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ATTRACTORS AND CHAIN RECURRENCE IN NONCOMPACT SPACE FOR SEMIGROUP OF CONTINUOUS MAPS

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Abstract. In [2, 3], Hurley extended Conley's theorem to noncompact spaces. Its significance is that it characterizes the chain recurrent set of a self-map in terms of attractors. Here, we consider an extension of Hurley's result for a semigroup of continuous maps. Earlier, in [7], we had extended Conley's theorem on compact space for a semigroup of continuous maps.

1. Introduction

A *continuous semigroup* is a set of (non-identity) continuous self maps, of a topological space X, closed under the composition. A semigroup G is said to be generated by a family $\{g_\alpha\}_\alpha$ if every element of G can be expressed as compositions of iterations of the elements of $\{g_\alpha\}_\alpha$. We denote this by $G =< g_\alpha >_\alpha$. The space X is assumed to be Hausdorff and first countable.

M. Hurley had shown that the chain recurrent set for a semiflow is the complement of the union of the set $B(A) \setminus A$, as A varies over the collection of attractors and $B(A)$ denotes the basin of attraction. This concept for flows on a compact space was originally introduced by Conley [1]. Later, in [4] Hurley extended this characterization for semiflows without the assumption of compactness. This paper aims to study the notion of attractors and chain recurrence for noncompact spaces in the context of a continuous semigroup such that the classical case turns out as a special case of this generalization. In [6, 7], we have studied the chain recurrent set and established the notion of attractors for a semigroup of continuous maps. Also, we extended the characterization of the chain recurrent set in terms of attractors on compact metric spaces. Here, we shall follow the treatment of Hurley for noncompact metric spaces and

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see how far this characterization applies in this setting of the semigroup of continuous maps.

In [2, 3, 4], Hurley generalized the definition of chain recurrence for noncompact spaces and expressed it in terms of attractors, analogous to the work of C. Conley. The equivalence of the two definitions follows from the following theorems of Hurley.

Theorem. [4] If (X,d) is a metric space and $f : X \to X$ is continuous, then the chain recurrent set $\mathsf{CR}(f)$, is the complement of the union of sets $B(A) \setminus A$ as A varies over the collection of attractors of f:

$$X \setminus \mathsf{CR}(f) = \bigcup_A [B(A) \setminus A]$$

where $B(A)$ denotes the basin of attraction of A.

Theorem. [4] Suppose that (X,d) is a metric space and $\Phi : [0,\infty) \times X \to X$ is a continuous semiflow. The chain recurrent set of Φ, denoted by $\mathsf{CR}(\Phi)$, is the complement of the union of sets $B(A) \setminus A$ as A varies over the collection of attractors of Φ and where $B(A)$ denotes the basin of attraction of A:

$$X \setminus \mathsf{CR}(\Phi) = \bigcup_A [B(A) \setminus A].$$

Our main result is the counterpart of the above theorem in the context of the action of a semigroup of continuous maps as follows:

Main Theorem. Let (X,d) be a metric space and G be an abelian semigroup. The chain recurrent set of G is the complement of the union of sets $B(A) \setminus A$ as A varies over the collection of attractors of G:

$$X \setminus \mathsf{CR}(G) = \bigcup_A [B(A) \setminus A].$$

Following Hurley [4] and the definition of a chain introduced in [6], we introduce the notion of a chain and a chain recurrent set for a continuous semigroup in the case of a noncompact space by using strictly positive continuous functions as follows: Let $P(X)$ denote the set of positive continuous functions on X, that is, $P(X) := \{\epsilon : X \to (0,\infty) : \epsilon \text{ is continuous}\}$.

Definition 1.1. Let G be a semigroup of continuous self-maps defined on a metric space (X,d). Let $a, b \in X$, $g \in G$ and $\epsilon \in P(X)$ be given. An *(ϵ, g)-chain* from a to b means a finite sequence $(a = x_1, \ldots, x_{n+1} = b; g_1, \ldots, g_n)$, where $x_i \in X$ and $g_i \in \widehat{G} = G \cup \{\text{identity}\}$ such that $d(g_i \circ g(x_i), x_{i+1}) < \epsilon(g_i \circ g(x_i))$, for each $i = 1, \ldots, n$.

Definition 1.2. A point $x \in X$ is called a *chain recurrent point* for G if for every $\epsilon \in P(X)$ and every $g \in G$ there exists an (ϵ, g)-chain from x to itself. The set of all chain recurrent points for G is denoted by $\mathsf{CR}(G)$.

The paper is organised as follows. In Section 2, we prove some characteristics of a chain recurrent set in noncompact spaces. In Section 3, we reformulate the definition of a basin of attraction in noncompact spaces and prove our main result.

2. Chain Recurrence in Noncompact Space

In this section, we study various properties satisfied by the chain recurrent set namely: the flow invariance property, the topological invariance property and the closure property. To establish these properties we need some lemmas. The following two lemmas have been proved by Hurley in [4].

Lemma 2.1. *Let* (X, d_X) *and* (Y, d_Y) *be two metric spaces and* $f : X \to Y$ *is a continuous map. Then, for each* $\epsilon \in P(Y)$ *there is a* $\delta \in P(X)$ *such that*

$$d_Y(f(x), f(y)) < \epsilon(f(x)) \quad \text{whenever} \quad d_X(x, y) < \delta(x).$$

Lemma 2.2. *If* $\epsilon \in P(X)$ *there is a* $\delta \in P(X)$ *with* $\delta < \epsilon/2$ *and satisfying* $\epsilon(y) > \epsilon(x)/2$ *whenever* $d(y, x) < \delta(x)$.

Here, we shall show the invariance of the set of chain recurrent points as defined in [6].

Theorem 2.3. *The chain recurrent set* $\mathsf{CR}(G)$ *is invariant under* G.

Proof. Let $x \in \mathsf{CR}(G)$ and g be any generator of G. Let $\epsilon \in P(X)$ be given and some $h \in G$. We shall construct an (ϵ, h)-chain from $g(x)$ to itself and it will follow that $g(x) \in \mathsf{CR}(G)$.

Since g is continuous, by Lemma 2.1, there exists a $\delta \in P(X)$ such that if $d(x, y) < \delta(x)$ then $d(g(x), g(y)) < \epsilon(g(x))$. Also $x \in \mathsf{CR}(G)$ implies there is a $(\delta, h \circ g)$-chain $(x = x_1, \dots, x_{n+1} = x; h_1, \dots, h_n)$ from x to itself. That is, for each $i = 1, \dots, n$, we have,

$$d(h_i \circ h \circ g(x_i), x_{i+1}) < \delta(h_i \circ h \circ g(x_i)).$$

Also, by the continuity of g and Lemma 2.1, we have,

$$d(g \circ h_i \circ h \circ g(x_i), g(x_{i+1})) < \epsilon(g \circ h_i \circ h \circ g(x_i)).$$

Hence, $(g(x), g(x_2), \dots, g(x_n), g(x); g \circ h_1, g \circ h_2, \dots, g \circ h_n)$ is an (ϵ, h)-chain from $g(x)$ to itself. □

Theorem 2.4. *The set of all chain recurrent points for* G *is a closed subset of* X.

Proof. Suppose $a \in X$ is a limit point of $\mathsf{CR}(G)$. Let $\epsilon \in P(X)$ and $g \in G$. Since g is continuous, by Lemma 2.1, there exists a $\delta_1 \in P(X)$ such that if $d(x, a) < \delta_1(a)$ then $d(g(x), g(a)) < \epsilon(g(a))$.

Also, by Lemma 2.2, for $\frac{\epsilon}{2} \in P(X)$ there is an $\epsilon' \in P(X)$ with $\epsilon' < \frac{1}{2}\frac{\epsilon}{2}$ and satisfies $\epsilon(y)/2 > \epsilon(x)/4$ whenever $d(y, x) < \epsilon'(x)$. Again, by Lemma 2.2, for $\epsilon' \in P(X)$ there is a $\delta_2 \in P(X)$ with $\delta_2 < \frac{\epsilon'}{2}$ and satisfies $\epsilon'(y) > \epsilon'(x)/2$ whenever $d(y, x) < \delta_2(x)$.

Define a function $\delta \in P(X)$ as $\delta(x) = \min\{\delta_1(x), \delta_2(x)\}$. Since a is a limit point of $\mathsf{CR}(G)$, there is $x \in \mathsf{CR}(G)$ such that $d(x, a) < \delta(a)$. Then $d(g(x), g(a)) < \epsilon(g(a))$ and hence $(a, g(x); \text{identity})$ is an (ϵ, g)-chain from a to $g(x)$.

Now, for $x \in \mathsf{CR}(G)$, there is a (δ, g^2)-chain $(x = x_1, \dots, x_{n+1} = x; g_1, \dots, g_n)$ from x to itself. Then $(g(x), x_2, \dots, x_n; g_1, g_2 g, \dots, g_{n-1} g)$ is an (ϵ, g)-chain from $g(x)$ to x_n. In particular, we have,

$$\begin{aligned} d(g_n g^2(x_n), x) &< \delta(g_n g^2(x_n)) \\ &\leq \delta_2(g_n g^2(x_n)) \\ &< \frac{\epsilon'}{2}(g_n g^2(x_n)) \\ &< \epsilon'(x) \\ &< \frac{\epsilon}{2}(g_n g^2(x_n)). \end{aligned}$$

Since

$$\begin{aligned} d(g_n g^2(x_n), a) &\leq d(g_n g^2(x_n), x) + d(x, a) \\ &< \delta_2(g_n g^2(x_n)) + \delta(a) \\ &< \epsilon'(x) + \epsilon'(x) \\ &< \epsilon(g_n g^2(x_n)), \end{aligned}$$

we have, $(x_n, a; g_n g)$ is an (ϵ, g)-chain from x_n to a. By transitivity via concatenating these (ϵ, g)-chains, there is an (ϵ, g)-chain from a to itself. □

In [5, 6], the notion of topological conjugacy of dynamics of two semigroups on two topological spaces is defined as follows: Let $G = \langle g_i \rangle_{i \in \Lambda}$ be a semigroup of continuous self maps of X and $\widetilde{G} = \langle \widetilde{g_i} \rangle_{i \in \Lambda}$ be a semigroup of continuous self maps of Y. Two dynamical systems (X, G) and $(Y, \widetilde{G})$ are said to be topologically conjugate if there exists a homeomorphism $\rho : X \to Y$ such that $\rho \circ g_i = \widetilde{g_i} \circ \rho$ for each $i \in \Lambda$.

Theorem 2.5. *Let (X, G) and $(Y, \widetilde{G})$ be two dynamical systems on the metric spaces (X, d_X) and (Y, d_Y). If a homeomorphism $\rho : X \to Y$ is a topological conjugacy then $\rho(\mathsf{CR}(G)) = \mathsf{CR}(\widetilde{G})$.*

Proof. Let $x \in \mathsf{CR}(G)$. Let $\epsilon \in P(Y)$ be given and some $\widetilde{g} \in \widetilde{G}$. First, we shall construct an $(\epsilon, \widetilde{g})$-chain from $y = \rho(x)$ to itself. Let $g \in G$ be such that $\rho \circ g = \widetilde{g} \circ \rho$.

Since $\rho : X \to Y$ is continuous, by Lemma 2.1, there exists a $\delta \in P(X)$ such that if $d_X(x_1, x_2) < \delta(x_1)$ then $d_Y(\rho(x_1), \rho(x_2)) < \epsilon(\rho(x_1))$, for $x_1, x_2 \in X$. Also $x \in \mathsf{CR}(G)$ implies there is a (δ, g)-chain $(x = x_1, \ldots, x_{n+1} = x; g_1, \ldots, g_n)$ from x to itself. That is, for each $i = 1, \ldots, n$, we have,

$$d_X(g_i \circ g(x_i), x_{i+1}) < \delta(g_i \circ g(x_i)).$$

For each $i = 1, \ldots, n+1$, let $y_i = \rho(x_i)$ and $\rho \circ g_i = \widetilde{g_i} \circ \rho$. Since

$$d_X(g_i \circ g(x_i), x_{i+1}) < \delta(g_i \circ g(x_i)),$$

we have,

$$\begin{aligned} d_Y(\widetilde{g_i} \circ \widetilde{g}(y_i), y_{i+1}) &= d_Y(\rho \circ g_i \circ g(x_i), \rho(x_{i+1})) \\ &< \epsilon(\rho \circ g_i \circ g(x_i)) \\ &= \epsilon(\widetilde{g_i} \circ \widetilde{g}(y_i)). \end{aligned}$$

Thus $(\rho(x) = y_1, \ldots, y_{n+1} = \rho(x); \widetilde{g}_1, \ldots, \widetilde{g}_n)$ is an $(\epsilon, \widetilde{g})$-chain from $\rho(x)$ to itself and hence $\rho(x) \in \mathsf{CR}(\widetilde{G})$. Therefore $\rho(\mathsf{CR}(G)) \subset \mathsf{CR}(\widetilde{G})$.

Conversely, let $y \in \mathsf{CR}(\widetilde{G})$. Since ρ^{-1} is a conjugacy from Y to X, we have $y \in \rho(\mathsf{CR}(G))$. Thus, $\rho(\mathsf{CR}(G)) = \mathsf{CR}(\widetilde{G})$. □

Thus $\mathsf{CR}(G)$ is invariant under topological conjugacy.

3. Conley's Theorem for Semigroup of Continuous Maps

In [7], we have introduced the notion of an attractor for a semigroup of continuous maps as follows:

Definition 3.1. An open subset U of X is said to be a *trapping region* for G if there exists an $h \in G$ such that for the set

$$\widetilde{U} := \{fh(x) : x \in U \text{ and } f \in \widehat{G}\}$$

we have $\mathsf{cl}(\widetilde{U}) \subset U$.

Definition 3.2. The *attractor* for G determined by a *trapping region* U for G is defined by

$$A := \{x \in X : \text{for every open subset } V \text{ of } X \text{ containing } x,\ V \cap f_k(h(U)) \neq \emptyset$$
$$\text{for infinitely many } k \in \mathbb{N}, \text{where } (f_k)_{k\in\mathbb{N}} \text{ is some unbounded sequence in } G\},$$

where $h \in G$ is as in the Definition 3.1.

The notion of unbounded sequence for the continuous semigroup was introduced in [5] as follows: A sequence of functions $(f_k)_{k\in\mathbb{N}}$ in $G = \langle g_\alpha \rangle_\alpha$ is said to be *unbounded* if there is

(1) a sequence $(n_k)_k$ of natural numbers with $n_k \to \infty$ as $k \to \infty$; and
(2) a generator $g_{\alpha_0} \in \{g_\alpha\}_\alpha$ such that each f_k consists of exactly n_k iterates of g_{α_0}, that is, $f_k = h_{n_k+1} \circ g_{\alpha_0} \circ h_{n_k} \circ g_{\alpha_0} \circ h_{n_k-1} \circ \cdots \circ h_2 \circ g_{\alpha_0} \circ h_1$, where each $h_i \in \widehat{G}$ and the functions h_i are independent of g_{α_0}.

In [7], the basin of attraction of an attractor is defined as the set of points whose omega limit set intersects with the attractor. Now, in the noncompact space, when the omega limit set can be empty, so a new definition in this case is given below.

Definition 3.3. The *basin of attraction* of an attractor A for G determined by a trapping region U is defined by

$$B(A) := \{x \in X : f(x) \in U \text{ for some } f \in G\}.$$

Clearly, $B(A)$ contains U. Also, when X is compact then the above definition is equivalent to that given in [7]: Indeed, if $x \in B(A)$ follows [7] then some $z \in \omega(x) \cap A$ and $A \subset U$. Let $(f_k)_{k\in\mathbb{N}}$ be an unbounded sequence in G such that $f_k(x) \to z$, as $k \to +\infty$. Then $f_k(x) \in U$ for large k. Conversely, if $x \in B(A)$ follows the above definition then $f(x) \in U$ for some $f \in G$. Let $(f_k)_{k\in\mathbb{N}}$ be an unbounded sequence in G and $h \in G$ be such that $\mathsf{cl}(\widetilde{U}) \subset U$. Since X is compact there is a $z \in X$ and a subsequence $(f_{k_l})_l$ such that $f_{k_l}hf(x) \to z$, as $l \to +\infty$ and hence $z \in \omega(x)$. For an open subset V of X containing z, there exists an $N \in \mathbb{N}$ such that for all $l \geq N$, $f_{k_l}hf(x) \in V$, hence $V \cap f_{k_l}h(U) \neq \emptyset$ and so $z \in A$. Therefore $z \in A \cap \omega(x)$.

We have noticed the following characteristics of an attractor in [7].

Proposition 3.4. [7] The attractor A determined by a trapping region U is invariant under G.

Proposition 3.5. [7] The attractor A determined by a trapping region U is a closed set.

Proposition 3.6. Let A be the attractor determined by a trapping region U. Then A is contained in U.

Proof. The result is vacuously true when U is empty.

Suppose U is a nonempty trapping region and $h \in G$ is such that $\mathsf{cl}(\widetilde{U}) \subset U$. Define $\eta \in P(X)$ by

$$\eta(x) = \frac{1}{2}\left(d(x, \mathsf{cl}(\widetilde{U})) + d(x, X \setminus U)\right).$$

Notice that $\eta(x)$ is always positive. For if $x \notin U$ then $d(x, \mathsf{cl}(\widetilde{U})) > 0$ and otherwise $d(x, X \setminus U) > 0$. Thus, if $y \in \mathsf{cl}(\widetilde{U})$ and $z \in X$ such that $d(y,z) < \eta(y)$ then $z \in U$.

Let $x \in A$. By Lemma 2.2, a $\delta \in P(X)$ exists corresponding to η. Then $B(x, \delta(x)) \cap f_k(h(U)) \neq \emptyset$ for infinitely many $k \in \mathbb{N}$, where $(f_k)_{k \in \mathbb{N}}$ is some unbounded sequence in G. Also,

$$f_k(h(U)) \subset \widetilde{U} \subset \mathsf{cl}(\widetilde{U}).$$

Therefore, $B(x, \delta(x)) \cap \mathsf{cl}(\widetilde{U}) \neq \emptyset$. Let $y \in B(x, \delta(x)) \cap \mathsf{cl}(\widetilde{U})$. Since

$$d(x,y) < \delta(x) < \eta(x)/2 < \eta(y),$$

we have $x \in U$. □

Proof of the Main theorem. Let A be an attractor of G and $p \in B(A) \setminus A$. Let U be a trapping region which determines A, and $h \in G$ is such that $\mathsf{cl}(\widetilde{U}) \subset U$. As in Proposition 3.6, there is $\eta \in P(X)$ such that if $y \in \mathsf{cl}(\widetilde{U})$ and $z \in X$ such that $d(y,z) < \eta(y)$ then $z \in U$. Without any loss of generality, we can take η to be $\min\{\eta, 1\}$.

For $p \in B(A)$, we have, $f(p) \in U$ for some $f \in G$. Let $(f_k)_{k \in \mathbb{N}}$ be an unbounded sequence in G. Since U is a trapping region, $f_k h f(p) \in U$. For each $k \in \mathbb{N}$, let $\hat{f}_k$ denotes $f_k h f$.

Now if $p \in \mathsf{CR}(G)$, then for each $k, m \in \mathbb{N}$, there exists an $(\eta/m, \hat{f}_k h)$-chain $(p = x_1, \ldots, x_{l_k+1} = p; h_1, \ldots, h_{l_k})$ from p to itself. Since G is abelian and $\hat{f}_k(p) \in U$, we have

$$h_1 \hat{f}_k h(x_1) \in h_1 h(U) \subset \widetilde{U} \subset \mathsf{cl}(\widetilde{U}).$$

Since

$$d(h_1 \hat{f}_k h(x_1), x_2) < \frac{1}{m}\eta(h_1 \hat{f}_k h(x_1)) \leq \eta(h_1 \hat{f}_k h(x_1)),$$

we have $x_2 \in U$. Similarly,

$$h_2 \hat{f}_k h(x_2) \in \widetilde{U} \subset \mathsf{cl}(\widetilde{U})$$

and $d(h_2 \hat{f}_k h(x_2), x_3) < \frac{1}{m}\eta(h_2 \hat{f}_k h(x_2)) \leq \eta(h_2 \hat{f}_k h(x_2))$ gives $x_3 \in U$. Following the previous arguments, we have $x_i \in U$ for every $i = 1, \ldots, l_k$. Since

$$d(h_{l_k} \hat{f}_k h(x_{l_k}), p) < \frac{1}{m}\eta(h_{l_k} \hat{f}_k h(x_{l_k})) \leq \frac{1}{m},$$

we have

$$B_{\frac{1}{m}}(p) \cap h_{l_k} \hat{f}_k h(U) \neq \emptyset,$$

where $B_{\frac{1}{m}}(p)$ is the open ball centered at p of radius $\frac{1}{m}$.

Now, the sequence of functions defined by

$$\bar{f}_k = h_{l_k} \hat{f}_k,$$

for all $k \in \mathbb{N}$, is an unbounded sequence in G. Also, for any open subset V of X containing p, there is $m \in \mathbb{N}$ such that $B_{\frac{1}{m}}(p) \subset V$ and an unbounded sequence $(\bar{f}_k)_{k\in\mathbb{N}}$ in G corresponding to the m. Thus, for all $k \in \mathbb{N}$

$$V \cap \bar{f}_k(h(U)) \supset B_{\frac{1}{m}}(p) \cap \bar{f}_k(h(U)) \neq \emptyset.$$

Hence $p \in A$, which is a contradiction. Therefore, if $p \in B(A) \setminus A$ then $p \notin \mathsf{CR}(G)$.

Conversely, let $p \notin \mathsf{CR}(G)$. Then there exists an $\epsilon \in P(X)$ and $h_1 \in G$, so there is no (ϵ, h_1)-chain from p to itself. Consider the set defined by

$$U := \{x \in X : \text{there is an } (\epsilon, h_1)\text{−chain from } p \text{ to } x\}.$$

Then $p \notin U$ and set

$$\widetilde{U} := \{fh_1(x) : x \in U \text{ and } f \in \widehat{G}\}.$$

Also, U is an open subset of X: for $x \in U$, there exists an (ϵ, h_1)-chain $(p = x_1, \ldots, x_{n+1} = x;\ f_1, \ldots, f_n)$ from p to x. Hence $x \in B(f_n h_1(x_n), \epsilon(f_n h_1(x_n)))$. If $y \in B(f_n h_1(x_n),\ \epsilon(f_n h_1(x_n)))$ then $(p = x_1, \ldots, x_{n+1} = y; f_1, \ldots, f_n)$ from p to y is an (ϵ, h_1)-chain and hence $y \in U$. Thus $B(f_n h_1(x_n), \epsilon(f_n h_1(x_n))) \subset U$ and U is open.

Let $y \in \mathsf{cl}(\widetilde{U})$. By Lemma 2.2, a $\delta \in P(X)$ exists corresponding to ϵ. Then for some $x \in U$ and $f \in \widehat{G}$, we have $fh_1(x) \in B(y, \delta(y))$. Also,

$$d(y, fh_1(x)) < \delta(y) < \frac{1}{2}\epsilon(y) < \epsilon(fh_1(x)).$$

Thus, $(x, y; f)$ is an (ϵ, h_1)-chain from x to y. By transitivity, there is an (ϵ, h_1)-chain from p to y. Hence $\mathsf{cl}(\widetilde{U}) \subset U$. Therefore, U is a trapping region for G.

Let A be an attractor determined by U. Since $p \notin U$ and $A \subset U$ (by Proposition 3.6), it follows that $p \notin A$.

Moreover, $(p, h_1(p); \text{identity})$ is an (ϵ, h_1)-chain from p to $h_1(p)$. Hence, $h_1(p) \in U$ implies $p \in B(A)$. Therefore, $p \in B(A) \setminus A$.

The following example will illustrate an application of the above theorem.

Example 3.7. Consider the polynomial mappings of the complex plane, given by

$$g_n : \mathbb{C} \to \mathbb{C}$$

$$g_n(z) = z^n.$$

We shall identify all the trapping regions and attractors for the dynamics of $G = \{g_n : n \in \mathbb{N} \setminus \{1\}\}$ on the phase space $\mathbb{C}$. The empty set and the phase space itself are trapping regions for G. Moreover, in this case, the corresponding attractor and basin of attraction coincide with the trapping regions respectively. Thus, we have $B(A) \setminus A = \emptyset$ for each attractor.

Next, we shall classify the attractors and basin of attraction for a proper nonempty trapping region of the phase space $\mathbb{C}$. For $g \in G$ and $|z| > 1$ we have $|g(z)| > |z|$ and for $|w| < 1$ we have $|g(w)| < |w|$. Firstly, we shall see that a trapping region intersecting the unit circle will be $\mathbb{C}$.

Let U be a trapping region for G intersecting the unit circle $\mathbb{S}^1$ and $h \in G$ be as in the Definition 3.1. Now some open arc of $\mathbb{S}^1$ lies inside U, that is, $A = \{e^{i\theta} : \theta \in (a, b)\} \subset U$. For $g_n \in G$, $g_n(A) = \{e^{in\theta} : \theta \in (a, b)\}$ and $g_n \circ h = g_m$ for some $m \in \mathbb{N}$. Here, we can find $n \in N$ sufficiently large so that $m(b-a) > 2\pi$. Thus,

$$\begin{aligned} g_n \circ h(A) &= g_m(A) \\ &= \{e^{im\theta} : \theta \in (a,b)\} \\ &= \{e^{i\phi} : \phi \in (ma, mb)\}. \end{aligned}$$

Hence $\mathbb{S}^1 \subset g_n \circ h(A) \subset \widetilde{U} \subset \mathsf{cl}(\widetilde{U}) \subset U$.

Since $\mathbb{S}^1$ is compact, we can find an open tubular neighbourhood of $\mathbb{S}^1$ in U as $T = \{z \in \mathbb{C} : c < |z| < d\}$ for some c, d where $0 < c < 1 < d < +\infty$. For, $g_n \in G$, $g_n(T) = \{z^n : c < |z| < d\} = \{w \in \mathbb{C} : c^n < |w| < d^n\}$. So the set $\widetilde{T} = \{fh(z) : z \in T,\ f \in \widehat{G}\} = \mathbb{C} \setminus \{0\}$. Now $\mathbb{C} = \mathsf{cl}(\widetilde{T}) \subset \mathsf{cl}(\widetilde{U}) \subset U$.

Hence a proper nonempty trapping region of the phase space $\mathbb{C}$ lies in the complement of the unit circle. Therefore, the attractors and basin of attraction for any proper trapping region correspond to one of the following types.

Type I: (Inside open unit disc) For each $r \in (0,1)$, the set $U_r = \{z \in \mathbb{C} : |z| < r\}$ is a trapping region for G. Since $z \in U_r$ implies $|g(z)| \leq |z|^2 < r$ for every $g \in G$. In particular, for $\widetilde{U}_r = \{g \circ g_2(z) : z \in U_r \text{ and } g \in \widehat{G}\}$, we have $\mathsf{cl}(\widetilde{U}_r) \subset U_r$. Here, for each $r \in (0,1)$, the attractor A_r is $\{0\}$ and the basin of attraction $B(A_r)$ is the set $\{z \in \mathbb{C} : |z| < 1\}$ for each r. Therefore, in this case, $B(A_r) \setminus A_r = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Moreover, any nonempty trapping region U in unit disc will contain the origin, for $0 \in \mathsf{cl}(\widetilde{U}) \subset U$. Again the attractor will be $\{0\}$ and the basin of attraction is the open unit disc, as $U_r \subset U$ for some $r \in (0,1)$.

Type II: (Outside closed unit disc) For each $r \in (1,\infty)$, the set $V_r = \{z \in \mathbb{C} : |z| > r\}$ is a trapping region for G. Now, for each $r \in (1,\infty)$ the attractor A_r is the empty set and the basin of attraction $B(A_r)$ is the set $\{z \in \mathbb{C} : |z| > 1\}$ for each r. Hence, $B(A_r) \setminus A_r = \{z \in \mathbb{C} : |z| > 1\}$. In-fact any nonempty trapping region V outside unit disc is the complement of a compact set containing closed unit disc. For if some $a \in V$ then an open ball $B(a,\rho) \subset V$. Consider the arc $A = \{|a|e^{i\theta} : \theta \in (\theta_1, \theta_2)\} \subset B(a,\rho)$. As above, by similar arguments, we see that for sufficiently large $n \in \mathbb{N}$, the circle $C = \{z : |z| = |a^n|\} \subset \widetilde{V} \subset V$. Again, we can find an open annulus around C in V as $T = \{z \in \mathbb{C} : c < |z| < d\}$ for some c, d where $1 < c < |a^n| < d < +\infty$. Further $\dfrac{d}{c} > 1$ and $\left(\dfrac{d}{c}\right)^m \to \infty$ as $m \to \infty$, there exists some $k \in \mathbb{N}$ such that $c < \left(\dfrac{d}{c}\right)^m$ for all $m > k$. Hence for all $m > k$, $c^m < c^{m+1} < d^m < d^{m+1}$ and $g_m(T) \cup g_{m+1}(T)$ is an annulus. Now for some $r > 1$ and $h \in G$ be as in the Definition 3.1, we have

$$V_r \subset \bigcup_{m>k} g_m \circ h(T) \subset \mathsf{cl}(\widetilde{V}) \subset V.$$

Thus $\mathbb{C} \setminus V$ has to be bounded and hence compact. For such a trapping region V, the attractor is the empty set and the basin of attraction is the complement of the closed unit disc as $V_r \subset V$ for some $r \in (1,\infty)$.

Type III: (Union of Type I and Type II) Another possible trapping region will be the union of Type I and Type II trapping regions. In this case, the attractor A will be $\{0\}$ and the basin of attraction will be complement of the unit circle in complex plane. Therefore, $B(A) \setminus A = \{z \in \mathbb{C} : |z| \neq 0, |z| \neq 1\}$.

Now by our main theorem, we have

$$\begin{aligned} X \setminus \mathsf{CR}(G) &= \bigcup_A [B(A) \setminus A] \\ &= \emptyset \cup \{z \in \mathbb{C} : 0 < |z| < 1\} \cup \{z \in \mathbb{C} : |z| > 1\} \cup \{z \in \mathbb{C} : |z| \neq 0, |z| \neq 1\} \\ &= \{z \in \mathbb{C} : |z| \neq 0, |z| \neq 1\}. \end{aligned}$$

Consequently, we have $\mathsf{CR}(G) = \{z \in \mathbb{C} : |z| = 0 \text{ or } |z| = 1\}$.

Note that we can directly find all the chain recurrent points for G and show that $\mathsf{CR}(G) = \{z \in \mathbb{C} : |z| = 0 \text{ or } |z| = 1\}$ as in Example 3.7 above. As 0 is a fixed point of G, hence a chain recurrent point for G.

For $0 < |z| < 1$, we shall show that z is not a chain recurrent point for G. Let $m, M \in \mathbb{N}$ be sufficiently large, so that $1/m$ is small enough such that $\big||g_{_M}(z)| - |z|\big| > \frac{1}{m}$. Let $\epsilon \in P(\mathbb{C})$ be bounded above by $1/m$. Then there is no $(\epsilon, g_{_M})$-chain from z to itself. If $(z = z_1, \ldots, z_{n+1}; h_1, \ldots, h_n)$ is an $(\epsilon, g_{_M})$-chain from z then $B(h_i \circ g_{_M}(z_i), \frac{1}{m}) \subset B(0, |z|)$ for all i. In particular, $z_{n+1} \in B(h_n \circ g_{_M}(z_n), \frac{1}{m}) \subset B(0, |z|)$ and hence $z \neq z_{n+1}$. Thus, z is not a chain recurrent point for G.

Also, by a similar argument, we see that for $|z| > 1$, z can not be a chain recurrent point for G. Taking m, M and ϵ as above satisfying $\big||g_{_M}(z)| - |z|\big| > \frac{1}{m}$. If $(z = z_1, \ldots, z_{k+1}; f_1, \ldots, f_k)$ is an $(\epsilon, g_{_M})$-chain from z then $B(f_i \circ g_{_M}(z_i), \frac{1}{m})$ is disjoint from the closed $B(0, |z|)$ for all i. In particular, $z_{k+1} \in B(f_k \circ g_{_M}(z_n), \frac{1}{m})$ and hence $z \neq z_{k+1}$. Thus, z is not a chain recurrent point for G.

Let $z_0 \in \mathbb{C}$ be such that $|z_0| = 1$, that is, $z_0 = \exp(i\pi\theta_0)$. Let $\epsilon \in P(\mathbb{C})$ and $n_0 \in \mathbb{N} \setminus \{1\}$. We shall construct an (ϵ, g_{n_0})-chain from z_0 to itself. For rationals are dense in reals, we can pick a $z_1 = \exp(i\pi p_1/q_1) \in B_\epsilon(g_{n_0}(z_0)) = \{z : |g_{n_0}(z_0) - z| < \epsilon(g_{n_0}(z_0))\}$, for some $p_1, q_1 \in \mathbb{N}$. Then,

$$\begin{aligned} g_{2q_1} g_{n_0}(z_1) &= \exp(i\pi p_1 n_0 2q_1/q_1) \\ &= 1. \end{aligned}$$

Since the unit circle is compact, there is $\epsilon_0 = \min\{\epsilon(z) : |z| = 1\}$. Let $p, q \in \mathbb{N}$ be such that $w = \exp(i\pi p/q) \in B_{\epsilon_0}(z_0) = \{z : |z_0 - z| < \epsilon_0\}$. There is an $N \in \mathbb{N}$ sufficiently large such that $z_2 = \exp(i\pi p/(q n_0 N)) \in B_\epsilon(1) = \{z : |1 - z| < \epsilon(1)\}$. Then $g_{_N} g_{n_0}(z_2) = w$ and $|w - z_0| < \epsilon_0 \leq \epsilon(w)$.

Therefore, $(z_0, z_1, z_2, z_0; \text{identity}, g_{2q_1}, g_{_N})$ is an (ϵ, g_{n_0})-chain from z_0 to itself. Thus z_0 is a chain recurrent point for G and we have $\mathsf{CR}(G) = \{z \in \mathbb{C} : |z| = 0 \text{ or } |z| = 1\}$.

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