = (R(\lambda, \mathscr{A})s)(\omega) = \int_0^\infty \mathrm{e}^{-\lambda t}(\mathscr{T}(t)s)(\omega)\ \mathrm{d}t = \int_0^\infty \mathrm{e}^{-\lambda t}T_\omega(t)s(\omega)\ \mathrm{d}t,$$

showing that $(A_\omega, \mathrm{D}(A_\omega))$ is indeed the generator for $(T_\omega(t))_{t\geq 0}$, cf. [20, Chapter II, Thm. 1.10]. This proves the first part of the statement.

(ii) $\Rightarrow$ (i): For the converse, assume that each operator $(A_\omega, \mathrm{D}(A_\omega))$ generates a C_0-semigroup $(T_\omega(t))_{t\geq 0}$ on X_ω satisfying $\|T_\omega(t)\| \leq M\mathrm{e}^{\alpha t}$ for all $t \geq 0$ and all $\omega \in \Omega$. From this we define an operator semigroup $(\mathscr{T}(t))_{t\geq 0}$ on $\Gamma_0(\mathcal{E})$ by

$$(\mathscr{T}(t)s)(\omega) := T_\omega(t)s(\omega), \quad t \geq 0,\ s \in \Gamma_0(\mathcal{E}),\ \omega \in \Omega.$$

Indeed, these operators are all defined on $\Gamma_0(\mathcal{E})$. To see this, let $s \in \Gamma_0(\mathcal{E})$, i.e., the mapping $s : \Omega \to \mathcal{E}$ is continuous, $\pi_{\mathcal{E}} \circ s = \mathrm{Id}_{\mathcal{E}}$ and vanishes at infinity. We need to show that $\mathscr{T}(t)s \in \Gamma_0(\mathcal{E})$ for all $t \geq 0$. By definition, we have $(\mathscr{T}(t)s)(\omega) := T_\omega(t)s(\omega)$ hence the continuity of the function $(\mathscr{T}(t)s)(\cdot)$ is directly linked to the continuity of the function $T_\omega(t)s(\cdot)$. However, the latter one is given by the assumption of (ii). Moreover, one has that $\|\mathscr{T}(t)\| \leq M\mathrm{e}^{\alpha t}$ for all $t \geq 0$ as it is also assumed that $\|T_\omega(t)\| \leq M\mathrm{e}^{\alpha t}$ for all $t \geq 0$ uniformly in ω. As $s : \Omega \to \mathcal{E}$ vanishes at infinity, also $(\mathscr{T}(t)s)(\cdot)$ does as $t \geq 0$ is fixed. Now, we observe that $(\mathscr{T}(t))_{t\geq 0}$ is a C_0-semigroup. Indeed, this holds true as each $s \in \Gamma_0(\mathcal{E})$ vanishes at infinity and $(\omega, t) \to T_\omega(t)s(\omega)$ is uniformly continuous on compact subsets. Finally, we determine the generator of the C_0-semigroup $(\mathscr{T}(t))_{t\geq 0}$. Assume that $(\mathscr{B}, \mathrm{D}(\mathscr{B}))$ is the generator of $(\mathscr{T}(t))_{t\geq 0}$, then for $s \in \mathrm{D}(\mathscr{B})$

$$(\mathscr{B}s)(\omega) = \lim_{t\to 0} \frac{\mathscr{T}(t)s(\omega) - s(\omega)}{t} = \lim_{t\to 0} \frac{T_\omega(t)s(\omega) - s(\omega)}{t}.$$

So, if $s \in \mathrm{D}(\mathscr{B})$, then $s(\omega) \in \mathrm{D}(A_\omega)$ and $(\mathscr{B}s)(\omega) = A_\omega s(\omega)$ for all $\omega \in \Omega$. Since $\mathscr{B}s \in \Gamma_0(\mathcal{E})$ we conclude that $s \in \mathrm{D}(\mathscr{A})$ and $\mathscr{A}s = \mathscr{B}s$ showing that $\mathscr{B} \subseteq \mathscr{A}$. On the other hand, if $s \in \mathrm{D}(\mathscr{A})$ we define $r \in \Gamma_0(\mathcal{E})$ by $r(s) := (\lambda - A_\omega)s(\omega)$, $\omega \in \Omega$, and observe that

$$s(\omega) = R(\lambda, A_\omega)r(\omega) = \int_0^\infty \mathrm{e}^{-\lambda t} T_\omega(t) r(\omega)\ \mathrm{d}t = (R(\lambda, \mathscr{B})r)(\omega),$$

implying that $s \in \mathrm{D}(\mathscr{B})$ and hence $\mathscr{A} \subseteq \mathscr{B}$, which concludes the proof. □

The theorem informally states that the well-posedness of a family of abstract Cauchy problems corresponds to the well-posedness of the corresponding abstract Cauchy problem on the associated Banach bundle. We will word this out for transport processes on changing networks in Section 4.

3.3. Stability on Banach bundles. We saw that due to Theorem 3.6 we are able to related C_0-semigroups on different Banach spaces to a C_0-semigroup on a single Banach bundle. However, Theorem 3.6 allows us to even have a look at long-term behavior. Recall the following definitions.

Definition 3.7. Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X. Then $(T(t))_{t\geq 0}$ is called *uniformly stable* if there exist $M \geq 1$ and $\alpha > 0$ such that $\|T(t)\| \leq M\mathrm{e}^{-\alpha t}$ for all $t \geq 0$.

Theorem 3.6 immediately yields the following result which we will state without proof.

Corollary 3.8. Let $(\mathcal{E}, \pi_{\mathcal{E}})$ be a Banach bundle over Ω with corresponding Banach spaces X_ω. Moreover, let $(A_\omega, \mathrm{D}(A_\omega))$ be an operator on X_ω for each $\omega \in \Omega$ and $(\mathscr{A}, \mathrm{D}(\mathscr{A}))$ be the operator on $\Gamma_0(\mathcal{E})$ defined by (3.1). Suppose that $\rho(\mathscr{A}) \cap \rho(A_\omega) \neq \varnothing$ for all $\omega \in \Omega$. Then the following assertions are equivalent:

(a) The operator $(\mathscr{A}, \mathrm{D}(\mathscr{A}))$ generates a uniformly stable C_0-semigroup $(\mathscr{T}(t))_{t\geq 0}$ on $\Gamma_0(\mathcal{E})$.
(b) Each operator $(A_\omega, \mathrm{D}(A_\omega))$ generates a uniformly stable C_0-semigroup $(T_\omega(t))_{t\geq 0}$ on X_ω satisfying $\|T_\omega(t)\| \leq M\mathrm{e}^{-\alpha t}$ for some $\alpha > 0$ and all $t \geq 0$ and $\omega \in \Omega$. Furthermore, the function $\Omega \times \mathbb{R}_+ \to \mathcal{E}$ defined by $(\omega, t) \mapsto T_\omega(t)s(\omega)$ is continuous for each $s \in \Gamma_0(\mathcal{E})$.

Remark 3.9. We observe that uniform stability of C_0-semigroups can also be characterized by means other than the exponential boundedness, e.g., spectral conditions. In particular, a comprehensive monograph on stability of operator semigroups (and single operators) has been published by Eisner [18]. Results on uniform stability can be found in [18, Chapter III, Sect. 2]. For example, a C_0-semigroup $(T(t))_{t\geq 0}$ is uniformly stable if and only if $\|T(t_0)\| < 1$ for some $t_0 > 0$ if and only if $\mathrm{r}(T(t_0)) < 1$ for some $t_0 > 0$. Here, we denote by $\mathrm{r}(T) < 1$ the spectral radius of a bounded linear operator T. Other characterizations in terms of the integrability of its orbits have for example been investigated by Datko [13] and Pazy [36, Thm. 4.4.1]. However, we will stay here in the framework of Banach bundles where we consider the operator semigroups on their own on each fiber.

3.4. Perturbations on Banach bundles. Beside the long-term study of operator semigroups on Banach bundles, we will also consider bounded perturbations on Banach bundles. Let $(\mathcal{E}, \pi_{\mathcal{E}})$ be a Banach bundle over Ω with corresponding Banach spaces X_ω and assume that $B_\omega \in \mathscr{L}(X_\omega)$ for each $\omega \in \Omega$, i.e., each B_ω is a bounded linear operator on X_ω for each $\omega \in \Omega$. From this we are able to construct an operator $\mathscr{B}$ on $\Gamma_0(\mathcal{E})$ by

$$(\mathscr{B}s)(\omega) := B_\omega s(\omega), \quad s \in \Gamma_0(\mathcal{E}),\ \omega \in \Omega. \tag{3.2}$$

However, in order to obtain that $\mathscr{B} \in \mathscr{L}(\Gamma_0(\mathcal{E}))$ we need a uniform bound on the operator norms of the family of operator $(B_\omega)_{\omega\in\Omega}$. Assume that there exists $K \geq 0$ such that $\|B_\omega\| \leq K$ for all $\omega \in \Omega$, then

$$\|\mathscr{B}s\|_{\Gamma_0(\mathcal{E})} = \sup_{\omega\in\Omega} \|B_\omega s(\omega)\| \leq K \cdot \sup_{\omega\in\Omega} \|s(\omega)\| = K \|s\|_{\Gamma_0(\mathcal{E})},$$

showing that under the above assumption one obtains again a bounded linear operator on $\Gamma_0(\mathcal{E})$. Having this in mind, we can prove the following result.

Proposition 3.10. Let $(\mathcal{E}, \pi_{\mathcal{E}})$ be a Banach bundle over Ω with corresponding Banach spaces X_ω. Moreover, let $(A_\omega, \mathrm{D}(A_\omega))$ be C_0-semigroup generators on X_ω for each $\omega \in \Omega$ and $(\mathscr{A}, \mathrm{D}(\mathscr{A}))$ be a C_0-semigroup generator on $\Gamma_0(\mathcal{E})$ defined by (3.1). Suppose that $\rho(\mathscr{A}) \cap \rho(A_\omega) \neq \varnothing$ for all $\omega \in \Omega$. Furthermore, assume that there are bounded linear operators B_ω on X_ω for each $\omega \in \Omega$ satisfying that there exists $K \geq 0$ such that $\|B_\omega\| \leq K$ for all $\omega \in \Omega$ and define the operator $\mathscr{B}$ on $\Gamma_0(\mathcal{E})$ as in (3.2). Then the following assertions are equivalent:

(a) The operator $(\mathscr{A} + \mathscr{B}, \mathrm{D}(\mathscr{A}))$ generates a C_0-semigroup $(\mathscr{S}(t))_{t\geq 0}$ on $\Gamma_0(\mathcal{E})$.

(b) Each operator $(A_\omega + B_\omega, \mathrm{D}(A_\omega))$ generates a C_0-semigroup $(S_\omega(t))_{t\geq 0}$ on X_ω satisfying $\|S_\omega(t)\| \leq M\mathrm{e}^{\alpha t}$ for some $\alpha \in \mathbb{R}$ and all $t \geq 0$ and $\omega \in \Omega$. Furthermore, the function $\Omega \times \mathbb{R}_+ \to \mathcal{E}$ defined by $(\omega, t) \mapsto T_\omega(t)s(\omega)$ is continuous for each $s \in \Gamma_0(\mathcal{E})$, where $(T_\omega(t))_{t\geq 0}$ is the C_0-semigroup on X_ω generated by $(A_\omega + B_\omega, \mathrm{D}(A_\omega))$ for $\omega \in \Omega$.

Proof. The proof of the result relies on the bounded perturbation theorem for C_0-semigroups, cf. [20, Chapter III, Thm. 1.3]. In fact, whenever $(A, \mathrm{D}(A))$ is a generator of a C_0-semigroup on a Banach space X and $B \in \mathscr{L}(X)$, then $(A + B, \mathrm{D}(A))$ generates a C_0-semigroup on X as well.

(a)⇒(b): Assume that $(\mathscr{A} + \mathscr{B}, \mathrm{D}(\mathscr{A}))$ generates a C_0-semigroup $(\mathscr{S}(t))_{t\geq 0}$ on $\Gamma_0(\mathcal{E})$. By Theorem 3.6 there exist C_0-semigroup generators $(C_\omega, \mathrm{D}(C_\omega))$ on X_ω for each $\omega \in \Omega$. Indeed, each operator C_ω is of the form $C_\omega = A_\omega + B_\omega$ as a consequence of [20, Chapter III, Cor. 1.7], i.e., the operator semigroup $(\mathscr{S}(t))_{t\geq 0}$ satisfies the variation of constant formula on $\Gamma_0(\mathcal{E})$

$$\mathscr{S}(t)s = \mathscr{T}(t)s + \int_0^t \mathscr{T}(t-\sigma)B\mathscr{S}(\sigma)s \,\mathrm{d}\sigma, \quad t \geq 0,\ s \in \Gamma_0(\mathcal{E}).$$

By following the lines of the proof of Theorem 3.6, this formula also has to hold on each space X_ω which concludes the argument again by using [20, Chapter III, Cor. 1.7].

(b)⇒(a): By assumption, each operator B_ω is a bounded linear operator on X_ω for $\omega \in \Omega$. By [20, Chapter III, Thm. 1.3] the operators $(A_\omega + B_\omega, \mathrm{D}(A_\omega))$ are generators on C_0-semigroups for each $\omega \in \Omega$. As the function $\Omega \times \mathbb{R}_+ \to \mathcal{E}$ defined by $(\omega, t) \mapsto T_\omega(t)s(\omega)$ is continuous for each $s \in \Gamma_0(\mathcal{E})$ and each single operator $(A_\omega, \mathrm{D}(A_\omega))$

generates a C_0-semigroup on X_ω for each $\omega \in \Omega$ we conclude by Theorem 3.6, that the operator $(\mathscr{A}, \mathrm{D}(\mathscr{A}))$ generates a C_0-semigroup on $\Gamma_0(\mathcal{E})$. Moreover, there exists $K \geq 0$ such that $\|B_\omega\| \leq K$ for all $\omega \in \Omega$. By the previous observations, the operator $\mathscr{B}$ as defined by (3.2) is a bounded linear operator on $\Gamma_0(\mathcal{E})$. Therefore, the operator $(\mathscr{A} + \mathscr{B}, \mathrm{D}(\mathscr{A}))$ generates a C_0-semigroup by [20, Chapter III, Thm. 1.3]. □

4. Example: Transport on changing networks

4.1. Graphs and networks. A *directed graphs* $G = (\mathrm{V}(G), \mathrm{E}(G))$, where $\mathrm{V}(G) = \{\mathrm{v}_i : \ i \in I\}$ is the set of *vertices* and $\mathrm{E}(G) = \{\mathrm{e}_j : \ j \in J\} \subseteq \mathrm{V} \times \mathrm{V}$ is the set of *directed edges* for some at most countable sets $I, J \subseteq \mathbb{N}$. For a directed edge $\mathrm{e} = (\mathrm{v}_i, \mathrm{v}_k)$, $i, k \in I$, we call v_i the *tail* and v_k the *head* of e. The edge e is an *outgoing edge* of the vertex v_i and an *incoming edge* for the vertex v_k. A graph G is called *simple* if there are neither loops nor multiple edges in G. We also assume that the graph G is *uniformly locally finite* meaning that each vertex has only finitely many outgoing edges and that the number of outgoing edges is uniformly bounded from above.

The *outgoing incidence matrix* of a given graph $G = (\mathrm{V}(G), \mathrm{E}(G))$ is a matrix $\Phi^- = (\Phi^-_{ij})$ defined by

$$\Phi^-_{ij} := \begin{cases} 1 & \text{if } \mathrm{v}_i \xrightarrow{\mathrm{e}_j} . \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

whereas the *incoming incidence matrix* $\Phi^+ = (\Phi^+_{ij})$ of $G = (\mathrm{V}(G), \mathrm{E}(G))$ is defined by

$$\Phi^+_{ij} := \begin{cases} 1 & \text{if } \xrightarrow{\mathrm{e}_j} \mathrm{v}_i, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

The *incidence matrix* Φ of the directed graph $G = (\mathrm{V}(G), \mathrm{E}(G))$ is defined by $\Phi := \Phi^+ - \Phi^-$ and describes the structure of the graph entirely. There are two other important matrices associated to a general graph and which are needed in what follows. The *transposed adjacency matrix* of the graph $G = (\mathrm{V}(G), \mathrm{E}(G))$ is defined by

$$\mathbb{A} := \Phi^+ \left(\Phi^-\right)^\top .$$

The nonzero entries of $\mathbb{A}$ correspond exactly to the edges of the graph, cf. [5, p. 280]. In fact, $\mathbb{A}$ can be described explicitly as

$$\mathbb{A}_{ij} := \begin{cases} 1 & \text{if } \mathrm{v}_j \xrightarrow{\mathrm{e}_k} \mathrm{v}_i, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

Last but not least we use the so-called *(transposed) adjacency matrix of the line graph* $\mathbb{B} = (\mathbb{B}_{ij})$ defined by $\mathbb{B} := (\Phi^-)^\top \Phi^+$. One can also give an explicit entrywise description as

$$\mathbb{B}_{ij} := \begin{cases} 1 & \text{if } \xrightarrow{\mathrm{e}_j} \mathrm{v}_k \xrightarrow{\mathrm{e}_i}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

Notice, that by the assumption of the uniform locally finiteness of the graph the matrix $\mathbb{B}$ is a bounded linear operator on $\ell^1 := \ell^1(J)$.

In what follows, we stick to the following mathematical setting. We identify every edge of our graph with the unit interval, $\mathrm{e}_j \equiv [0, 1]$ for each $j \in J$, and parametrize it contrary to the direction of the flow, if $c_j > 0$, $j \in J$, so that it is assumed to have its tail at the endpoint 1 and its head at the endpoint 0, i.e., the material flows from 1 to 0. With this assumption, we stay within the framework introduced by B. Dorn, M. Kramar Fijavž and E. Sikolya, see for example [26, 15]. For simplicity we use the notation $\mathrm{e}_j(1)$ and $\mathrm{e}_j(0)$ for the tail and the head, respectively. In this way we obtain a *metric graph*.

4.2. Flows in networks. In what follows, we only treat the infinite graph case, since the finite one is included. Let $G = (\mathrm{V}(G), \mathrm{E}(G))$ be a simple and locally finite graph and consider the following partial differential equation on the graph G for $i \in I$, $j \in J$:

$$\begin{cases} \dfrac{\partial}{\partial t} w_j(t,x) = c_j \dfrac{\partial}{\partial x} w_j(t,x), & x \in (0,1)\,,\ t \geq 0, \\ w_j(x,0) = f_j(x), & x \in (0,1)\,, \\ \Phi^-_{ij} w_j(1,t) = \displaystyle\sum_{k\in\mathbb{N}} \Phi^+_{ik} w_k(0,t), & t \geq 0. \end{cases} \tag{PDE}$$

In what follows, we assume that

$$m \leq c_j \leq M, \quad j \in J. \tag{4.5}$$

The space $X := \mathrm{L}^1\left([0,1], \ell^1\right)$ equipped with the norm given by

$$\|f\| := \int_0^1 \|f(s)\|_{\ell^1} \, \mathrm{d}s$$

is a Banach space. On X we introduce the operator $(A, \mathrm{D}(A))$ defined by

$$A := \operatorname{diag}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right), \quad \mathrm{D}(A) := \left\{f \in \mathrm{W}^{1,1}\left([0,1], \ell^1\right) :\ f(1) = \mathbb{B}_C f(0)\right\}, \tag{4.6}$$

where

$$\mathbb{B}_C := C^{-1}\mathbb{B}C \quad \text{and} \quad C := \operatorname{diag}(c_j)\,. \tag{4.7}$$

Notice, that (4.5) assures, that the operator $\mathbb{B}_C$ is bounded on $\ell^1 := \ell^1(J)$. It is well-known that the corresponding abstract Cauchy problem on the Banach space $X = \mathrm{L}^1\left([0,1], \ell^1\right)$ is equivalent to the partial differential equation (PDE), see for example [15, Prop. 3.1] and [26] for the finite graph case.

Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of simple (but possibly) infinite (uniformly locally finite) graphs. In comparison with [8] we do not need to assume that we have a (categorical) limit of graphs. One each of these graphs we consider a transport process which yields an differential operator $(A_n, \mathrm{D}(A_n))$ as described by (4.6) on a Banach space X_n given by

$$X_n = \begin{cases} \mathrm{L}^1\left([0,1], \mathbb{C}^{|\mathrm{E}(G_n)|}\right), & \text{if } G_n \text{ is finite}, \\ \mathrm{L}^1\left([0,1], \ell^1\right), & \text{otherwise}. \end{cases}$$

Here $|\mathrm{E}(G_n)|$ denotes the number of edges of the graph G_n.

Now, we use the theory of Banach bundles. For more generality, assume that Ω is a locally compact Hausdorff space and that we have a family of Banach spaces $(X_\omega)_{\omega\in\Omega}$. Define the space $\mathcal{E}$ by

$$\mathcal{E} := \coprod_{\omega\in\Omega} X_\omega = \bigcup_{\omega\in\Omega} \{(\omega, x) :\ x \in X_\omega\}\,.$$

We equip $\mathcal{E}$ with the finest topology making the canonical injections $\iota_\omega : X_\omega \to \mathcal{E}$, defined by $\iota_\omega(x) := (\omega, x)$, $\omega \in \Omega$, $x \in X_\omega$, continuous. It follows, that a set $U \subseteq \mathcal{E}$ is open if and only if $\iota_\omega^{-1}(U)$ is open in X_ω for each $\omega \in \Omega$. Define $\pi_{\mathcal{E}} : \mathcal{E} \to \Omega$ by $\pi_{\mathcal{E}}(\omega, x) := \omega$, i.e., roughly speaking the map $\pi_{\mathcal{E}}$ reads off the index of the space where x is an element of. We want to show that $(\mathcal{E}, \pi_{\mathcal{E}})$ is actually a continuous Banach bundle.

Lemma 4.1. *The map $\pi_{\mathcal{E}} : \mathcal{E} \to \Omega$ is continuous.*

Proof. Recall, from the definition of continuous maps from topological disjoint unions, that the map $\pi_{\mathcal{E}}$ is continuous if and only if the maps $\pi_{\mathcal{E}} \circ \iota_\omega : X_\omega \to \Omega$ are continuous for each $\omega \in \Omega$. To see that, let $\omega \in \Omega$ be arbitrary and let $(x_\alpha)_{\alpha \in A}$ be a net in X_ω such that $x_\alpha \to x$. Observe that $\pi_{\mathcal{E}}(\iota_\omega(x_\alpha)) = \pi_{\mathcal{E}}(\iota_\omega(x)) = \omega$ for each $\alpha \in A$, hence $\pi_{\mathcal{E}}(\iota_\omega(x_\alpha)) \to \pi_{\mathcal{E}}(\iota_\omega(x))$ showing that $\pi_{\mathcal{E}}$ is continuous. □

Proposition 4.2. The map $+ : \mathcal{E} \times_{\mathbb{R}_{\geq 0}} \mathcal{E} \to \mathcal{E}$ defined by $(x, y) \mapsto x +_{\pi_{\mathcal{E}}(x)} y$ is continuous, where $\mathcal{E} \times_{\mathbb{R}_{\geq 0}} \mathcal{E} := \{(x, y) \in \mathcal{E} \times \mathcal{E} :\ \pi_{\mathcal{E}}(x) = \pi_{\mathcal{E}}(y)\}$.

Proof. Let $(x_\alpha, y_\alpha)_{\alpha \in A}$ be a net in $\mathcal{E} \times_{\mathbb{R}_{\geq 0}} \mathcal{E} \to \mathcal{E}$ converging to $(x, y) \in \mathcal{E} \times_{\mathbb{R}_{\geq 0}} \mathcal{E} \to \mathcal{E}$. In particular, we find $\omega_0 \in \Omega$ such that $(x, y) \in X_{\omega_0} \times X_{\omega_0}$ and $\alpha_0 \in A$ such that $(x_\alpha, y_\alpha) \in X_{\omega_0} \times X_{\omega_0}$ for all $\alpha \succ \alpha_0$ and $(x_\alpha, y_\alpha)_{\alpha \succ \alpha_0}$ is convergent to (x, y) in $X_{\omega_0} \times X_{\omega_0}$. In other words, one has that

$$\|x_\alpha - x\|_{X_{\omega_0}} + \|y_\alpha - y\|_{X_{\omega_0}} \to 0, \quad \alpha \succ \alpha_0.$$

Moreover, we conclude that $x_\alpha + y_\alpha \in X_{\omega_0} \times X_{\omega_0}$ for all $\alpha \succ \alpha_0$ and that

$$\|(x_\alpha + y_\alpha) - (x + y)\|_{X_{\omega_0}} \leq \|x_\alpha - x\|_{X_{\omega_0}} + \|y_\alpha - y\|_{X_{\omega_0}} \to 0, \quad \alpha \succ \alpha_0,$$

showing that $x_\alpha + y_\alpha \to x + y$. □

By using the same techniques we obtain also the following result.

Proposition 4.3. The map $\cdot : \mathbb{R} \times \mathcal{E} \to \mathcal{E}$ defined by $(\lambda, f) \mapsto \lambda \cdot_{\pi(f)} f$ is continuous.

Proposition 4.4. The map $\|\cdot\| : \mathcal{E} \to \mathbb{R}_{\geq 0}$ defined by $x \mapsto \|x\|_{X_{\pi(x)}}$ is continuous.

Proof. Let $(x_\alpha)_{\alpha \in A}$ be a net in $\mathcal{E}$ converging to x. By definition, there exists $\omega_0 \in \Omega$ and $\alpha_0 \in A$ such that $x, x_\alpha \in X_{\omega_0}$ for all $\alpha \succ \alpha_0$ and $(x_\alpha)_{\alpha \succ \alpha_0}$ converges to x in X_{ω_0}, i.e., $\|x_\alpha - x\|_{X_{\omega_0}} \to 0$ where $\alpha \succ \alpha_0$. Now observe that for $\alpha \succ \alpha_0$ one obtains by the reverse triangle inequality that $\left| \|x_\alpha\|_{X_{\omega_0}} - \|x\|_{X_{\omega_0}} \right| \leq \|x_\alpha - x\|_{X_{\omega_0}}$ which proves the assertion. □

Now, we have to stick back to our original example, where we have a sequence $(X_n)_{n \in \mathbb{N}}$ of Banach spaces instead of a family of Banach spaces labeled by an arbitrary locally compact Hausdorff space Ω. In particular, we make a specific choice of Ω by taking $\Omega := \mathbb{N}$ equipped with the discrete metric. In order to prove the last property we need for $(\mathcal{E}, \pi_{\mathcal{E}})$ to be a Banach bundle, we make use of the equivalent formulation noticed in Remark 3.2. We need this since we have the following property which holds true for the case $\Omega = \mathbb{N}$: if $(\psi_\alpha)_{\alpha \in A}$ is a net in $\mathbb{N}$ such that $\psi_\alpha \to \psi$, then there exists $\alpha_0 \in A$ such that $\psi_\alpha = \psi$ for all $\alpha \succ \alpha_0$. The following result is now a direct consequence of the fact that every set is open in $\Omega = \mathbb{N}$.

Lemma 4.5. *$\pi_{\mathcal{E}} : \mathcal{E} \to \mathbb{N}$ is an open map.*

We conclude by summarizing the previous results

Theorem 4.6. *The pair $(\mathcal{E}, \pi_{\mathcal{E}})$ is a continuous Banach bundle.*

Proof. Since we already proved Lemma 4.5, Lemma 4.2, Proposition 4.3 and Proposition 4.4, it suffices to show the following. Let $\xi \in \mathbb{N}$ be arbitrary and $(x_\alpha)_{\alpha\in A}$ be a net in $\mathcal{E}$ such that $\|x_\alpha\|_{X_{\pi(x_\alpha)}} \to 0$ and $\pi_{\mathcal{E}}(x_\alpha) \to \xi$. Then we have to show that $x_\alpha \to 0_\xi$ in $\mathcal{E}$. By our previous observations about convergence in $\Omega = \mathbb{N}$ we conclude that the condition $\pi_{\mathcal{E}}(x_\alpha) \to \xi$ implies that there exists $\alpha_0 \in A$ such that $\pi_{\mathcal{E}}(x_\alpha) = \xi$ for all $\alpha \succ \alpha_0$. This observation together with the assumption $\|x_\alpha\|_{X_{\pi(x_\alpha)}} \to 0$ shows that $x_\alpha \to 0_\xi$ in $\mathcal{E}$. $\square$

Since we have a (continuous) Banach bundle, we are able to construct the corresponding section space $\Gamma_0(\mathcal{E})$. By [5, Cor. 18.13] one has that $\lambda \in \rho(A_n)$ whenever $\mathrm{Re}(\lambda) > 0$. We observe, that instead of requiring continuous dependence of the semigroups in Theorem 3.6, it suffices that the map $\omega \mapsto R(\lambda, A_\omega)s(\omega)$ is continuous for $\omega \in \Omega$ and $s \in \Gamma_0(\mathcal{E})$ and uniformly bounded as one can apply a Trotter–Kato approximation argument, see also [23, Rem. 3.6]. We use this in our situation, as we know exactly how the resolvents look like, cf. [5, Prop. 18.12]. In fact, it suffices that all velocity functions $c_j(\cdot)$ are continuous and that there exists $m, M > 0$ such that $m \le c_j(t) \le M$ for all $t \ge 0$. Following [5, Sect. 18.2] we conclude that $\rho(\mathscr{A}) \cap \rho(A_n) \neq \varnothing$ for each $n \in \mathbb{N}$. Since $\mathbb{N}$ has a discrete structure, every function $s : \mathbb{N} \to \mathcal{E}$ is continuous and $\Gamma_0(\mathcal{E})$ consists of functions $s : \mathbb{N} \to \mathcal{E}$ such that $\pi_{\mathcal{E}} \circ s = \mathrm{Id}_{\mathbb{N}}$ and the values of s vanish for all but finite data. This is due to the fact that $K \subseteq \mathbb{N}$ compact if and only if K is finite. Since it is known, that every operator $(A_n, \mathrm{D}(A_n))$ generates a C_0-semigroup on X_n for each $n \in \mathbb{N}$, cf. [5, Cor. 18.15], we conclude from Theorem 3.6 that also the abstract Cauchy problem on $\Gamma_0(\mathcal{E})$ is well-posed.

Conclusion

In this paper we showed that evolution equations on changing networks can be studied by means of (continuous) Banach bundles. A priori, the concept of Banach bundles does not have to do with networks (or graphs). However, as soon as we add the dynamical component by means of evolution equations, one is in the situation of Banach spaces and operator semigroups as it was first presented by Kramar and Sikolya [26] and later on by many other authors. The changing character of the networks with the corresponding evolution equations gives then naturally rise to the Banach bundle approach as the underlying Banach space changes as soon as the network changes. In order to be able to analyze the whole situation, one can then make use of operator semigroup theory as the Banach bundle yields the space of continuous sections vanishing at infinity $\Gamma_0(\mathcal{E})$. In fact, whenever one can prove that one has well-posedness on $\Gamma_0(\mathcal{E})$ this yields well-posedness of each single dynamical process. This also justifies the need and the use of the theory developed in this paper.

Acknowledgment

The author want to thank the anonymous referee for the fruitful and valuable feedback which helped to improve the paper as a whole. The author also wants to thank H. Kreidler for the fruitful discussion regarding the proof of Remark 3.2. This work is also based on the research supported by the National Research Foundation of South Africa (Grant number: SRUG220317136). It is acknowledged that opinions, findings and conclusions or recommendations expressed in any publication generated by this supported research is that of the author(s). The National Research Foundation accepts no liability whatsoever in this regard.

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