Poincare Journal of Analysis & Applications
Vol. 11, No. 2 (2024), 211-229

DOI: 10.46753/pjaa.2024.v011i02.010
Online Published on 13. 12. 2024

MODULI OF CONTINUITY AND WILSON WAVELET APPROXIMATION OF SOLUTION FUNCTIONS OF FREDHOLM INTEGRAL EQUATIONS WITH BOUNDED M^{th} DERIVATIVES

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Date of Receiving	:	22. 10. 2024
Date of Acceptance	:	02. 12. 2024

Abstract. Wilson wavelets have been discussed in this paper. An analysis has been conducted on the orthonormality property of Wilson wavelets. A verification has been done on the Wilson wavelet series of functions. The Wilson wavelet approximation of a function f whose M^{th} derivative $f^{(M)}$ belonging to Hölder's class of order $0 < \alpha \leq 1$ has been determined. The moduli of continuity has been estimated for solution function f of Fredholm integral equation. A method to solve integral equations by Wilson wavelet technique has been proposed. It is observed that the solutions of Fredholm integral equation obtained by the proposed method are almost same to their exact solutions. This illustrates the effectiveness of the proposed method.

1. Introduction

In recent years, wavelets have found their ways into different fields of science and engineering such as signal analysis for waveform representation, segmentation, image manipulation and numerical analysis[11]. In wavelet analysis, wavelet functions approximate a function with less nonzero wavelet coefficients in comparison to Fourier analysis. It studies the functions which are decomposed into basic functions and these are achieved by dilation and translation of mother wavelet.

In modern analysis, orthogonal functions and wavelets have been used to approximate a function by partial sums of its wavelet series. The main reason behind using orthogonal

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2010 *Mathematics Subject Classification.* 42C40, 45B05, 65T60, 65R20.
Key words and phrases. Hölder's class, Operational matrix, Wilson wavelet, Wavelet approximation.
Communicated by: Suman Panwar
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basis is that it reduces these problems to solving a system of linear algebraic equations by truncated approximation series

$$(S_{2^k,\hat{m}}(f))(t) = \sum_{n=0}^{2^k-1} \sum_{m=-2r-1}^{-2r} c_{n,m}\psi_{n,m}(t) \equiv C^T\psi(t) \tag{1.1}$$

of Wilson wavelet series

$$f(t) = \sum_{n=0}^{\infty} \sum_{m=-2r-1}^{-2r} c_{n,m}\psi_{n,m}(t) \tag{1.2}$$

of $f \in L^2[0,1)$, where C and $\psi(t)$ are $m = 2^{k+1}$ column vectors, $\hat{m} = (-2r-1)+(-2r)$.

There are several problems of science and technology in the form of integral equations. There are various methods to find numerical solution of integral equations [1][2]. Sometimes it is difficult to find solution of integral equation analytically. Hence, there is a need of approximate solution of such problems by Wilson wavelet method .

In this paper, a method to solve Fredholm integral equations has been introduced and an approximate solution of this integral equation using Wilson wavelets have been computed [12]. In order to find that, the convergence of Wilson wavelet series has been analyzed. A new operational matrix of integration for the basis functions has been obtained to eliminate the integral operation and to reduce the problem into solving a system of algebraic equations [4, 9, 10, 7].

The paper is organised as follows : The Wilson wavelets, their orthonormality, and other basic definitions required for subsequent developments are covered in Section 2. In Section 3, the convergence analysis of Wilson wavelet series has been studied. In Section 4, Estimators for Wilson wavelet approximations of functions belonging to $H^\alpha[0,1)$ class, having M^{th} derivative bounded along with their moduli of continuity has been investigated. In Section 5, a method to solve Fredholm integral equations by Wilson wavelets has been developed and the effectiveness of this method has been observed by examples. In Section 6, the conclusion of the research work has been mentioned.

2. Preliminaries

To begin, we will recall some well-known definitions of wavelets and function spaces that will be employed subsequently:

Definition 2.1. Function of $H^\alpha[0,1)$ class
A continuous function $g \in H^\alpha[0,1)$, if it satisfies the inequality:

$$|g(x) - g(y)| = O(|x-y|^\alpha) \;\; \forall\; x, y \in [0,1) \text{ and } 0 < \alpha \leq 1.$$

For details, one can see [13].

Definition 2.2. Modulus of Continuity
The Modulus of continuity of a function $g \in L^2[0,1)$ is defined as:

$$\begin{aligned} W(g,\delta) &= \sup_{0<h\leq\delta} ||g(\cdot + h) - g(\cdot)||_2 \\ &= \sup_{0<h\leq\delta} (\int_0^1 |g(t+h) - g(t)|^2 dt)^{\frac{1}{2}}. \end{aligned}$$

It is notable that $W(f,\delta)$ is a non-decreasing function of δ and $W(f,\delta) \to 0$ as $\delta \to 0$. For more details, one can refer to [4].

2.1. The Wilson Wavelets. Wilson introduced a system of basis function as follows:

$$\psi_{n,m}(t) = \begin{cases} \epsilon_n cos(2n\pi t)\omega(t - \frac{m}{2}), & \text{m is even,} \\ \sqrt{2} sin(2(n+1)\pi t)\omega(t - \frac{m+1}{2}), & \text{m is odd.} \end{cases} \tag{2.1}$$

where

$$\epsilon_n = \begin{cases} 1 & , \ n = 0 \\ \sqrt{2} & , \ n \in \mathbb{N} \end{cases}$$

with a smooth well-localized window function ω [8, 1]. The positive and negative frequencies are the focal points of this system's components. From this, an orthonormal system was built using a series of basis functions, which eventually came to be known as the "Wilson basis"[5]. Now we consider $\omega = \chi_{[r,r+1)}$, $r \in \mathbb{Z}$ in Eq. (2.1), i.e.

$$\psi_{n,m}(t) = \begin{cases} \epsilon_n cos(2n\pi t)\chi_{[r,r+1)}(t - \frac{m}{2}), & \text{m is even} \\ \sqrt{2} sin(2(n+1)\pi t)\chi_{[r,r+1)}(t - \frac{m+1}{2}), & \text{m is odd.} \end{cases} \tag{2.2}$$

where

$$\epsilon_n = \begin{cases} 1 & , \ n = 0 \\ \sqrt{2} & , \ n \in \mathbb{N} \end{cases}$$

The set $\{\psi_{nm}(\centerdot)|m \in \mathbb{Z}, n \in \mathbb{N} \cup \{0\}\}$ is a tight frame for $L^2(\mathbb{R})$ with bound 1 [8]. The set $\{\psi_{nm}(\centerdot)|m \in \{-2r-1, -2r\}\}, n \in \mathbb{N}\cup\{0\}$, for $r \in \mathbb{Z}$ in Eq.(2.2) is an orthonormal basis for $L^2[0,1)$. This is called the Wilson wavelets.

2.2. Wilson Wavelet Series Approximation. Any square integrable function $f(t)$ defined on $[0,1)$ can be expanded in terms of the Wilson wavelets as:

$$f(t) = \sum_{n=0}^{\infty} c_{n,-2r}\psi_{n,-2r}(t) + \sum_{n=0}^{\infty} c_{n,-2r-1}\psi_{n,-2r-1}(t) \tag{2.3}$$

where $c_{n,m} =< f(t), \psi_{n,m}(t) >$, $m \in \{-2r-1, -2r\}$ and $< \ldots >$ denotes the inner product on $L^2[0,1)$. If the infinite series in Eq. (2.3) is truncated, then it can be written as:

$$S_{2^k-1,M}(f)(t) = \sum_{n=0}^{2^k-1} \sum_{m=-2r-1}^{-2r} c_{n,m}\psi_{n,m}(t) \triangleq C^T\psi(t) \tag{2.4}$$

where C and $\psi(t)$ are $\hat{m} = 2^{k+1}$ column vectors given by:

$$C \triangleq [c_{0,-2r-1}, c_{0,-2r}, c_{1,-2r-1}, c_{1,-2r} \ldots, c_{2^k-1,-2r-1}, c_{2^k-1,-2r}]^T \tag{2.5}$$

$$\textit{and} \quad \psi(t) \triangleq [\psi_{0,-2r-1}(t), \psi_{0,-2r}(t), \psi_{1,-2r-1}(t), \ldots, \psi_{2^k-1,-2r-1}(t), \psi_{2^k-1,-2r}(t)]^T \tag{2.6}$$

The Wilson wavelet approximation $E_{2^k,\hat{m}}(f)$, where $\hat{m} = (-2r-1) + (-2r)$ of a function $f \in L^2[0,1)$ by $(2^k, (-2r-1)+(-2r))^{th}$ partial sums $S_{2^k,(-2r-1)+(-2r)}(f)$ of its Wilson wavelet series is given by:

$$E_{2^k,\hat{m}}(f) = \min_{S_{2^k,\hat{m}}(f)} ||f - S_{2^k,\hat{m}}||_2,$$

where

$$||f||_2 = \Big(\int_0^1 |f(t)|^2 dt \Big)^{\frac{1}{2}}.$$

If $E_{2^k,\hat{m}}(f) \to 0$ as $k, \hat{m} \to \infty$, then $E_{2^k,\hat{m}}(f)$ is called the most accurate estimation of f [14].

2.3. Orthonormality of Wilson wavelets. Here, the property of orthonormality of the set $\{\psi_{n,m}(\centerdot) | m \in \mathbb{Z}, n \in \mathbb{N} \cup \{0\}\}$ of Wilson wavelets has been investigated in the following proposition.

Proposition 2.3. The set $\{\psi_{n,m}(\centerdot) | m \in \mathbb{Z}, n \in \mathbb{N} \cup \{0\}\}$ of Wilson wavelet series is orthonormal, i.e ;

$$< \psi_{n,m}, \psi_{n',m'} > = \int_0^1 \psi_{n,m}(t)\psi_{n',m'}(t)dt = \delta_{n,n'}\delta_{m,m'} \tag{2.7}$$

where

$$\delta_{n,n'} = \begin{cases} 1 & , n = n' \\ 0 & , n \neq n' \end{cases} \quad , \quad \delta_{m,m'} = \begin{cases} 1 & , m = m' \\ 0 & , m \neq m' \end{cases}$$

Proof. **Case 1**: Let $n = n'$, $< \psi_{n,m}, \psi_{n,m'} >= \int_0^1 \psi_{n,m}(t)\overline{\psi_{n,m'}(t)}dt$
For $m = m'$,

$$< \psi_{n,m}, \psi_{n,m} >= \int_0^1 \epsilon_n^2 cos^2(2n\pi t)dt = 1.$$

For $m \neq m'$, let $m = -2r-1$, $m' = -2r$

$$< \psi_{n,m}, \psi_{n,m'} >= \int_0^1 \sqrt{2}\ \epsilon_n^2 sin(2(n+1)\pi t) cos^2(2n\pi t)dt = 0.$$

Hence,

$$< \psi_{n,m}, \psi_{n,m'} >= \delta_{m,m'}.$$

Case 2: Let $m = m'$, $< \psi_{n,m}, \psi_{n',m} >= \int_0^1 \psi_{n,m}(t)\overline{\psi_{n',m}(t)}dt$
For $n = n'$,

$$< \psi_{n,m}, \psi_{n,m} >= \int_0^1 \epsilon_n^2 cos^2(2n\pi t)dt = 1, \quad \text{when m is even,}$$

$$< \psi_{n,m}, \psi_{n,m} >= \int_0^1 2 sin^2(2(n+1)\pi t)dt = 1, \ \text{when m is odd.}$$

For $n \neq n'$,

$$< \psi_{n,m}, \psi_{n',m} >= \int_0^1 \epsilon_n^2 cos(2n\pi t) cos(2n'\pi t)dt = 0, \ \text{when m is even,}$$

$$< \psi_{n,m}, \psi_{n',m} >= \int_0^1 2\ sin(2(n+1)\pi t) sin(2(n'+1)\pi t)dt = 0, \quad \text{when m is odd.}$$

Hence,

$$< \psi_{n,m}, \psi_{n',m} >= \delta_{n,n'}.$$

Case 3: Let $n \neq n'$ and $m \neq m'$, so let $m = -2r - 1$ and $m' = -2r$

$$< \psi_{n,m}, \psi_{n',m'} >= \int_0^1 \sqrt{2} sin(2(n+1)\pi t)\epsilon_n cos(2n'\pi t)dt = 0.$$

Hence,

$$< \psi_{n,m}, \psi_{n',m'} >= \delta_{n,n'}\delta_{m,m'}.$$

Therefore, the set $\{\psi_{n,m}(\centerdot)|m \in \mathbb{Z}, n \in \mathbb{N} \cup \{0\}\}$ is orthonormal. □

2.4. Verification of Wilson Wavelet series. The verification of Wilson wavelet series is presented as follows:

Example 2.4. For $f(t) = 1$ for all t $\in [0, 1)$, $||f||_2^2 = 1$.

$\psi_{0,-2r} = 1$, $c_{0,-2r} =< f, \psi_{0,-2r} >= \int_0^1 f(t)dt = 1$,

$\psi_{0,-2r-1} = \sqrt{2} sin(2\pi t)$, $c_{0,-2r-1} =< f, \psi_{0,-2r-1} >= \int_0^1 \sqrt{2} f(t) sin 2\pi t dt = 0$

$c_{n,-2r} = \int_0^1 \sqrt{2} cos(2n\pi(t)) = 0$ and $c_{n,-2r-1} = \int_0^1 \sqrt{2} sin 2(n+1)\pi t dt = 0$, $\forall\ n \geq 1$.

$$\begin{aligned}
\text{Then, } f(t) &= \sum_{n=0}^{\infty} (c_{n.-2r}\psi_{n,-2r}(t) + c_{n,-2r-1}\psi_{n,-2r-1}(t)), \\
\|f\|_2^2 &= \sum_{n=0}^{\infty} (|c_{n,-2r}|^2 + |c_{n,-2r-1}|^2) \\
&= (|c_{0,-2r}|^2 + |c_{0,-2r-1}|^2) + \sum_{n=1}^{\infty} (|c_{n,-2r}|^2 + |c_{n,-2r-1}|^2) = 1.
\end{aligned}$$

Example 2.5. For $f(t) = t + 1$ for all t $\in [0, 1)$.

$$\begin{aligned}
||f||_2^2 &=< f, f >= \int_0^1 ((t+1)^2)dt = \frac{7}{3}. \\
c_{0,-2r} &=< f, \psi_{0,-2r} >= \int_0^1 (t+1)dt = \frac{3}{2}, \\
c_{0,-2r-1} &= \int_0^1 \sqrt{2}\sin(2\pi t)(t+1)dt = \frac{-1}{\sqrt{2}\pi}, \\
c_{n,-2r} &= \int_0^1 \sqrt{2}(t+1)cos(2n\pi t)dt = 0, \\
c_{n,-2r-1} &= \int_0^1 \sqrt{2}(t+1)sin(2(n+1)\pi t)dt = \frac{-1}{\sqrt{2}(n+1)\pi}\ \forall\ n \geq 1. \\
||f||_2^2 &= |c_{0,-2r}|^2 + |c_{0,-2r-1}|^2 + \sum_{n=1}^{\infty} \left(|c_{n,-2r}|^2 + |c_{n,-2r-1}|^2\right)
\end{aligned}$$

$$= \frac{9}{4} + \frac{1}{2\pi^2} + \sum_{n=1}^{\infty}\Big(0 + \frac{1}{2(n+1)^2\pi^2}\Big) = \frac{9}{4} + \frac{1}{2\pi^2} + \frac{1}{2\pi^2}\Big(\frac{\pi^2}{6} - 1\Big) = \frac{7}{3}.$$

Hence, the series is verified for $f(t) = 1,\ t+1$ for all t $\in [0,1)$. This is carried out to demonstrate the existence of the wavelet series.

2.5. Operational Matrix of Integration. In this section, the operational matrix of integration for Wilson wavelets has been constructed. For $r = 0$ and $n = 0, 1, 2, 3$

$$\psi = [\psi_{0,-1} \quad \psi_{0,0} \quad \psi_{1,-1} \quad \psi_{1,0} \quad \psi_{2,-1} \quad \psi_{2,0} \quad \psi_{3,-1} \quad \psi_{3,0}]$$

The eight Wilson wavelet basis functions on the interval $[0,1)$ are given as:

$$\begin{aligned} \psi_{0,-1} &= \sqrt{2}sin(2\pi t); \quad \psi_{0,0} = 1; \quad && \psi_{1,-1} = \sqrt{2}sin(4\pi t);\\ \psi_{1,0} &= \sqrt{2}cos(2\pi t); \quad \psi_{2,-1} = \sqrt{2}sin(6\pi t); \quad && \psi_{2,0} = \sqrt{2}cos(4\pi t);\\ \psi_{3,-1} &= \sqrt{2}sin(8\pi t); \quad \psi_{3,0} = \sqrt{2}cos(6\pi t). \end{aligned}$$

We obtain the operational matrix P satisfying $\int_0^t \psi(y)dy = P\psi(t)$ has been obtained as follows :

$$\begin{aligned}
\int_0^t \psi_{0,-1}(y)dy &= \frac{1-cos(2\pi t)}{\sqrt{2}\pi} = [0 \quad \frac{1}{\sqrt{2}\pi} \quad 0 \quad \frac{-1}{2\pi} \quad 0 \quad 0 \quad 0 \quad 0]\psi(t);\\
\int_0^t \psi_{0,0}(y)dy &= [\frac{-1}{\sqrt{2}\pi} \quad \frac{1}{2} \quad \frac{-1}{2\sqrt{2}\pi} \quad 0 \quad \frac{-1}{3\sqrt{2}\pi} \quad 0 \quad \frac{-1}{4\sqrt{2}\pi} \quad 0]\psi(t);\\
\int_0^t \psi_{1,-1}(y)dy &= \frac{1-cos(4\pi t)}{2\sqrt{2}\pi} = [0 \quad \frac{1}{2\sqrt{2}\pi} \quad 0 \quad 0 \quad 0 \frac{-1}{4\pi} \quad 0 \quad 0]\psi(t);\\
\int_0^t \psi_{1,0}(y)dy &= \frac{sin(2\pi t)}{\sqrt{2}\pi} = [\frac{1}{2\pi} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]\psi(t);\\
\int_0^t \psi_{2,-1}(y)dy &= \frac{1-cos(6\pi t)}{3\sqrt{2}\pi} = [0 \quad \frac{1}{3\sqrt{2}\pi} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \frac{-1}{6\pi}]\psi(t);\\
\int_0^t \psi_{2,0}(y)dy &= \frac{sin(4\pi t)}{2\sqrt{2}\pi} = [0 \quad 0 \quad \frac{1}{4\pi} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]\psi(t);\\
\int_0^t \psi_{3,-1}(y)dy &= \frac{1-cos(8\pi t)}{4\sqrt{2}\pi} = [0 \quad \frac{1}{4\sqrt{2}\pi} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]\psi(t) \ ;\\
\text{and} \quad \int_0^t \psi_{3,0}(y)dy &= \frac{sin(6\pi t)}{3\sqrt{2}\pi} = [0 \quad 0 \quad 0 \quad 0 \quad \frac{1}{6\pi} \quad 0 \quad 0 \quad 0]\psi(t).
\end{aligned}$$

Hence, the Wilson wavelet operational matrix of integration $P_{8\times 8}$ is given as:

$$P_{8\times8}=\begin{bmatrix} 0 & \frac{1}{\sqrt{2}\pi} & 0 & \frac{-1}{2\pi} & 0 & 0 & 0 & 0\\ \frac{-1}{\sqrt{2}\pi} & \frac{1}{2} & \frac{-1}{2\sqrt{2}\pi} & 0 & \frac{-1}{3\sqrt{2}\pi} & 0 & \frac{-1}{4\sqrt{2}\pi} & 0\\ 0 & \frac{1}{2\sqrt{2}\pi} & 0 & 0 & 0 & \frac{-1}{4\pi} & 0 & 0\\ \frac{1}{2\pi} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{1}{3\sqrt{2}\pi} & 0 & 0 & 0 & 0 & 0 & \frac{-1}{6\pi}\\ 0 & 0 & \frac{1}{4\pi} & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{1}{4\sqrt{2}\pi} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{6\pi} & 0 & 0 & 0 \end{bmatrix}_{8\times8}$$

3. Convergence Analysis

In this section, the convergence analysis of the Wilson wavelet expansion of a solution function f of Fredholm integral equation is to be discussed.

Theorem 3.1. *If f is a solution of Fredholm integral equation such that each of $f, f^{(1)}, f^{(2)}, ..., f^{(M)}$ is bounded in $[0,1)$ i.e. $|f^{(q)}(t)| \leq C_f > 0$, for $q = 0,1,2,..,M$, then the Wilson wavelet series* (2.3) *of f converges absolutely in norm $||\centerdot||_2$.*

Proof. Let $r \in \mathbb{Z}$, by considering Wilson wavelet series (2.3) we get

$$|c_{0,-2r}| = |\int_0^1 f(t)dt| \leq \int_0^1 |f(t)dt| < \infty,$$

$$c_{0,-2r-1} = \int_0^1 f(t)\ \psi_{0,-2r-1}(t)dt = \int_0^1 \sqrt{2}\ f(t)\sin(2\pi t)dt.$$

$$\begin{aligned}|c_{0,-2r-1}| &= |\int_0^1 \sqrt{2} f(t)\sin(2\pi t)dt|\\ &\leq \sqrt{2}\int_0^1 |f(t)||\sin(2\pi t)|dt\\ &\leq \sqrt{2}\ |\int_0^1 f^2(t)dt|^{\frac{1}{2}}\ |\int_0^1 \sin^2(2\pi t)dt|^{\frac{1}{2}},\ \text{by Cauchy schwarz inequality}\\ &\leq \sqrt{2}\ ||f||_2\Big(\int_0^1 \Big(\frac{1-\cos(4\pi t)}{2}\Big)dt\Big)^{\frac{1}{2}}\\ &= \sqrt{2}\ ||f||_2 . \frac{1}{\sqrt{2}} = ||f||_2 < \infty\ .\end{aligned}$$

$$\begin{aligned}c_{n,-2r} &= \sqrt{2}\int_0^1 f(t)\cos(2n\pi t)dt,\ \text{integrating by parts}\\ &= \frac{\sqrt{2}}{2n\pi}\int_0^1 f'(t)\sin(2n\pi t)dt\end{aligned}$$

$$= \frac{\sqrt{2}}{(2n\pi)^2}\int_0^1 f''(t)\cos(2n\pi t).$$

Now repeating the above process of integrating by parts $M(\geq 2)$ times,

$$c_{n,-2r} = \frac{\sqrt{2}}{(2n\pi)^M}\int_0^1 f^{(M)}(t)\sin(2n\pi t)dt, \quad \text{when M is odd integer.}$$

$$|c_{n,-2r}| \leq \frac{\sqrt{2}}{(2n\pi)^M}\int_0^1 |f^{(M)}(t)||\sin(2n\pi t)|dt$$

$$\leq \frac{\sqrt{2}C_f}{(2n\pi)^M}\int_0^1 |\sin(2n\pi t)|dt \leq \frac{\sqrt{2}C_f}{(2n\pi)^M}.$$

Similarly, if M is even integer then

$$|c_{n,-2r}| \leq \frac{\sqrt{2}C_f}{(2n\pi)^M}.$$

Hence,

$$|c_{n,-2r}| \leq \frac{C_f}{\sqrt{2}.2^{M-1}.n^M.\pi^M} \quad \forall\, n \in \mathbb{N}\ . \tag{3.1}$$

$$c_{n,-2r-1} = \int_0^1 f(t)\psi_{n,-2r-1}(t)dt$$

$$= \sqrt{2}\int_0^1 f(t)sin(2(n+1)\pi t).\omega(t+r)dt$$

$$= \sqrt{2}\int_0^1 f(t)sin(2(n+1)\pi t)dt, \quad \text{integrating by parts}$$

$$= \frac{\sqrt{2}}{2(n+1)\pi}\int_0^1 f'(t)cos(2(n+1)\pi t)dt.$$

Now repeating the above process of integrating by parts M (≥ 1) times we get,

$$c_{n,-2r-1} = \frac{\sqrt{2}}{(2(n+1)\pi)^M}\int_0^1 f^{(M)}(t)cos(2(n+1)\pi t)dt, \ \text{when M is odd integer}$$

$$|c_{n,-2r-1}| \leq \frac{\sqrt{2}}{(2(n+1)\pi)^M}\int_0^1 |f^{(M)}(t)||\cos(2(n+1)\pi t)|dt$$

$$\leq \frac{\sqrt{2}C_f}{(2(n+1)\pi)^M}\int_0^1 |\cos(2(n+1)\pi t)|dt$$

$$\leq \frac{\sqrt{2}C_f}{(2(n+1)\pi)^M}.$$

Similarly, if M is even positive integer then

$$|c_{n,-2r-1}| \leq \frac{\sqrt{2}C_f}{(2n\pi)^M}.$$

Hence,

$$|c_{n,-2r-1}| \leq \frac{\sqrt{2}C_f}{2^M(n+1)^M\pi^M} = \frac{C_f}{\sqrt{2}2^{M-1}(n+1)^M\pi^M} \ . \tag{3.2}$$

By applying Cauchy integral test we get,

$$\begin{aligned}
||\sum_{n=0}^{\infty}\sum_{m=-2r-1}^{-2r} c_{n,m}\psi_{n,m}||_2 &= ||\sum_{m=-2r-1}^{-2r} c_{0,m}\psi_{0,m} + \sum_{n=1}^{\infty}\sum_{m=-2r-1}^{-2r} c_{n,m}\psi_{n,m}||_2 \\
&\leq \sum_{n=1}^{\infty}|c_{n,-2r-1}| + \sum_{n=1}^{\infty}|c_{n,-2r}| \\
&\leq \frac{C_f}{\sqrt{2}2^{M-1}\pi^M}\Big(\sum_{n=1}^{\infty}\frac{1}{n^M} + \sum_{n=1}^{\infty}\frac{1}{(n+1)^M}\Big) \\
&\leq \frac{C_f}{\sqrt{2}2^{M-1}\pi^M}\Big(1 + \int_1^{\infty}\frac{dn}{n^M} + \frac{1}{2} + \int_1^{\infty}\frac{dn}{(n+1)^M}\Big) \\
&\leq \frac{C_f}{\sqrt{2}2^{M-1}\pi^M}\Big(\frac{3}{2} + \frac{1}{M-1} + \frac{1}{(M-1)2^{M-1}}\Big) < \infty.
\end{aligned}$$

This implies that this series is absolutely convergent in $||\cdot||_2$ and hence it is convergent for $M \geq 2$. □

Theorem 3.2. *If f is the solution of Fredholm integral equation such that each of $f, f^{(1)}, f^{(2)}, \ldots, f^{(M)}$ belongs to $H^\alpha[0,1)$ then its Wilson wavelet series* (2.3) *of f converges absolutely.*

Proof. Let $r \in \mathbb{Z}$, by considering the Wilson wavelets series we get,

$$\begin{aligned}
|c_{0,-2r}| &= |\int_0^1 f(t)\psi_{0,-2r}(t)dt| \leq \int_0^1 |f(t)|dt < \infty, \\
|c_{0,-2r-1}| &= |\int_0^1 f(t)\psi_{0,-2r-1}(t)dt| = |\int_0^1 \sqrt{2}f(t)\sin(2\pi t dt)| < \infty.
\end{aligned}$$

Following the proof of Theorem (3.1),

$$\begin{aligned}
c_{n,-2r} &= \frac{\sqrt{2}}{(2n\pi)^M}\int_0^1 f^{(M)}(t)\sin(2n\pi t)dt, \quad \text{when M is odd integer} \\
&= \frac{\sqrt{2}}{(2n\pi)^M}\Big(\int_0^1 (f^{(M)}(t) - f^{(M)}(t_0))sin(2n\pi t)dt \\
&\quad + f^{(M)}(t_0)\int_0^1 sin(2n\pi t)dt\Big) \\
&= \frac{\sqrt{2}}{(2n\pi)^M}\int_0^1 \Big(f^{(M)}(t) - f^{(M)}(t_0)\Big)sin(2n\pi t)dt
\end{aligned}$$

$$= \frac{\sqrt{2}}{(2n\pi)^M} \sum_{p=1}^{n} \int_{\frac{(p-1)}{n}}^{\frac{p}{n}} \Big(f^{(M)}(t) - f^{(M)}(t_0)\Big) sin(2n\pi t)dt$$

$$|c_{n,-2r}| \leq \frac{\sqrt{2}}{(2n\pi)^{(M)}} \sum_{p=1}^{n} \int_{\frac{(p-1)}{n}}^{\frac{p}{n}} \Big|f^{(M)}(t) - f^{(M)}(t_0)\Big| \Big|\sin(2n\pi t)\Big| dt$$

$$\leq \frac{\sqrt{2}C_f}{(2n\pi)^M} \frac{1}{n^\alpha} . \frac{1}{n}, \ f \in H^\alpha[0,1)$$

$$= \frac{\sqrt{2}C_f}{2^M \pi^M n^{M+\alpha+1}} \ .$$

Hence,

$$|c_{n,-2r}| \leq \frac{\sqrt{2}C_f}{2^M \pi^M n^{M+\alpha+1}} \ . \tag{3.3}$$

Similarly,

$$|c_{n,-2r-1}| \leq \frac{\sqrt{2}C_f}{2^M (n+1)^M \pi^M n^{\alpha+1}} \ . \tag{3.4}$$

Applying Cauchy integral test, we get

$$||\sum_{n=0}^{\infty} \sum_{m=-2r-1}^{-2r} c_{nm}\psi_{nm}(t)||_2 = ||\sum_{m=-2r-1}^{-2r} c_{0,m}\psi_{0,m}(t) + \sum_{n=1}^{\infty} \sum_{m=-2r-1}^{-2r} c_{nm}\psi_{nm}(t)||_2$$

$$\leq \sum_{n=1}^{\infty} |c_{n,-2r-1}| + \sum_{n=1}^{\infty} |c_{n,-2r}|$$

$$\leq \frac{2\sqrt{2}C_f}{2^M \pi^M} \sum_{n=1}^{\infty} \frac{1}{n^{M+\alpha+1}}, \ (n+1) > n$$

$$\leq \frac{2\sqrt{2}C_f}{2^M \pi^M} \Big(1 + \int_1^\infty \frac{dn}{n^{M+\alpha+1}}\Big)$$

$$\leq \frac{2\sqrt{2}C_f}{2^M \pi^M} \Big(1 + \frac{1}{M+\alpha}\Big) < \infty \ .$$

Hence, the series $\sum_{n=0}^{\infty} \sum_{m=-2r-1}^{-2r} c_{nm}\psi_{nm}(t)$ is absolutely convergent . □

Theorem 3.3. *If f is the exact solution of the Fredholm integral equation then the Wilson wavelet series (2.3) of f converges uniformly to f.*

Proof. Let $f(t) = \sum_{n=0}^{\infty} \sum_{m=-2r-1}^{-2r} c_{n,m}\psi_{n,m}(t)$

$$< f, f >$$

$$= \Big\langle \sum_{n=0}^{\infty} \sum_{m=-2r-1}^{-2r} c_{n,m}\psi_{n,m}(t), \sum_{n'=0}^{\infty} \sum_{m'=-2r-1}^{-2r} c_{n',m'}\psi_{n',m'}(t) \Big\rangle$$

$$= \sum_{n=0}^{2^k-1} \sum_{m=-2r-1}^{-2r} |c_{n,m}|^2 ||\psi_{n,m}||_2^2 + \sum_{n=2^k}^{\infty} \sum_{m=-2r-1}^{-2r} |c_{n,m}|^2 ||\psi_{n,m}||_2^2$$
$$= \sum_{n=0}^{2^k-1} \sum_{m=-2r-1}^{-2r} |c_{n,m}|^2$$

Thus, $\sum_{n=0}^{2^k-1} \sum_{m=-2r-1}^{-2r} |c_{n,m}|^2 =< f, f >= \int_0^1 |f|^2 dt < \infty \ , \ f \in L^2[0,1).$

Therefore, the series $\sum_{n=0}^{2^k-1} \sum_{m=-2r-1}^{-2r} c_{n,m}\psi_{n,m}(t)$ is convergent.
By Bessel's Inequality, $\sum_{n=0}^{2^k-1} \sum_{m=-2r-1}^{-2r} |c_{n,m}|^2 \leq ||f||_2^2 < \infty$.
Let $\left(S_{2^k,\hat{m}} f\right)(t) = \sum_{n=0}^{2^k-1} \sum_{m=-2r-1}^{-2r} c_{n,m}\psi_{n,m}(t) \ , \ \hat{m} \ = (-2r-1) + (-2r)$.
For $k' > k$,

$$\left|\left|(S_{2^{k'},\hat{m}} f) - (S_{2^k,\hat{m}} f)\right|\right|_2^2 = \left|\left| \sum_{n=2^k}^{2^{k'}-1} \sum_{m=-2r-1}^{-2r} c_{n,m}\psi_{n,m} \right|\right|_2^2$$
$$\leq \sum_{n=2^k}^{2^{k'}-1} \sum_{m=-2r-1}^{-2r} |c_{n,m}|^2 ||\psi_{n,m}||_2^2$$
$$= \sum_{n=2^k}^{2^{k'}-1} \sum_{m=-2r-1}^{-2r} |c_{n,m}|^2 \to 0 \ as \ \ k \to \infty \ \& \ k' \to \infty.$$

Hence, $(S_{2^k,\hat{m}} f)$ is a Cauchy sequence in $L^2[0,1)$.
Since $L^2[0,1)$ is a Banach space, therefore the Cauchy sequence $(S_{2^k,\hat{m}} f)$ converges to a function $g \in L^2[0,1)$ such that

$$g(t) = \lim_{k\to\infty} (S_{2^k,\hat{m}} f) = \lim_{k\to\infty} \sum_{n=0}^{2^k} c_{n,m}\psi_{n,m}(t).$$

Now, it is needed to show that f(t)=g(t).

$$< g(t) - f(t), \psi_{n,m}(t) > =< g(t), \psi_{n,m}(t) > - < f(t), \psi_{n,m}(t) >$$
$$= lim_{k\to\infty} < (S_{2^k} f), \psi_{n,m}(t) > -c_{n,m}$$
$$= c_{n,m} - c_{n,m} = 0.$$

Therefore, $g = f$.
Hence, the series $\sum_{n=0}^{2^k-1} \sum_{m=-2r-1}^{-2r} c_{n,m}\psi_{n,m}(t)$ converges uniformly to $f(t)$ as $k \to \infty$. $\square$

4. Main Results

In this section, the approximations of solutions f of Fredholm integral equations by $(2^k,\hat{m})^{th}$ partial sums $S_{2^k,\hat{m}}(f)$ have been determined in the following form:

Theorem 4.1. *If each of $f,f^{(1)},f^{(2)},..,f^{(M)}$ is bounded in [0,1) i.e. $|f^M(t)|\leq C_f$, and its Wilson wavelet series* (2.3) *having* $(2^k,\hat{m})^{th}$ *partial sums* (2.4) *then, approximation $E^{(1)}_{2^k\hat{m}}(f)$ satisfies:*

$$E^{(1)}_{2^k,\hat{m}}(f)=min_{S_{2^k,\hat{m}}}||f-(S_{2^k,\hat{m}}f)||=O\Big(\frac{1}{2^{(M-1)(k+1)}}\Big),\ M>1.$$

Proof. If each of $f,f^{(1)},f^{(2)},..,f^{(M)}$ is bounded then following the proof of Theorem (3.1), we get

$$|c_{n,-2r}|\leq\frac{C_f}{\sqrt{2}.2^{M-1}.n^M.\pi^M},\ |c_{n,-2r-1}|\leq\frac{C_f}{\sqrt{2}2^{M-1}(n+1)^M\pi^M}$$

$$\begin{aligned}
&f(t)-(S_{2^k-1,\hat{m}}f)(t)\\
&=\sum_{n=1}^{\infty}\sum_{m=-2r-1}^{-2r}c_{n,m}\psi_{n,m}(t)-\sum_{n=1}^{2^k-1}\sum_{m=-2r-1}^{-2r}c_{n,m}\psi_{n,m}(t)\\
&=\Big(\sum_{n=1}^{2^k-1}+\sum_{n=2^k}^{\infty}\Big)\sum_{m=-2r-1}^{-2r}c_{n,m}\psi_{n,m}(t)-\sum_{n=1}^{2^k-1}\sum_{m=-2r-1}^{-2r}c_{n,m}\psi_{n,m}(t)\\
&=\sum_{n=1}^{2^k-1}\sum_{m=-2r-1}^{-2r}c_{n,m}\psi_{n,m}(t)+\sum_{n=2^k}^{\infty}\sum_{m=-2r-1}^{-2r}c_{n,m}\psi_{n,m}(t)\\
&\quad-\sum_{n=1}^{2^k-1}\sum_{m=-2r-1}^{-2r}c_{n,m}\psi_{n,m}(t)\\
&=\sum_{n=2^k}^{\infty}\sum_{m=-2r-1}^{-2r}c_{n,m}\psi_{n,m}(t).
\end{aligned}$$

On applying the Minkowski's inequality and Cauchy integral test we get,

$$\begin{aligned}
&||f-(S_{2^k,\hat{m}}f)||_2\\
&\leq\sum_{n=2^k}^{\infty}\sum_{m=-2r-1}^{-2r}|c_{n,m}|||\psi_{n,m}||_2\\
&\leq\frac{C_f}{\sqrt{2}2^{M-1}\pi^M}\sum_{n=2^k}^{\infty}(\frac{1}{n^M}+\frac{1}{(n+1)^M})
\end{aligned}$$

$$\leq \frac{C_f}{\sqrt{2}2^{M-1}\pi^M}\Big[\frac{1}{(2^k)^M}+\int_{2^k}^{\infty}\frac{dn}{n^M}+\frac{1}{(2^k+1)^M}+\int_{2^k}^{\infty}\frac{dn}{(n+1)^M}\Big]$$
$$\leq \frac{C_f}{\sqrt{2}2^{M-1}\pi^M}\Big[1+\frac{1}{M-1}\Big]\Big[\frac{2}{(2^k)^{M-1}}+\frac{2}{(2^k+1)^{M-1}}\Big]$$
$$\leq \frac{4C_f}{\sqrt{2}2^{M-1}\pi^M}\Big(1+\frac{1}{M-1}\Big)\frac{1}{(2^k)^{M-1}},\ (2^k+1)\geq(2^k)$$
$$\leq \frac{4C_f}{\sqrt{2}\pi^M}\Big(1+\frac{1}{M-1}\Big)\frac{1}{2^{(M-1)(1+k)}},\ \text{for } M>1$$
$$= A\Big(\frac{1}{2^{(M-1)(k+1)}}\Big),\quad A=\Big(\frac{4C_f}{\sqrt{2}\pi^M}\Big(1+\frac{1}{(M-1)}\Big)\Big)$$
$$= O\Big(\frac{1}{2^{(M-1)(k+1)}}\Big).$$

Hence, $E^{(1)}_{2^k,\hat{m}}(f)=min_{S_{2^k,\hat{m}}}||f-(S_{2^k,\hat{m}}f)||$

$$= O\Big(\frac{1}{2^{(M-1)(k+1)}}\Big),\ M>1.$$

□

Theorem 4.2. *If each of $f, f^{(1)}, f^{(2)}, .., f^{(M)}$ belongs to $H^{\alpha}[0,1)$ class, $E^{(2)}_{2^k,\hat{m}}$ satisfies*

$$E^{(2)}_{2^k,\hat{m}}(f)=min_{S_{2^k,\hat{m}}}||f-(S_{2^k,\hat{m}}f)||_2=O\Big(\frac{1}{2^{k(M+\alpha)}}\Big).$$

Proof. Following the proof of Theorem 3.2 we get,

$$|c_{n,-2r}|\leq\frac{\sqrt{2}C_f}{2^M\pi^M n^{M+\alpha+1}}\quad,\qquad |c_{n,-2r-1}|\leq\frac{\sqrt{2}C_f}{2^M(n+1)^M\pi^M n^{\alpha+1}}.$$

Applying the Minkowski's inequality and Cauchy integral test we get,

$$||f-(S_{2^k,\hat{m}}f)||_2\leq\sum_{n=2^k}^{\infty}\sum_{m=-2r-1}^{-2r}|c_{n,m}|||\psi_{n,m}||_2$$
$$=\sum_{n=2^k}^{\infty}(|c_{n,-2r-1}|)+|c_{n,-2r}|)$$
$$\leq\frac{2\sqrt{2}C_f}{2^M\pi^{(M)}}\sum_{n=2^k}^{\infty}\frac{1}{n^{(M+\alpha+1)}}\ ,\ (n+1)\geq n$$
$$=\frac{2\sqrt{2}C_f}{2^M\pi^{(M)}}\left[\frac{1}{(2^k)^{(M+\alpha+1)}}+\int_{2^k}^{\infty}\frac{dn}{n^{(M+\alpha+1)}}\right]$$
$$\leq\frac{4\sqrt{2}C_f}{2^M\pi^{(M)}}\Big(1+\frac{1}{M+\alpha}\Big)\frac{1}{(2^k)^{(M+\alpha)}}$$

$$\leq A_f \times \frac{1}{2^{k(M+\alpha)}},\ A_f = \Big(\frac{4\sqrt{2}C_f}{2^M\pi^M}\Big(1+\frac{1}{(M+\alpha)}\Big)\Big).$$

Hence, $E^{(2)}_{2^k,\hat{m}}(f) = min_{S_{2^k,\hat{m}}}||f-(S_{2^k,\hat{m}}f)||_2 = O\Big(\frac{1}{2^{k(M+\alpha)}}\Big).$

□

Theorem 4.3. *If M^{th} derivative of solution f of Fredholm Integral equation is bounded i.e. $|f^p(t)| \leq C_f > 0,\ p = 0,1,2,..,M$ and its Wilson Wavelet series (2.3) having $(2^k,\hat{m})^{th}$ partial sums $(S_{2^k,\hat{m}}f)(t)$ (2.4) then the moduli of continuity of $f - S_{2^k,\hat{m}}(f)$ satisfies:*

$$W^{(1)}(f - S_{2^k,\hat{m}}(f), \frac{1}{2^k}) = \sup_{0<h\leq\frac{1}{2^k}} ||(f - S_{2^k,\hat{m}}(f))(\cdot + h) - (f - S_{2^k,\hat{m}}(f)(\cdot))||_2$$
$$= O\Big(\frac{1}{2^{(M-1)(\alpha+1)}}\Big), M > 1.$$

Proof. Following the proof of Theorem (4.1),

$$||f - S_{2^k,\hat{m}}(f)||_2 = O\Big(\frac{1}{2^{(M-1)(k+1)}}\Big),\ M > 1\ .$$

Then,

$$\begin{aligned} W^{(1)}(f - S_{2^k,\hat{m}}(f), \frac{1}{2^k}) &= \sup_{0<h\leq\frac{1}{2^k}} ||(f - S_{2^k,\hat{m}}(f))(t + h) - (f - S_{2^k,\hat{m}}(f)(t))||_2 \\ &\leq ||f - S_{2^k,\hat{m}}(f)||_2 + ||f - S_{2^k,\hat{m}}(f)||_2 \\ &= 2||f - S_{2^k,\hat{m}}||_2 = 2.E^{(1)}_{2^k,\hat{m}}(f) \\ &= 2.O\Big(\frac{1}{2^{(M-1)(k+1)}}\Big) = O\Big(\frac{1}{2^{(M-1)(k+1)}}\Big), M > 1\ . \end{aligned}$$

□

Theorem 4.4. *If the solution function f of Fredholm Integral equation is such that its M^{th} derivative, i.e, $f^{(M)} \in H^\alpha[0,1)$ then*

$$W^{(2)}(f - S_{2^k,\hat{m}}(f), \frac{1}{2^k}) = O\Big(\frac{1}{2^{k(M+\alpha)}}\Big).$$

Proof. Following the proof of Theorem (4.2) and Theorem (4.3),

$$||f - S_{2^k,\hat{m}}(f)||_2 = O\Big(\frac{1}{2^{k(M+\alpha)}}\Big).$$
$$\text{and}\ \ W^{(2)}(f - S_{2^k,\hat{m}}(f), \frac{1}{2^k}) = O\Big(\frac{1}{2^{k(M+\alpha)}}\Big)\ .$$

□

5. A method to solve Fredholm integral equations

Consider the Fredholm Integral Equations of second kind as:

$$f(x) = g(x) + a + \int_0^1 K(x,t)f(t)dt \ \ , \ 0 \leq x, t < 1 \ . \tag{5.1}$$

Approximating $f(x), g(x), a, K(x,t)$ using Wilson wavelet as:

$$f(x) = C^T\psi(x) \ , \ g(x) = d^T\psi(x) \ , \ a = b^T\psi(x) \ , \ K(x,t) = \psi^T(x)K\psi(t) \tag{5.2}$$

where C and $\psi(x)$ are defined similarly to Eq. 2.5 and Eq. 2.6.
Also K is a $2^{k+1} \times 2^{k+1}$ matrix, where the elements of K are as follows :

$$\int_0^1 \int_0^1 \psi_{ni}(x)\psi_{lj}(t)K(x,t)dtdx,$$

$$n = 0, 1, \ldots, 2^k - 1, \ i = -1, 0, \ l = 0, 1, \ldots, 2^k - 1, \ j = -1, 0.$$

Moreover, each of d and b is $2^{k+1} \times 1$ matrix that is defined similarly to Eq. (2.5) and the elements of d can be obtained by:

$$\int_0^1 g(x)\psi_{n,m}(x)dx \ , \ n = 0, \ldots, 2^k - 1, \ m = -1, 0.$$

Then by replacing the values of Eq.(6.2) in Eq.(6.1) ,

$$C^T\psi(x) = d^T\psi(x) + b^T\psi(x) + \int_0^1 \psi(x)^T K\psi(t)\psi(t)^T C dt \ .$$

Orthonormality of Wilson Wavelets implies that :

$$\psi(x)^T C = \psi(x)^T d + \psi(x)^T b + \psi(x)^T KC \ . \tag{5.3}$$

Multiplying above Eq. (5.3) by $\psi(x)$ and integrating on [0,1) , C is obtained as:

$$C = (I - K)^{-1}(d + b), \text{ where I is identity matrix.}$$

Putting the value of C in $f(x) = C^T\psi(x)$, the solution of Fredholm integral Eq. 5.1 is obtained by this process. For details, one can refer to [6].

Example 5.1. Consider the Fredholm integral equation :

$$f(x) = cos(4\pi x) + 1 + \int_0^1 tf(t)dt \qquad 0 \leq x, t < 1. \tag{5.4}$$

Its exact solution is $f(x) = cos(4\pi x) + 2$ and belongs to $H^\alpha[0,1)$. Following the algorithm discussed above, $g(x) = cos(4\pi x), a = 1, K(x,t) = t$.

$$\text{Let} \quad f(x) = C^T\psi(x), \quad g(x) = d^T\psi(x), \quad a = b^T\psi(x)$$
$$\text{and} \ \ K(x,t) = \psi(x)^T K\psi(t).$$

$$\psi(x)^T = \sqrt{2} \begin{bmatrix} sin(2\pi x) & \frac{1}{\sqrt{2}} & sin(4\pi x) & cos(2\pi x) & sin(6\pi x) \\ cos(4\pi x) & sin(8\pi x) & cos(6\pi x) & & \end{bmatrix}$$

$$d^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix} \quad , \quad a^T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(d+a)^T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0.70710678 & 0 & 0 \end{bmatrix}.$$

Now,

$$C^T = [(I-K)^{-1}(d+a)]^T = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0.70710678 & 0 & 0 \end{bmatrix},$$

$$f(x) = C^T\psi(x) = 2 + cos(4\pi x).$$

The solution of Eq. (5.4) obtained by Wilson wavelet coincides with its exact solution as depicted in Figure 1.

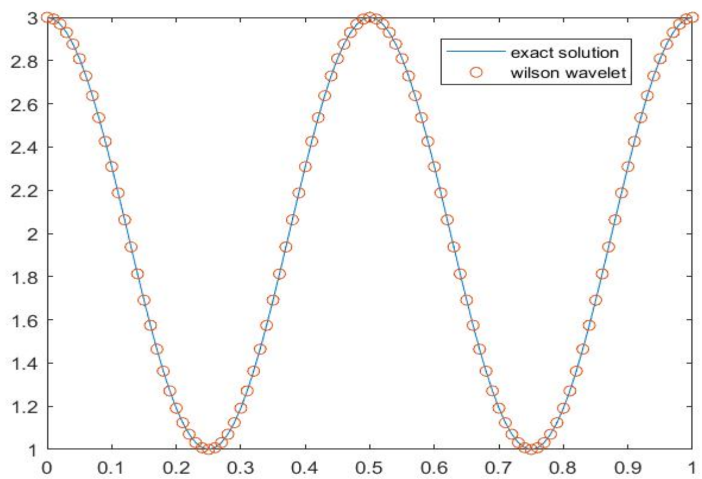

Figure 1. The graph represents the exact and the approximated solution of Eq. (5.4).

Example 5.2.

$$f(x) = x + \int_0^1 (tf(t)+2)dt \qquad 0 \le x, t < 1. \tag{5.5}$$

This is the Fredholm Integral Equation and its exact solution is $f(x) = x + \frac{14}{3}$.

$$\begin{aligned} f(x+t) - f(x) &= (x + t + \frac{14}{3}) - (x + \frac{14}{3}) \\ &= t \\ &\le |t|^\alpha,\ 0 \le \alpha < 1 \\ &= O(|t|^\alpha) \end{aligned}$$

Hence, f(t) $\in H^\alpha[0,1)$ and the outcomes flow from the aforementioned theorems. Following the procedure adopted in Example 5.1, the obtained solution is almost similiar to its exact solution. Table 1 presents the comparison between the exact and numerical solution and is plotted in Figure 2.

TABLE 1. The comparison of exact solution and wilson wavelet solution

x	Exact solution	Wilson wavelet solution	Absolute error
0	4.66666666	5.144241945	0.47757528
0.1	4.76666666	4.65809410	0.108572553
0.2	4.86666666	4.75803512	0.10863154
0.3	4.96666666	4.92174347	0.04492319
0.4	5.06666666	5.05437366	0.012293
0.5	5.16666666	5.14424194	0.02245806
0.6	5.26666666	5.23411022	0.03255644
0.7	5.36666666	5.36674041	0.00007375
0.8	5.46666666	5.40247294	0.06419372
0.9	5.56666666	5.63038979	0.06372313
1	5.66666666	5.14424194	0.52242472

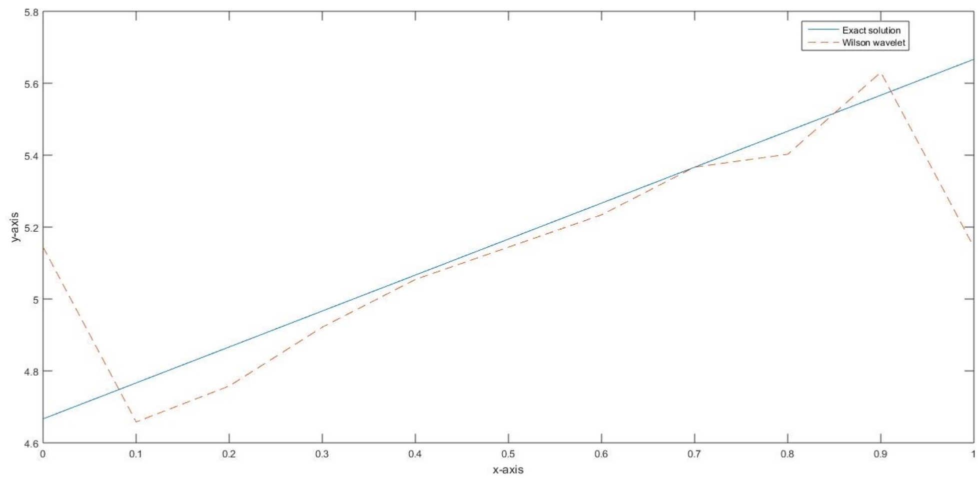

FIGURE 2. The graph represents the exact and the approximated solution of Eq. (5.5).

6. Conclusions

Following conclusions are derived from this research paper.
(1) In Theorem (3.1) and Theorem 3.2, the convergence analysis of the solutions of Fredholm integral equation have been studied by Wilson wavelet method.

(2) The approximations of solutions of these equations by their partial sums of Wilson wavelet series have been estimated when solution $f \in H^{\alpha}[0,1)$ as well as $|f^{(M)}(t)| \leq C_f > 0, \forall\, t \in [0,1)$.
(3) In Theorems (4.1) and (4.2),

$$E^{(1)}_{2^k,\hat{m}}(f) = O\Big(\frac{1}{2^{(M-1)(k+1)}}\Big) \to 0 \ as\ k \to \infty$$

and

$$E^{(2)}_{2^k,\hat{m}}(f) = O\Big(\frac{1}{2^{k(M+\alpha)}}\Big) \to 0 \ as\ k \to \infty\ .$$

(4) In Theorems (4.3) and (4.4), the moduli of continuity has been estimated as following

$$W^{(1)}(f - S_{2^k,\hat{m}}(f), \frac{1}{2^k}) = O\Big(\frac{1}{2^{(M-1)(k+1)}}\Big) \to 0 \ as\ k \to \infty.$$

$$W^{(2)}(f - S_{2^k,\hat{m}}(f), \frac{1}{2^k}) = O\Big(\frac{1}{2^{k(M+\alpha)}}\Big) \to 0 \ as\ k \to \infty.$$

(5) From Theorems (4.1),(4.2),(4.3),(4.4), it is notable that

$$W^{(1)}\Big((f - S_{2^k,\hat{m}}(f), \frac{1}{2^k})\Big) \leq 2.E^{(1)}_{2^k,\hat{m}}(f)$$

and

$$W^{(2)}\Big((f - S_{2^k,\hat{m}}(f), \frac{1}{2^k})\Big) \leq 2.E^{(2)}_{2^k,\hat{m}}(f).$$

Hence, Moduli of continuity $W^{(1)}\Big((f - S_{2^k,\hat{m}}(f)\Big)$ and $W^{(2)}\Big((f - S_{2^k,\hat{m}}(f)\Big)$ are superior and finer than $E^{(1)}_{2^k,\hat{m}}(f)$ and $E^{(2)}_{2^k,\hat{m}}(f)$ respectively .
(6) The operational matrix of integration has been derived .
(7) A Wilson wavelet-based approach for solving Fredholm integral equations has been presented. The approach is predicated on reducing the system into a collection of algebraic equations.
(8) The exact solutions and the solution obtained by proposed method based on Wilson wavelet method of Fredholm integral equations are almost identical. This illustrates the applicability and accuracy of proposed technique to solve Fredholm integral equations.

Acknowledgements

Shyam Lal, one of the authors, is thankful to DST - CIMS for encouragement to this work. Upasana Vats, one of the authors, is grateful to U.G.C (India) for providing financial assistance for her research work. The authors express their gratitude to the anonymous reviewer for dedicating his time to provide insightful feedback and recommendations that enhance the way this study is presented.

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