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SPECTRAL CUT-OFF REGULARIZATIONS FOR AN ILL-POSED NON-LINEAR ABSTRACT CAUCHY PROBLEM

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Abstract. We focus on nonlinear abstract Cauchy problem $u_t(t) + Au(t) = f(t, u(t))$, $0 \leq t < \tau$ with $u(\tau) = \phi$, where A is a densely defined positive self-adjoint unbounded operator acting on a Hilbert space. This problem is ill-posed in the sense that minor variations in the final value can lead to a significant deviations in the solution. Solution of the considered problem satisfies ill-posed integral equation. Spectral cut-off is used to obtain regularized approximations for the solution of the integral equation. Under the assumptions on the solution $u(\cdot)$ instated of conditions on the final value ϕ and the function $f(\cdot, u(\cdot))$ as in Jana and Nair (2019), we obtain error estimates for exact as well as noisy final value ϕ. Also we investigate stability analysis under appropriate parameter choice strategies. This work extends the work of Nam (2010) from the operator A having discrete spectrum to continuous spectrum. Estimates obtained in this work include the estimates obtained in Tuan (2010), Jana and Nair (2016, 2019) together with various other results present in the existing literature.

1. Introduction

Let H be a Hilbert space over real or complex field, and let $A : D(A) \subset H \to H$ be a densely defined positive self-adjoint unbounded operator. For $\tau > 0$ and $\phi \in H$, consider the problem of solving the *nonlinear Cauchy final value problem*, denoted briefly as nonlinear FVP,

$$u_t(t) + Au(t) = f(t, u(t)), \qquad 0 \leq t < \tau \tag{1.1}$$

$$u(\tau) = \phi, \tag{1.2}$$

where $f : [0, \tau] \times H \to H$ is a Borel measurable function such that for any continuous function $v : [0, \tau] \to H$, the function $s \mapsto f(s, v(s))$ belongs to $L^1([0, \tau]; H)$. It is well known that the above problem is ill-posed in the sense that a solution may not exist and if a solution exists, then small perturbations in ϕ can lead to large deviations in

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the solution. Some regularization method has to be employed to get stable approximate solutions. There are many regularization methods in the literature for Cauchy FVP.

Some of the authors used *quasi-reversibility method* which was introduced by Lattes and Lions (cf. [8]) for linear homogeneous FVP and it is based on an approximation of A by a family of bounded operators. This method was also used by other authors for solving linear homogeneous FVP (see e.g. Boussetila and Rebbani [1], Miller [9], Showlter [14]) and linear nonhomogeneous FVP (see e.g. Jana and Nair [6]) and nonlinear nonhomogeneous FVP (Trong and Tuan [16], Tuan, Trong and Quan [17]).

Another method is *quasi-boundary value method* which is based on considering a perturbation in final value (see e.g. Clark and Oppenheimer [2], Denche and Bessila [3], Denche and Djezzar [4] for homogeneous FVP).

Some of the authors used *spectral cut-off regularization method* which is based on truncation of the spectral representation of certain functions of the operator A. This method was used by many authors for linear homogeneous FVP (see e.g. Tuan [15]), linear nonhomogeneous FVP (see e.g., Tuan and Trong [18], Jana and Nair [5], Nair [11]) and nonlinear nonhomogeneous FVP (see e.g., Jana and Nair [7] and Nam[10]).

In this paper, we shall consider *spectral cut-off regularization* method for nonlinear nonhomogeneous FVP (1.1)-(1.2). If $u(\cdot)$ is a solution of the considered problem (1.1)-(1.2), then it satisfies integral equation

$$u(t) = e^{(\tau-t)A}\Big(\phi - \int_t^\tau e^{-(\tau-s)A} f(s,u(s))ds\Big), \quad 0 \le t \le \tau. \tag{1.3}$$

Here, for any $t > 0$, e^{tA} is the operator defined by

$$e^{tA} := \int_0^\infty e^{t\lambda} dE_\lambda,$$

where $\{E_\lambda : \lambda \ge 0\}$ is the resolution of identity of A. Since $e^{(\tau-t)A}$ is an unbounded operator for $0 \le t < \tau$, it is clear from (1.3) that the dependence of $u(\cdot)$ on ϕ is not continuous. To obtain a stable approximation for $u(\cdot)$, some regulation method has to be employed. For this purpose, we consider $u^\alpha(t,\phi)$ as regularized solution which is the solution (whenever exists) of the well-posed integral equation, obtained by truncating the spectral representation of the operator $e^{(\tau-t)A}$ at the level α,

$$\begin{aligned} u^\alpha(t,\phi) &= e^{(\tau-t)A}\chi_{[0,\alpha]}(A)\Big(\phi - \int_t^\tau e^{-(\tau-s)A} f(s,u^\alpha(s,\phi))ds\Big) \\ &= e^{(\tau-t)A}\chi_{[0,\alpha]}(A)\phi - \int_t^\tau e^{(s-t)A}\chi_{[0,\alpha]}(A) f(s,u^\alpha(s,\phi))ds. \end{aligned}$$

By spectral representation, the above integral equation can be written as

$$u^\alpha(t,\phi) = \int_0^\alpha e^{(\tau-t)\lambda} dE_\lambda \phi - \int_t^\tau \int_0^\alpha e^{(s-t)\lambda} dE_\lambda f(s,u^\alpha(s,\phi))ds. \tag{1.4}$$

In [7], it is shown that for each $\alpha > 0$ and $\phi \in H$, there exists a unique function $u^\alpha(\cdot,\phi)$ satisfying the integral equation (1.4), provided $f(\cdot,\cdot)$ satisfies the following conditions:

(i) the function

$$s \mapsto f(s,\varphi) \text{ belongs to } L^1([0,\tau];H) \text{ for some } \varphi \in H \text{ and} \tag{1.5}$$

(ii) there exists $\kappa > 0$ such that

$$\|f(s,\varphi_1) - f(s,\varphi_2)\| \le \kappa \|\varphi_1 - \varphi_2\| \tag{1.6}$$

for all $\varphi_1, \varphi_2 \in H$ and for all $s \in [0,\tau]$.

The conditions (1.5)-(1.6) imply that the function $s \mapsto f(s, v(s))$ belongs to $L^1([0,\tau]; H)$ for any continuous function $v : [0,\tau] \to H$ (see Lemma 3.1 in [7]).

In [7], Jana and Nair derived error estimates under conditions on the final value ϕ and the function $f(\cdot, u(\cdot))$. In this paper, we derive estimates under the assumptions on the solution $u(\cdot)$ itself. In fact, we derive error estimates under the assumption that $u(\cdot)$ satisfies

$$\int_0^\infty h^2(\lambda) e^{2\lambda t} d\|E_\lambda u(t)\|^2 \le \rho_h^2 \quad \forall t \in [0,\tau], \tag{1.7}$$

for some $\rho_h > 0$, where $h : (0,\infty) \to (0,\infty)$ is a monotonically increasing continuous or piecewise continuous function. We have shown that

$$\|u(t) - u^\alpha(t)\| \le \rho_h e^{\tau\kappa} \, \frac{e^{-t\alpha}}{h(\alpha)}, \quad 0 \le t < \tau.$$

To get error estimate under a weaker condition on $u(\cdot)$ than (1.7), we assume that

$$\int_0^\infty h^2(\lambda) e^{2\lambda \min\{t,\gamma\}} d\|E_\lambda u(t)\|^2 \le \rho_h^2 \quad \forall t \in [0,\tau], \tag{1.8}$$

for some $\rho_h > 0$ and $\gamma \in (0,\tau)$. In this case, we divide the interval $[0,\tau]$ into subintervals $[t_n, t_{n+1}]$ for $n = 0,1,\cdots,n_0$, for some $n_0 \in \mathbb{N}$ in such a way that $t_0 = 0, t_1 = \gamma$ and $t_n \ge t_{n+1} - \gamma$ and modify the regularized solution as $\tilde{u}^\alpha : [0,\tau] \to H$ such that $\tilde{u}^\alpha(t) = u_n^\alpha(t, \phi_n)$ on $[t_n, t_{n+1})$ for $n = 0,1,\cdots,n_0$, together with $\tilde{u}^\alpha(\tau) = u_{n_0}^\alpha(\tau,\phi)$. Here $u_n^\alpha(\cdot,\phi_n)$ is the solution of the integral equation

$$u_n^\alpha(t,\phi_n) = \int_0^\alpha e^{(t_{n+1}-t)\lambda} dE_\lambda \phi_n - \int_t^{t_{n+1}} \int_0^\alpha e^{(s-t)\lambda} dE_\lambda f(s, u^\alpha(s,\phi_n)) ds, \quad t_n \le t \le t_{n+1}$$

with $\phi_n = u(t_{n+1})$ for $n = 0,1,\cdots,n_0$. We have shown that for $n = 0,1,\cdots,n_0$,

$$\|u(t) - \tilde{u}^\alpha(t)\| \le \rho_h e^{\tau\kappa} \, \frac{e^{-(t+\gamma-t_{n+1})\alpha}}{h(\alpha)},$$

whenever $t_n \le t < t_{n+1}$.

We also derive error estimates when data ϕ is noisy with noise level $\varepsilon > 0$, that is, ϕ^ε is given in place of ϕ with

$$\|\phi - \phi^\varepsilon\| \le \varepsilon.$$

The estimates are derived under appropriate parameter choice strategies. Our results, in this paper, extend the results obtained in the recent work of Nam [10], which was in the setting with A having discrete spectrum. Also, our results include results obtained in Tuan [15], Jana and Nair [7], [5], as particular cases.

In Section 2, we give preliminaries result require for our analysis. In Section 3, we consider regularized solution and derive error estimates when final value ϕ is exact. For noisy final value ϕ, stability analysis is carried out under appropriate parameter choice strategies. We deduce many results in the literature as special cases.

2. Preliminaries

Let $A : D(A) \subset H \to H$ be a densely defined positive self-adjoint unbounded operator on the Hilbert space H. Recall from spectral theorem (cf. Yosida [19]) that for any piecewise continuous function $g : [0, \infty) \to \mathbb{R}$, the operator $g(A)$ is defined by

$$g(A)\varphi := \int_0^\infty g(\lambda) dE_\lambda \varphi, \quad \varphi \in D(g(A)),$$

where

$$D(g(A)) := \Big\{\varphi \in H : \int_0^\infty g(\lambda)^2 d\|E_\lambda \varphi\|^2 < \infty\Big\}.$$

In particular, for $t \geq 0$, the operator e^{tA} is given by

$$e^{tA}\varphi = \int_0^\infty e^{\lambda t} dE_\lambda \varphi \quad \forall \varphi \in D(e^{tA})$$

where

$$D(e^{tA}) = \{\varphi \in H : \int_0^\infty e^{2\lambda t} d\|E_\lambda \varphi\|^2 < \infty\}.$$

Also, we can see that for $t \geq 0$

$$e^{-tA} = \int_0^\infty e^{-t\lambda} dE_\lambda,$$

where

$$D(e^{-tA}) := \{\varphi \in H : \int_0^\infty e^{-2\lambda t} d\|E_\lambda \varphi\|^2 < \infty\} = H.$$

Also, we note that $\{e^{-tA} : t \geq 0\}$ is a family of bounded linear operators on H which is a strongly continuous (or C_0) semigroup generated by $-A$ (cf. Pazy [12]), and $\|e^{-tA}\| \leq 1$ for all $t \geq 0$.

We shall use the following lemma which is a consequence of the well-known Gronwall's inequality (cf. Perko [13]).

Lemma 2.1. *Let $g, \beta \in C([0, \tau]; \mathbb{R})$ and satisfy*

$$g(t) \leq \beta(t) + \kappa \int_t^\tau g(s) ds \tag{2.1}$$

for some $\kappa > 0$, then for $t \in [0, \tau]$

$$g(t) \leq \beta(t) + \kappa \int_t^\tau e^{\kappa(s-t)}\, \beta(s) ds.$$

Moreover, if β is non-decreasing, then $g(t) \leq \beta(t)\, e^{(\tau-t)\kappa}$.

3. Main Results

3.1. Regularization and Error Analysis. Let $u(\cdot)$ be a solution (if exists) of the nonlinear FVP given by (1.1)-(1.2). Then, it satisfies following equation

$$u(t) = e^{(\tau-t)A}\Big(\phi - \int_t^\tau e^{-(\tau-s)A} f(s, u(s)) ds\Big), \quad 0 \leq t \leq \tau, \tag{3.1}$$

where $f : [0, \tau] \times H \to H$ is a Borel measurable function such that for any continuous function $v : [0, \tau] \to H$, the function $s \mapsto f(s, v(s))$ belongs to $L^1([0, \tau]; H)$.

Since $e^{(\tau-t)A}$ is an unbounded operator, it is clear from (3.1) that the dependence of $u(\cdot)$ on ϕ is not continuous. To obtain a stable approximation for $u(\cdot)$, some regulation method has to be employed. For this purpose, we define $u^\alpha(t,\phi)$ as the solution of the integral equation

$$u^\alpha(t,\phi) = \int_0^\alpha e^{(\tau-t)\lambda} dE_\lambda \phi - \int_t^\tau \int_0^\alpha e^{(s-t)\lambda} dE_\lambda f(s, u^\alpha(s,\phi))ds, \quad 0 \le t \le \tau. \tag{3.2}$$

Existence of the solution of the integral equation (3.2) is ensured when f satisfies conditions (1.5)-(1.6) (see Theorem 4.2 in Jana and Nair [7]).

3.1.1. *Error analysis with exact data.* To derive error estimates, we shall make use of the following lemma whose proof is given in Lemma 4.1 of [7].

Lemma 3.1. *Let $0 \le t \le s \le \tau$ and $\varphi \in H$. Then, for $\alpha > 0$*

$$\Big\| \int_0^\alpha e^{(s-t)\lambda} dE_\lambda \varphi \Big\| \le e^{(s-t)\alpha} \|\varphi\|.$$

Theorem 3.2. *Let $f : [0,\tau] \times H \to H$ be a Borel measurable function satisfying conditions (1.5)-(1.6). Let $u(\cdot)$ be the solution of (1.1)-(1.2) and $u^\alpha(\cdot) := u^\alpha(\cdot,\phi)$ be the solution of the integral equation (3.2).*

(i) *If $u(\cdot)$ satisfies (1.7), then for $t \in [0,\tau)$, $\lim_{\alpha\to\infty} u^\alpha(t) = u(t)$ and*

$$\|u(t) - u^\alpha(t)\| \le \rho_h e^{\tau\kappa} \frac{e^{-t\alpha}}{h(\alpha)}, \qquad 0 \le t < \tau.$$

(ii) *If $u(\cdot)$ satisfies (1.8), then*

$$\|u(t) - u^\alpha(t)\| \le \rho_h e^{\tau\kappa} \frac{e^{(\tau-t-\gamma)\alpha}}{h(\alpha)}, \qquad 0 \le t < \tau.$$

Proof. For $t \in [0,\tau)$,

$$u(t) - u^\alpha(t) = \chi_{(\alpha,\infty)}(A)u(t) + \Big(\chi_{[0,\alpha]}(A)u(t) - u^\alpha(t)\Big).$$

Therefore,

$$\|u(t) - u^\alpha(t)\| \le \|\chi_{(\alpha,\infty)}(A)u(t)\| + \|\chi_{[0,\alpha]}(A)u(t) - u^\alpha(t)\|. \tag{3.3}$$

Now,

$$\begin{aligned} \chi_{[0,\alpha]}(A)u(t) &= \chi_{[0,\alpha]}(A)e^{(\tau-t)A}\Big(\phi - \int_t^\tau e^{-(\tau-s)A} f(s,u(s))ds\Big) \\ &= e^{(\tau-t)A}\chi_{[0,\alpha]}(A)\phi - \int_t^\tau e^{(s-t)A}\chi_{[0,\alpha]}(A) f(s,u(s))ds \\ &= \int_0^\alpha e^{(\tau-t)\lambda} dE_\lambda \phi - \int_t^\tau \int_0^\alpha e^{(s-t)\lambda} dE_\lambda f(s,u(s))ds. \end{aligned}$$

Therefore

$$\chi_{[0,\alpha]}(A)u(t) - u^\alpha(t) = \int_t^\tau \int_0^\alpha e^{(s-t)\lambda} dE_\lambda \Big(f(s,u^\alpha(s)) - f(s,u(s))\Big)ds.$$

By using Lemma 3.1 and condition (1.6), we get

$$\begin{aligned}\|\chi_{[0,\alpha]}(A)u(t)-u^\alpha(t)\| &\le \int_t^\tau \Big\| \int_0^\alpha e^{(s-t)\lambda} dE_\lambda \Big(f(s,u^\alpha(s))-f(s,u(s))\Big)\Big\| ds\\ &\le \int_t^\tau e^{(s-t)\alpha}\|f(s,u^\alpha(s))-f(s,u(s))\|ds\\ &\le \int_t^\tau \kappa e^{(s-t)\alpha}\|u^\alpha(s)-u(s)\|ds\end{aligned}$$

i.e.,

$$\|\chi_{[0,\alpha]}(A)u(t)-u^\alpha(t)\| \le e^{-t\alpha}\kappa \int_t^\tau e^{s\alpha}\|u(s)-u^\alpha(s)\|ds. \tag{3.4}$$

From (3.3), by using (3.4), we get

$$e^{t\alpha}\|u(t)-u^\alpha(t)\| \le e^{t\alpha}\|\chi_{(\alpha,\infty)}(A)u(t)\| + \kappa \int_t^\tau e^{s\alpha}\|u(s)-u^\alpha(s)\|ds. \tag{3.5}$$

Now, by Lemma 2.1, we have

$$e^{t\alpha}\|u(t)-u^\alpha(t)\| \le e^{t\alpha}\|\chi_{(\alpha,\infty)}(A)u(t)\| + \kappa \int_t^\tau e^{(s-t)\kappa}e^{s\alpha}\|\chi_{(\alpha,\infty)}(A)u(s)\|ds$$

i.e.,

$$\|u(t)-u^\alpha(t)\| \le \|\chi_{(\alpha,\infty)}(A)u(t)\| + \kappa \int_t^\tau e^{(s-t)\kappa}e^{(s-t)\alpha}\|\chi_{(\alpha,\infty)}(A)u(s)\|ds. \tag{3.6}$$

(i) Let $u(\cdot)$ satisfies condition (1.7). Now, for $t\in[0,\tau]$

$$\begin{aligned}\|\chi_{(\alpha,\infty)}(A)u(t)\|^2 &= \int_\alpha^\infty d\|E_\lambda u(t)\|^2\\ &= \int_\alpha^\infty \frac{h^2(\lambda)e^{2t\lambda}}{h^2(\lambda)e^{2t\lambda}} d\|E_\lambda u(t)\|^2\\ &\le \frac{e^{-2t\alpha}}{h^2(\alpha)}\int_\alpha^\infty h^2(\lambda)e^{2t\lambda}d\|E_\lambda u(t)\|^2.\end{aligned}$$

Hence,

$$\|\chi_{(\alpha,\infty)}(A)u(t)\| \le \frac{e^{-t\alpha}}{h(\alpha)}\|h(A)e^{tA}\chi_{(\alpha,\infty)}(A)u(t)\| \le \rho_h \frac{e^{-t\alpha}}{h(\alpha)} \quad \forall t\in[0,\tau]. \tag{3.7}$$

Note that

$$\lim_{\alpha\to\infty} e^{t\alpha}\|\chi_{(\alpha,\infty)}(A)u(t)\| = 0 \quad \text{and} \quad e^{t\alpha}\|\chi_{(\alpha,\infty)}(A)u(t)\| \le \rho_h.$$

Now, by using *dominated convergence theorem*, from (3.6), we get $\lim_{\alpha\to\infty} u^\alpha(t) = u(t)$ for each $t\in[0,\tau)$. Now using (3.7), from (3.6), we have

$$\|u(t)-u^\alpha(t)\| \le \rho_h \frac{e^{-t\alpha}}{h(\alpha)}\Big(1+\kappa\int_t^\tau e^{(s-t)\kappa}ds\Big) = \rho_h e^{(\tau-t)\kappa}\frac{e^{-t\alpha}}{h(\alpha)} \le \rho_h e^{\tau\kappa}\frac{e^{-t\alpha}}{h(\alpha)}.$$

(ii) Let $u(\cdot)$ satisfies condition (1.8). For $0<\gamma<\tau$, (1.8) takes the form

$$\int_0^\infty h^2(\lambda)e^{2\gamma\lambda}d\|E_\lambda u(t)\|^2 \le \rho_h^2 \quad \forall t\in[\gamma,\tau],$$

and

$$\int_0^\infty h^2(\lambda)e^{2\lambda t}d\|E_\lambda u(t)\|^2 \le \rho_h^2 \quad \forall t \in [0,\gamma].$$

Now, for $t \in [\gamma, \tau]$

$$\begin{aligned}\|\chi_{(\alpha,\infty)}(A)u(t)\|^2 &= \int_\alpha^\infty \frac{h^2(\lambda)e^{2\lambda\gamma}}{h^2(\lambda)e^{2\lambda\gamma}}d\|E_\lambda u(t)\|^2 \\ &\le \frac{e^{-2\alpha\gamma}}{h^2(\alpha)}\int_\alpha^\infty h^2(\lambda)e^{2\lambda\gamma}d\|E_\lambda u(t)\|^2.\end{aligned}$$

i.e.,

$$e^{t\alpha}\|\chi_{(\alpha,\infty)}(A)u(t)\| \le \rho_h\frac{e^{(t-\gamma)\alpha}}{h(\alpha)} \le \rho_h\frac{e^{(\tau-\gamma)\alpha}}{h(\alpha)}. \tag{3.8}$$

Now by (3.7), we get

$$e^{t\alpha}\|\chi_{(\alpha,\infty)}(A)u(t)\| \le \frac{\rho_h}{h(\alpha)} \quad \forall t \in [0,\gamma].$$

Hence,

$$e^{t\alpha}\|\chi_{(\alpha,\infty)}(A)u(t)\| \le \rho_h\frac{e^{(\tau-\gamma)\alpha}}{h(\alpha)} \quad \forall t \in [0,\tau]. \tag{3.9}$$

By using (3.9), from (3.5), we get

$$e^{t\alpha}\|u(t)-u^\alpha(t)\| \le \rho_h\frac{e^{(\tau-\gamma)\alpha}}{h(\alpha)} + \kappa\int_t^\tau e^{s\alpha}\|u^\alpha(s)-u(s)\|ds.$$

By Lemma 2.1, we get

$$e^{t\alpha}\|u(t)-u^\alpha(t)\| \le \rho_h e^{(\tau-t)\kappa}\frac{e^{(\tau-\gamma)\alpha}}{h(\alpha)}.$$

Hence, we get the required results.

□

Remark 3.3. If $u(\cdot)$ satisfies condition (1.8) for $\gamma \in (0,\tau)$, estimate in Theorem 3.2 (ii), ensures $\lim_{\alpha\to\infty} u^\alpha(t) = u(t)$, $t \in (\tau-\gamma,\tau)$. If h is unbounded, $\lim_{\alpha\to\infty} u^\alpha(t) = u(t)$, $t \in [a,\tau)$ for some $a \in [0,\tau-\gamma]$. ♢

Corollary 3.4. Let $h : (0,\infty) \to (0,\infty)$ is a monotonically increasing piecewise continuous function. If $u(\cdot)$ satisfies condition

$$\int_0^\infty h^2(\lambda)e^{2\lambda\gamma}d\|E_\lambda u(t)\|^2 \le \rho_h^2 \quad \forall t \in [0,\tau], \tag{3.10}$$

for some $\rho_h > 0$ and $\gamma > 0$, then

$$\|u(t)-u^\alpha(t)\| \le e^{\tau\kappa}\rho_h\frac{e^{(\tau-\gamma-t)\alpha}}{h(\alpha)}, \quad 0 \le t < \tau.$$

Proof. For $0 < \gamma < \tau$, condition (3.10) implies that $u(\cdot)$ satisfies condition (1.8) and hence by Theorem 3.2(ii), we get required estimate. For $\gamma \ge \tau$, condition (3.10) implies that $u(\cdot)$ satisfies condition (1.7) for $h(\lambda)e^{(\gamma-\tau)\lambda}$ in place of $h(\lambda)$ and hence by Theorem 3.2(i), we get required estimate. □

Remark 3.5. In [7], we consider nonlinear FVP and obtain same convergence rate as in Theorem 3.2 (i) under the condition

$$\int_0^\infty h^2(\lambda)e^{2\lambda\tau}d\|E_\lambda\phi\|^2 < \infty \quad \text{and} \quad \int_0^\tau \Big\| \int_0^\infty h(\lambda)e^{s\lambda}dE_\lambda f(s,u(s))\Big\| ds < \infty \quad (3.11)$$

In [5], we consider liner nonhomogeneous FVP and same convergence rate in Theorem 3.2 (i) under the condition (3.11) where $f(t,u(t))$ replace by $f(t)$. The condition (3.11) imply condition (1.7). In fact, by using condition (3.11), from (3.1), we have

$$h(A)e^{tA}u(t) = h(A)e^{\tau A}\phi - \int_t^\tau h(A)e^{sA}f(s,u(s)ds.$$

Now, taking norm, we get

$$\|h(A)e^{tA}u(t)\| \leq \|h(A)e^{\tau A}\phi\| + \int_0^\tau \|h(A)e^{sA}f(s,u(s)\|ds < \infty \quad \forall t \in [0,\tau].$$

$\diamondsuit$

Remark 3.6. Observe that estimate in Theorem 3.2(ii), does not ensure convergence on $[0,\tau-\gamma]$ when h is a bounded function. If h is such that $\frac{e^{(\tau-\gamma-t)\alpha}}{h(\alpha)} \to 0$ does not hold, for all $t \in [0,a]$, for some $a \in (0,\tau)$, then the estimate in Theorem 3.2(ii), does not ensure convergence on $[0,\tau)$. In such situation, we divide the interval $[0,\tau]$ into subintervals $[t_n, t_{n+1}]$ for $n = 0,1,\cdots,n_0$, for some $n_0 \in \mathbb{N}$ in such a way that $t_n \geq t_{n+1} - \gamma$ and modify the regularized solution and obtain estimates which ensure convergence in the following theorem. One can chose n_0, $t_0, t_1, \cdots, t_{n_0+1}$ as follows: Let $0 < \theta < \gamma < \tau$ and $n_0 = \inf\{n : (n+1)\gamma > \tau + \theta\}$. Let $t_0 = 0, t_1 = \gamma$, $t_n = \tau - (n_0 + 1 - n)\Big(\gamma - \frac{\theta}{n_0-1}\Big)$ for $n = 2,3,\cdots,n_0$ and $t_{n_0+1} = \tau$. $\diamondsuit$

Theorem 3.7. *Let $f : [0,\tau] \times H \to H$ be a Borel measurable function satisfying conditions (1.5)-(1.6). Let $u(\cdot)$ be the solution of (1.1)-(1.2) and satisfies condition (1.8). Let n_0, $t_0, t_1, \cdots, t_{n_0+1}$ be as in Remark 3.6. Let $\tilde{u}^\alpha : [0,\tau] \to H$ be such that $\tilde{u}^\alpha(t) = u_n^\alpha(t,\phi^n)$ on $[t_n, t_{n+1})$ for $n = 0,1,\cdots,n_0$, together with $\tilde{u}^\alpha(\tau) = u^\alpha_{n_0}(\tau,\phi)$, where $u_n^\alpha(\cdot,\phi^n)$ is the solution of the integral equation*

$$u_n^\alpha(t,\phi^n) = \int_0^\alpha e^{(t_{n+1}-t)\lambda}dE_\lambda\phi^n - \int_t^{t_{n+1}}\int_0^\alpha e^{(s-t)\lambda}dE_\lambda f(s,u^\alpha(s,\phi^n))ds, \quad t_n \leq t \leq t_{n+1}$$

and $\phi^n = u(t_{n+1})$ for $n = n_0,\cdots,1,0$. Then,

$$\|u(t) - \tilde{u}^\alpha(t)\| \leq \rho_h e^{\tau\kappa}\frac{e^{(t_{n+1}-t-\gamma)\alpha}}{h(\alpha)}, \quad t_n \leq t < t_{n+1} \quad (3.12)$$

for $n = n_0,\cdots,1,0$.

Proof. Since $u(\cdot)$ is a solution of (1.1)-(1.2), it satisfies following equation

$$u(t) = e^{(t_{n+1}-t)A}\Big(u(t_{n+1}) - \int_t^{t_{n+1}} e^{-(t_{n+1}-s)A}f(s,u(s))ds\Big), \quad t_n \leq t \leq t_{n+1}$$

for $n = n_0,\cdots,1,0$.

Now, by similar argument as in the proof of Corollary 3.4, we get estimate (3.12) for $n = n_0,\cdots,2,1$. Again, by similar argument as in the proof of Theorem 3.2(i), we get required estimate on $t \in [t_0,t_1)$.

$\square$

3.1.2. *Error analysis with noisy data.* Let the given data is noisy with noise level $\epsilon > 0$, that is, ϕ_ε is given instead of ϕ with

$$\|\phi - \phi_\varepsilon\| \leq \varepsilon.$$

Let $u(\cdot)$ be the solution of (1.1)-(1.2) and $u_\varepsilon^\alpha(\cdot) := u^\alpha(\cdot, \phi_\varepsilon)$ be the solution of the integral equation (3.2) for $\phi = \phi_\varepsilon$, that is,

$$u_\varepsilon^\alpha(t) = \int_0^\alpha e^{(\tau-t)\lambda} dE_\lambda \phi - \int_t^\tau \int_0^\alpha e^{(s-t)\lambda} dE_\lambda f(s, u_\varepsilon^\alpha(s)) ds, \quad 0 \leq t \leq \tau$$

for $\alpha > 0$. To derive error estimates with noisy data, we shall make use of the following result whose proof is similar to the proof of the Theorem 4.4 in [7].

Theorem 3.8. *Let $\phi_1, \phi_2 \in H$ and $0 \leq t_n < t_{n+1} \leq \tau$. Let $f : [0,\tau] \times H \to H$ be a Borel measurable function satisfying conditions (1.5)-(1.6). Let $u_n^\alpha(\cdot, \phi_1)$ and $u_n^\alpha(\cdot, \phi_2)$ be the solution of the integral equation*

$$u_n^\alpha(t, \phi) = \int_0^\alpha e^{(t_{n+1}-t)\lambda} dE_\lambda \phi - \int_t^{t_{n+1}} \int_0^\alpha e^{(s-t)\lambda} dE_\lambda f(s, u_n^\alpha(s, \phi)) ds, \quad t_n \leq t \leq t_{n+1}$$

corresponding to ϕ_1 and ϕ_2 respectively. Then,

$$\|u_n^\alpha(t, \phi_1) - u_n^\alpha(t, \phi_2)\| \leq e^{\tau\kappa} e^{(t_{n+1}-t)\alpha} \|\phi_1 - \phi_2\|, \qquad t_n \leq t \leq t_{n+1}.$$

Theorem 3.9. *Let $\phi \in H$ and $f : [0,\tau] \times H \to H$ be a Borel measurable function satisfying conditions (1.5)-(1.6). Let $\tilde{\rho_h} = \max\{1, \rho_h\} e^{\tau\kappa}$.*

(i) *If $u(\cdot)$ satisfies condition (1.7), then we get*

$$\|u(t) - u_\varepsilon^\alpha(t)\| \leq \tilde{\rho_h} \Big(e^{(\tau-t)\alpha} \varepsilon + \frac{e^{-t\alpha}}{h(\alpha)} \Big), \qquad 0 \leq t < \tau.$$

(ii) *If $u(\cdot)$ satisfies condition (1.8), then we get*

$$\|u(t) - u_\varepsilon^\alpha(t)\| \leq \tilde{\rho_h} \Big(e^{(\tau-t)\alpha} \varepsilon + \frac{e^{(\tau-t-\gamma)\alpha}}{h(\alpha)} \Big), \qquad 0 \leq t < \tau.$$

Proof. Let $t \in [0, \tau)$. By using Theorem 3.8, we get

$$\begin{aligned} \|u(t) - u_\varepsilon^\alpha(t)\| &\leq \|u(t) - u^\alpha(t)\| + \|u^\alpha(t) - u_\varepsilon^\alpha(t)\| \\ &\leq \|u(t) - u^\alpha(t)\| + e^{\tau\kappa} e^{(\tau-t)\alpha} \|\phi - \phi_\varepsilon\| \end{aligned}$$

Since $\|\phi - \phi_\varepsilon\| \leq \varepsilon$, therefore

$$\|u(t) - u_\varepsilon^\alpha(t)\| \leq \|u(t) - u^\alpha(t)\| + e^{\tau\kappa} e^{(\tau-t)\alpha} \varepsilon. \tag{3.13}$$

(i) The required estimate follows from (3.13) by using Theorem 3.2 (i).
(ii) The required estimate follows from (3.13) by using Theorem 3.2 (ii).

□

Corollary 3.10. Let $f : [0,\tau] \times H \to H$ be a Borel measurable function satisfies conditions (1.5)-(1.6). The following results hold.

(i) Let $u(\cdot)$ satisfies the condition

$$\int_0^\infty e^{2\lambda t} d\|E_\lambda u(t)\|^2 \leq \rho_h^2 \quad \forall t \in [0, \tau].$$

Then

$$\|u(t) - u_\varepsilon^\alpha(t)\| \leq \tilde{\rho_h} \Big(e^{(\tau-t)\alpha} \varepsilon + e^{-t\alpha} \Big), \quad 0 \leq t < \tau.$$

(ii) Let $u(\cdot)$ satisfies the condition

$$\int_0^\infty \lambda^{2p} e^{2\lambda t} d\|E_\lambda u(t)\|^2 \leq \rho_h^2 \quad \forall t \in [0,\tau].$$

Then

$$\|u(t) - u_\varepsilon^\alpha(t)\| \leq \tilde{\rho_h}\Big(e^{(\tau-t)\alpha}\varepsilon + \frac{e^{-t\alpha}}{\alpha^p}\Big), \quad 0 \leq t < \tau.$$

(iii) Let $u(\cdot)$ satisfies the condition

$$\int_0^\infty e^{2\lambda(q+t)} d\|E_\lambda u(t)\|^2 \leq \rho_h^2, \quad \forall t \in [0,\tau].$$

Then

$$\|u(t) - u_\varepsilon^\alpha(t)\| \leq \tilde{\rho_h}\Big(e^{(\tau-t)\alpha}\varepsilon + e^{-(q+t)\alpha}\Big), \quad 0 \leq t < \tau.$$

(iv) Let $u(\cdot)$ satisfies the condition

$$\int_0^\infty e^{2\lambda\gamma} d\|E_\lambda u(t)\|^2 \leq \rho_h^2 \quad \forall t \in [0,\tau],$$

where $\gamma > 0$. Then

$$\|u(t) - u_\varepsilon^\alpha(t)\| \leq \tilde{\rho_h}\Big(e^{(\tau-t)\alpha}\varepsilon + e^{(\tau-t-\gamma)\alpha}\Big), \quad 0 \leq t < \tau.$$

Proof. The estimates in (i),(ii) and (iii) follow from Theorem 3.2(i) by taking (i) $h(\lambda) = 1$, (ii) $h(\lambda) = \lambda^p$ and (iii) $h(\lambda) = e^{q\lambda}$, respectively and the estimates in (iv) follows by using Corollary 3.4 with $h(\lambda) = 1$ in (3.10) and Theorem 3.8. □

3.1.3. *Error analysis with parameter choice strategy.* From Theorem 3.9, we have estimates

(i)

$$\|u(t) - u_\varepsilon^\alpha(t)\| \leq \tilde{\rho_h}\Big(e^{(\tau-t)\alpha}\varepsilon + \frac{e^{-t\alpha}}{h(\alpha)}\Big)$$

(ii)

$$\|u(t) - u_\varepsilon^\alpha(t)\| \leq \tilde{\rho_h}\Big(e^{(\tau-t)\alpha}\varepsilon + \frac{e^{(\tau-t-\gamma)\alpha}}{h(\alpha)}\Big).$$

Note that for fixed $\varepsilon > 0$ and $t \in [0,\tau)$

$$e^{(\tau-t)\alpha}\varepsilon + \frac{e^{-t\alpha}}{h(\alpha)} \to \infty \quad \text{and} \quad e^{(\tau-t)\alpha}\varepsilon + \frac{e^{(\tau-t-\gamma)\alpha}}{h(\alpha)} \to \infty \quad \text{as} \quad \alpha \to \infty.$$

We would like to find $\alpha = \alpha^\varepsilon$ depending on ε such that

$$e^{(\tau-t)\alpha^\varepsilon}\varepsilon + \frac{e^{-t\alpha^\varepsilon}}{h(\alpha^\varepsilon)} \to 0 \quad \text{and} \quad e^{(\tau-t)\alpha^\varepsilon}\varepsilon + \frac{e^{(\tau-t-\gamma)\alpha^\varepsilon}}{h(\alpha^\varepsilon)} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

We shall make use of the following lemma.

Lemma 3.11. *Let $h : (0,\infty) \to (0,\infty)$ be a piecewise continuous, monotonically increasing function and $l > 0$. For $t \in [0,l)$ and $\alpha > 0$, let*

$$\psi_{l,t}^\varepsilon(\alpha) = e^{(l-t)\alpha}\varepsilon + \frac{e^{-t\alpha}}{h(\alpha)} \ \textit{and}\ \beta_{l,t,\gamma}^\varepsilon(\alpha) = e^{(l-t)\alpha}\varepsilon + \frac{e^{(l-\gamma-t)\alpha}}{h(\alpha)}$$

for some $\gamma > 0$. *Then there exists* $\alpha^{\varepsilon}_{l,t} > 0$ *and* $\alpha^{\varepsilon}_{l,t,\gamma} > 0$ *such that*

$$\psi^{\varepsilon}_{l,t}(\alpha^{\varepsilon}_{l,t}) \leq 2 \inf_{\alpha>0} \psi^{\varepsilon}_{l,t}(\alpha) \quad \textit{and} \quad \beta^{\varepsilon}_{l,t,\gamma}(\alpha^{\varepsilon}_{l,t,\gamma}) \leq 2 \inf_{\alpha>0} \beta^{\varepsilon}_{l,t,\gamma}(\alpha).$$

Also, the following results hold.

(i) *For* $t \in (0, l)$, $\psi^{\varepsilon}_{l,t}(\alpha^{\varepsilon}_{l,t}) \to 0$ *as* $\varepsilon \to 0$.
Further, if h *is unbounded, then* $\psi^{\varepsilon}_{l,t}(\alpha^{\varepsilon}_{l,t}) \to 0$ *as* $\varepsilon \to 0$ *for* $t = 0$.

(ii) *For* $\gamma \in (0, \tau]$ *and* $t \in (\max\{l - \gamma, 0\}, l)$, $\beta^{\varepsilon}_{l,t,\gamma}(\alpha^{\varepsilon}_{l,t,\gamma}) \to 0$ *as* $\varepsilon \to 0$.
Further, if $\gamma > l$ *or* g *is unbounded, then* $\beta^{\varepsilon}_{l,t,\gamma}(\alpha^{\varepsilon}_{l,t,\gamma}) \to 0$ *as* $\varepsilon \to 0$ *for* $t \in [\max\{l - \gamma, 0\}, l)$.

Proof. Since $\psi^{\varepsilon}_{l,t}(\alpha) \geq \varepsilon$ and $\beta^{\varepsilon}_{l,t,\gamma}(\alpha) \geq \varepsilon$, $\inf_{\alpha>0} \psi^{\varepsilon}_{l,t}(\alpha)$ and $\inf_{\alpha>0} \beta^{\varepsilon}_{l,t,\gamma}(\alpha)$ exist. There exist sequences $\{\alpha^{\varepsilon,n}_{l,t}\}_{n=1}^{\infty}$ and $\{\alpha^{\varepsilon,n}_{l,t,\gamma}\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \psi^{\varepsilon}_{l,t}(\alpha^{\varepsilon,n}_{l,t}) = \inf_{\alpha>0} \psi^{\varepsilon}_{l,t}(\alpha) \text{ and } \lim_{n\to\infty} \beta^{\varepsilon}_{l,t}(\alpha^{\varepsilon,n}_{l,t,\gamma}) = \inf_{\alpha>0} \beta^{\varepsilon}_{l,t,\gamma}(\alpha).$$

Hence there exist $\alpha^{\varepsilon}_{l,t} > 0$ and $\alpha^{\varepsilon}_{l,t,\gamma} > 0$ such that

$$\psi^{\varepsilon}_{l,t}(\alpha^{\varepsilon}_{l,t}) \leq 2 \inf_{\alpha>0} \psi^{\varepsilon}_{l,t}(\alpha) \quad \text{and} \quad \beta^{\varepsilon}_{l,t,\gamma}(\alpha^{\varepsilon}_{l,t,\gamma}) \leq 2 \inf_{\alpha>0} \beta^{\varepsilon}_{l,t,\gamma}(\alpha).$$

(i) Note that

$$\psi^{\varepsilon}_{l,t}(\alpha^{\varepsilon}_{l,t}) \leq 2 \inf_{\alpha>0} \psi^{\varepsilon}_{l,t}(\alpha) \leq 2\, \psi^{\varepsilon}_{l,t}(\alpha) \quad \forall \alpha > 0.$$

In particular, taking $\alpha^{\varepsilon} = \frac{\theta}{l} \ln(\frac{1}{\varepsilon})$ for some $\theta \in (0, 1)$, we have

$$\psi^{\varepsilon}_{l,t}(\alpha^{\varepsilon}_{l,t}) \leq 2\, \psi^{\varepsilon}_{l,t}(\alpha^{\varepsilon}) = 2\Big(\varepsilon^{1-\frac{\theta(l-t)}{l}} + \frac{\varepsilon^{\frac{\theta t}{l}}}{h(\alpha^{\varepsilon})}\Big). \tag{3.14}$$

Since $0 < \frac{\theta(l-t)}{l} < 1$, hence results follow from (3.14).

(ii) Note that

$$\beta^{\varepsilon}_{l,t,\gamma}(\alpha^{\varepsilon}_{l,t,\gamma}) \leq 2 \inf_{\alpha>0} \beta^{\varepsilon}_{l,t,\gamma}(\alpha) \leq 2\, \beta^{\varepsilon}_{l,t,\gamma}(\alpha) \quad \forall \alpha > 0.$$

In particular, taking $\alpha^{\varepsilon} = \frac{\theta}{l} \ln(\frac{1}{\varepsilon})$ for some $\theta \in (0, 1)$, we have

$$\beta^{\varepsilon}_{l,t,\gamma}(\alpha^{\varepsilon}_{l,t,\gamma}) \leq 2\, \beta^{\varepsilon}_{l,t,\gamma}(\alpha^{\varepsilon}) = 2\, \Big(\varepsilon^{1-\frac{\theta(l-t)}{l}} + \frac{\varepsilon^{\frac{\theta}{l}(\gamma+t-l)}}{h(\alpha^{\varepsilon})}\Big). \tag{3.15}$$

Hence results follow from (3.15).

□

Proof of the following theorem is obvious from Theorem 3.9, by using Lemma 3.11.

Theorem 3.12. *Let* $f : [0, \tau] \times H \to H$ *be a Borel measurable function satisfying conditions (1.5)-(1.6) and* $u(\cdot)$ *be the solution of (1.1)-(1.2). Let* $\psi^{\varepsilon}_{\tau,t}(\cdot)$, $\beta^{\varepsilon}_{\tau,t,\gamma}(\cdot)$, $\alpha^{\varepsilon}_{\tau,t}$ *and* $\alpha^{\varepsilon}_{\tau,t,\gamma}$ *be as in the Lemma 3.11 and* $\tilde{\rho_h} = \max\{\rho_h, 1\}e^{\tau\kappa}$.

(i) *If* $u(\cdot)$ *satisfies condition (1.7), then for* $\alpha = \alpha^{\varepsilon}_{\tau,t}$

$$\|u(t) - u^{\alpha^{\varepsilon}_{\tau,t}}(t)\| \leq \tilde{\rho_h}\ \psi^{\varepsilon}_{\tau,t}(\alpha^{\varepsilon}_{\tau,t}), \qquad 0 \leq t < \tau.$$

(ii) *If* $u(\cdot)$ *satisfies condition (1.8), then for* $\alpha = \alpha^{\varepsilon}_{\tau,t,\gamma}$

$$\|u(t) - u^{\alpha^{\varepsilon}_{\tau,t,\gamma}}(t)\| \leq \tilde{\rho_h}\ \beta^{\varepsilon}_{\tau,t,\gamma}(\alpha^{\varepsilon}_{\tau,t,\gamma}), \qquad 0 \leq t < \tau.$$

Remark 3.13.

(i) By finding the critical value $\alpha^\varepsilon_{\tau,t} = \frac{1}{q+\tau}\log\big(\frac{q+t}{(\tau-t)\varepsilon}\big)$ of the function $\psi^\varepsilon_{\tau,t}(\cdot)$ for $h(\lambda) = e^{q\lambda}$ and $0 \le t < \tau$ with $t+q > 0$ in Lemma 3.11(i), we get estimate from Theorem 3.12(i),

$$\|u(t) - u^{\alpha^\varepsilon_{\tau,t}}(t)\| \le \tilde{\rho_h}\,\Big(\frac{q+\tau}{\tau-t}\Big)\Big(\frac{\tau-t}{q+t}\Big)^{\frac{q+t}{q+\tau}}\varepsilon^{\frac{q+t}{q+\tau}}. \tag{3.16}$$

Since $\big(\frac{q+\tau}{\tau-t}\big)\big(\frac{\tau-t}{q+t}\big)^{\frac{q+t}{q+\tau}} \le 2$,

$$\|u(t) - u^{\alpha^\varepsilon_{\tau,t}}(t)\| \le 2\tilde{\rho_h}\,\varepsilon^{\frac{q+t}{q+\tau}}. \tag{3.17}$$

(ii) For some $a \in (0,1)$, taking $\alpha^\varepsilon_{\tau,t} = \frac{a}{\tau}\log\big(\frac{1}{\varepsilon}\big)$, we get estimate from Corollary 3.10(ii)

$$\|u(t) - u^{\alpha^\varepsilon_{\tau,t}}(t)\| \le \tilde{\rho_h}\Big(\varepsilon^{1-a} + \varepsilon^{\frac{at}{\tau-t}}\big(\frac{\tau-t}{a}\big)^p\Big(\log\big(\frac{1}{\varepsilon}\big)\Big)^{-p}\Big), \quad 0 \le t < \tau. \tag{3.18}$$

Also, by taking $\alpha^\varepsilon_{\tau,t} = \frac{1}{\tau}\ln(\frac{1}{\varepsilon})$, we get estimate from Corollary 3.10(ii)

$$\|u(t) - u^{\alpha^\varepsilon_{\tau,t}}(t)\| \le \tilde{\rho_h}\varepsilon^{\frac{t}{\tau}}\Big(1 + \tau^p\Big(\ln\big(\frac{1}{\varepsilon}\big)\Big)^{-p}\Big), \quad 0 \le t < \tau. \tag{3.19}$$

(iii) By finding the critical value $\alpha^\varepsilon_{\tau,t,\gamma} = \frac{1}{\gamma}\log\big(\frac{t-\tau+\gamma}{(\tau-t)\varepsilon}\big)$ of the function $\beta^\varepsilon_{\tau,t,\gamma}(\cdot)$ for $h(\lambda) = 1$ and $\tau - \gamma < t < \tau$ in Lemma 3.11(ii), we get estimate from Theorem 3.12(ii),

$$\|u(t) - u^{\alpha^\varepsilon_{\tau,t,\gamma}}(t)\| \le \tilde{\rho_g}\,\Big(\frac{\gamma}{t+\gamma-\tau}\Big)\Big(\frac{\gamma+t-\tau}{\tau-t}\Big)^{\frac{\tau-t}{\gamma}}\,\varepsilon^{\frac{\gamma+t-\tau}{\gamma}}. \tag{3.20}$$

Since $\big(\frac{\gamma}{t+\gamma-\tau}\big)\big(\frac{\gamma+t-\tau}{\tau-t}\big)^{\frac{\tau-t}{\gamma}} \le 2$,

$$\|u(t) - u^{\alpha^\varepsilon_{\tau,t}}(t)\| \le 2\tilde{\rho_h}\,\varepsilon^{\frac{\gamma+t-\tau}{\gamma}}. \tag{3.21}$$

$\diamondsuit$

Remark 3.14. The estimates obtained in Remark 5.11 (i),(ii) in [7] under condition (3.11) for $h(\lambda) = e^{q\lambda}$, $q \ge 0$ and $h(\lambda) = \lambda^p$, $p > 0$ respectively, are same as the estimates obtained in Remark 3.13 (i),(ii) under the condition (1.7) for $h(\lambda) = e^{q\lambda}$, $q \ge 0$ and $h(\lambda) = \lambda^p$, $p > 0$ respectively. Condition (3.11) imply condition (1.7) (see Remark 3.5). Also, when noise in the final value ϕ, the estimate obtained in Theorem 3.10, 3.11, 3.12 in [5] are same as the estimates obtained (3.16),(3.17) for $q = 0$; (3.16),(3.17) and (3.18),(3.19) under the condition (3.11) where $f(t, u(t))$ is replaced by $f(t)$. By using quasi-reversibility method, estimate in (3.16) is obtained in Theorem 4.16 in [6] when $f(t, u(t))$ is replaced by $f(t)$ and noise is only in final value ϕ. $\diamondsuit$

Remark 3.15. Taking $\alpha := \alpha^\varepsilon = \frac{1}{l}\log(\frac{1}{\varepsilon})$ in Corollary 3.10(i), (ii), (iv), for $t \in [0,\tau)$, we get the estimates

$$\|u(t) - u^\alpha_\varepsilon(t)\| \le 2\tilde{\rho_h}\varepsilon^{\frac{t}{l}}, \tag{3.22}$$

$$\|u(t) - u^\alpha_\varepsilon(t)\| \le 2\max\{1, l^p\}\tilde{\rho_h}\max\{\big(\log(\frac{1}{\varepsilon})\big)^{-p}, \varepsilon^{\frac{l-\tau}{l}}\}\varepsilon^{\frac{t}{l}}, \tag{3.23}$$

and

$$\|u(t) - u^\alpha_\varepsilon(t)\| \le 2\tilde{\rho_h}\max\{\varepsilon^{\frac{l-\tau}{l}}, \varepsilon^{\frac{\gamma-\tau}{l}}\}\varepsilon^{\frac{t}{l}}, \tag{3.24}$$

respectively.

$\diamondsuit$

Remark 3.16. In [10], Nam consider (1.1)-(1.2) with the operator A having discrete spectrum and obtained estimates (3.22),(3.23), (3.24) in Lemma 1 (i),(ii), (iii) respectively, under same condition on $u(\cdot)$. In this case, Nam results are particular cases of our results. $\diamondsuit$

Remark 3.17. Estimate obtained in Theorem 3.12(ii) ensures convergence for all $t \in [0, \tau)$ depending on smoothness of h. Suppose h is such that estimate in Theorem 3.12(ii) does not ensure convergence on interval $[0, a)$ for some $a \in (0, \tau)$ (one such h is $h(\lambda) = 1$). In such case, motivated from Nam [10], taking t_n for $n = 0, 1, \cdots, n_0, n_0 + 1$ as in Remark 3.6, we modify the regularized solution and compute estimate on $[t_{n_0}, \tau)$ and using resulting estimate at t_{n_0}, we calculate estimate on $[t_{n_0-1}, t_{n_0})$ and so on in the following theorem. $\diamondsuit$

Theorem 3.18. *Let $f : [0, \tau] \times H \to H$ be a Borel measurable function satisfying conditions (1.5)-(1.6). Let $u(\cdot)$ be the solution of (1.1)-(1.2) satisfying (1.8). Let all the notations of the form $\psi^\varepsilon_{l,t}(\cdot)$, $\alpha^\varepsilon_{l,t}$, $\beta^\varepsilon_{l,t,\gamma}(\cdot)$ and $\alpha^\varepsilon_{l,t,\gamma}$ be as in the Lemma 3.11 and $\tilde{\rho}_h = \max\{1, \rho_h\} e^{\tau\kappa}$. Let t_n for $n = 0, 1, \cdots, n_0, n_0 + 1$ be as in Remark 3.6. Let $\tilde{u}^\alpha_\varepsilon : [0, \tau] \to H$ be such that $\tilde{u}^\alpha_\varepsilon(t) = u^\alpha_{n,\varepsilon}(t, \phi^n_\varepsilon)$ on $[t_n, t_{n+1})$ for $n = 0, 1, \cdots, n_0$, with $\tilde{u}^\alpha_\varepsilon(\tau) = u^\alpha_{n_0,\varepsilon}(\tau, \phi^{n_0}_\varepsilon)$, where $u^\alpha_{n,\varepsilon}(\cdot, \phi^n_\varepsilon)$ is the solution of the integral equation*

$$u^\alpha_{n,\varepsilon}(t, \phi^n_\varepsilon) = \int_0^\alpha e^{(t_{n+1}-t)\lambda} dE_\lambda \phi^n_\varepsilon - \int_t^{t_{n+1}} \int_0^\alpha e^{(s-t)\lambda} dE_\lambda f(s, u^\alpha_{n,\varepsilon}(s, \phi^n_\varepsilon)) ds, \quad t_n \le t \le t_{n+1},$$

and $\phi^{n_0}_\varepsilon = \phi_\varepsilon$, $\varepsilon_{n_0} = \varepsilon$, $\phi^{n-1}_\varepsilon = u^{\beta^{\varepsilon_n}_{t_{n+1},t_n,\gamma}}_{n,\varepsilon}(t_n, \phi^n_\varepsilon)$, $\varepsilon_{n-1} = \beta^{\varepsilon_n}_{t_{n+1},t_n,\gamma}(\alpha^{\varepsilon_n}_{t_{n+1},t_n,\gamma})$ for $n = n_0, (n_0 - 1), \cdots, 1$. Then, for $t \in [t_n, t_{n+1})$ and $\alpha := \alpha^{\varepsilon_n}_{t_{n+1},t,\gamma}$, we have estimate

$$\|u(t) - \tilde{u}^\alpha_\varepsilon(t)\| \le \tilde{\rho}_h \ \beta^{\varepsilon_n}_{t_{n+1},t,\gamma}(\alpha),$$

for $n = n_0, n_0 - 1, \cdots, 1, 0$.

Proof. For $t_1 \le t \le t_{n_0+1}$, condition (1.8) takes the form

$$\int_0^\infty h^2(\lambda) e^{2\gamma\lambda} d\|E_\lambda u(t)\|^2 \le \rho_h^2, \quad \forall t \in [t_1, t_{n_0+1}]$$

which is of the form (3.10). Let $\tilde{u}^\alpha(\cdot)$ be the function as in the Theorem 3.7.

For $t \in [t_{n_0}, t_{n_0+1})$, from Theorem 3.7 and 3.8, we have

$$\|u(t) - \tilde{u}^\alpha(t)\| \le \rho_h e^{\tau\kappa} \ \frac{e^{(t_{n_0+1}-t-\gamma)\alpha}}{h(\alpha)}$$

and

$$\|\tilde{u}^\alpha(t) - \tilde{u}^\alpha_\varepsilon(t)\| \le e^{\tau\kappa} e^{(t_{n_0+1}-t)\alpha} \|u(t_{n_0+1}) - \phi_\varepsilon\|,$$

respectively. Since $\|u(t_{n_0+1}) - \phi_\varepsilon\| \le \varepsilon$, by similar argument as in Theorem 3.9, we get

$$\|u(t) - \tilde{u}^\alpha_\varepsilon(t)\| \le \tilde{\rho}_h \ \Big(e^{(t_{n_0+1}-t)\alpha} \varepsilon_{n_0} + \frac{e^{(t_{n_0+1}-t-\gamma)\alpha}}{h(\alpha)} \Big), \quad t_{n_0} \le t < t_{n_0+1}. \qquad (3.25)$$

Now, choosing the parameter $\alpha := \alpha^{\varepsilon_{n_0}}_{t_{n_0+1},t,\gamma}$ and taking the function $\beta^{\varepsilon_{n_0}}_{t_{n_0+1},t,\gamma}(\cdot)$ as in Lemma 3.11, we get estimate from (3.25)

$$\|u(t) - \tilde{u}^\alpha_\varepsilon(t)\| \le \tilde{\rho}_h \beta^{\varepsilon_{n_0}}_{t_{n_0+1},t,\gamma}(\alpha), \quad t_{n_0} \le t < t_{n_0+1}.$$

In particular, for $t = t_{n_0}$, we have

$$\|u(t_{n_0}) - \tilde{u}_\varepsilon^{\alpha^{\varepsilon_{n_0}}_{t_{n_0+1},t_{n_0},\gamma}}(t_{n_0})\| \le \tilde{\rho_h} \beta^{\varepsilon_{n_0}}_{t_{n_0+1},t_{n_0},\gamma}(\alpha^{\varepsilon_{n_0}}_{t_{n_0+1},t_{n_0},\gamma}).$$

i.e,

$$\|u(t_{n_0}) - \phi_\varepsilon^{n_0-1}\| \le \tilde{\rho_h}\varepsilon_{n_0-1}. \tag{3.26}$$

Now by using (3.26) and applying same process as above for finding error estimate on $[t_{n_0}, t_{n_0+1})$, we get required estimate on $[t_{n_0-1}, t_{n_0})$. Again using estimate at $t = t_{n_0-1}$, applying same process of finding error estimate on $[t_{n_0}, t_{n_0+1})$, we get required estimate on $[t_{n_0-2}, t_{n_0-1})$. Similarly, we get required estimate on $[t_n, t_{n+1}]$ for $n = n_0 - 3, \cdots, 1, 0$. □

Remark 3.19. Let $u(\cdot)$ satisfies condition (1.8) for $h(\lambda) = 1$. Let $t_0, t_1, \cdots, t_{n_0+1}$ and $\tilde{u}_\varepsilon^\alpha(\cdot)$ be as in Theorem 3.18. Let $\varepsilon_{n_0+1} = \varepsilon$ and $\varepsilon_n = \varepsilon_{n+1}^{\frac{\gamma+t_n-t_{n+1}}{\gamma}}$, i.e,

$$\varepsilon_n = \varepsilon^{\frac{(\gamma+t_{n_0}-t_{n_0+1})(\gamma+t_{n_0-1}-t_{n_0})\cdots(\gamma+t_n-t_{n+1})}{\gamma^{n_0+1-n}}}$$

for $n = n_0, n_0 - 1, \cdots, 1, 0$. Then, for $t \in [t_n, t_{n+1})$ and $\alpha := \frac{1}{\gamma}\log\left(\frac{t-t_{n+1}-\gamma}{(t_{n+1}-\gamma)\varepsilon_{n+1}}\right)$, we get

$$\|u(t) - \tilde{u}_\varepsilon^\alpha(t)\| \le 2\tilde{\rho_h}\varepsilon_n$$

for $n = n_0, n_0 - 1, \cdots, 1, 0$.

♢

Remark 3.20. Nam proved Theorem 3 in [10] by using Lemma 1. In Remark 3.15, we obtained estimates wich are of same order as in Lemma 1 (cf. Nam [10]). Same theorem as Theorem 3 in [10] can be stated in this paper and proof of that theorem will be same as the proof of Theorem 3 in [10] by using Remark 3.15. ♢

Conclusion: We consider spectral cut-off regularization as regularization method for nonlinear nonhomogeneous FVP. We derive estimates under the assumptions on the solution $u(\cdot)$ instated of conditions on the final value ϕ and the function $f(\cdot, u(\cdot))$ as in Jana and Nair [7]. We derive error estimates for exact as well as noisy final value ϕ. Appropriate parameter choice strategies are considered when the final value ϕ is noisy. This work extends the work of Nam [10] from the self adjoint positive operator A having discrete spectrum to any self-adjoint positive operator A. Also results obtained in Jana and Nair [5], [7] are particular cases of this paper when noise is only in the final value. This work extends the work of Tuan [15] from homogeneous FVP to nonhomogeneous nonlinear FVP and generalizes many other results in literature.

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