Poincare Journal of Analysis & Applications
Vol. 11, No. 2 (2024), 247-251
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DOI: 10.46753/pjaa.2024.v011i02.012
Online Published on 29. 12. 2024

ON MAZUR PROPERTY IN BANACH SPACES WITH PRI

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Date of Receiving : 17. 09. 2024
Date of Revision : 23. 12. 2024
Date of Acceptance : 23. 12. 2024

Abstract. Mazur property is proved in some Banach spaces having projectional resolutions of identity (PRI). As a consequence these Banach spaces (having PRI) have Pettis integration property.

1. Introduction and Preliminaries

The concept of projectional resolutions of the identity (in short, PRI) has played an important tool in the study of geometric properties of Banach spaces like Gelfand-Philips property, having weak* sequentially compact and weak* angelic dual unit balls, existance of Markusévic bases and Odell-Rosenthal type results. In this paper, we shall prove another important property namely Mazur property in some Banach spaces having PRI. As a consequence, these spaces have Pettis Integration property.

An infinite dimensional Banach space is denoted by E and let E^* and E^{**} be its first and second dual respectively. The cardinality of the smallest dense set in E is denoted by dens E. The first ordinal with cardinality $\aleph_0$ is denoted by ω; the other symbols used for ordinals are α, β, λ and μ. Let the first ordinal with cardinality dens E is denoted by μ. A long sequence $\{p_\alpha\}_{\omega\le\alpha\le\mu}$ of projections on a Banach space E is said to be a projectional resolutions of the identity (PRI) of E, if the following conditions are satisfied:

$$
\begin{array}{rll}
(i) & \|p_\alpha\| = 1 & (\omega \le \alpha \le \mu) \\
(ii) & p_\alpha \cdot p_\beta = p_\beta \cdot p_\alpha = p_\beta & (\omega \le \beta \le \alpha \le \mu) \\
(iii) & \text{dens } p_\alpha(E) \le \text{card } \alpha & (\omega \le \alpha \le \mu) \\
(iv) & p_\alpha(E) = \overline{\cup\{p_{\beta+1}(E) : \omega \le \beta < \alpha\}} & (\omega \le \alpha \le \mu) \\
(v) & p_\mu = I. &
\end{array}
$$

2010 *Mathematics Subject Classification.* 46B10, 46B15.
Key words and phrases. Mazur property, PRI.
Communicated by: Ravindra K. Bisht

If E has a PRI $\{p_\alpha\}_{\omega\le\alpha\le\mu}$, then the map $\alpha\mapsto p_\alpha(x)$ from $[\omega,\mu]$ to E is continuous for each $x\in E$, and if $\beta\le\mu$ is limit ordinal then $p_\beta(x)=\lim_{\alpha<\beta}p_\alpha(x)$. Also, the subspace $p_\alpha(E)$ is denoted by E_α, for each $\omega\le\alpha<\mu$ and we note that E^*_α and E^{**}_α are isomorphic to $p^*_\alpha(E^*)$ and $p^{**}_\alpha(E^*)$, respectively. The concept of PRI was introduced by Amir and Lindenstrauss [1] wherein existence of a PRI in every WCG Banach space was established. One may refer to ([13], Chapter 19 and 20) and ([2], Chapter 6) for more study on PRI.

A PRI $\{p_\alpha\}_{\omega\le\alpha\le\mu}$ of E is defined to be of type I if

$$\overline{\cup\{p^*_\alpha(E^*):\omega\le\alpha<\mu\}}=E^*,$$

and called shrinking if $\{p^*_\alpha\}_{\omega\le\alpha\le\mu}$ is a PRI of E^*. (see [6, 12]). We note that a shrinking PRI is always of type I. John and Zizler [8, 9] proved the existence of a shrinking PRI in a WCG Banach space with a WCG dual. Vaśak proved the existence of a shrinking PRI in a more general setting when both E and E^* are WCD. Fabian [4] showed that WCD Asplund spaces admit shrinking PRI and these spaces are WCG. For some more results on existence of a shrinking PRI one may refer to [12].

Definition 1.1. A Banach space E is said to be in the class $\mathcal{P}^*$ if its has a PRI $\{p_\alpha\}_{\omega\le\alpha\le\mu}$ of type I such that $p_\alpha(E)$ and $(p_{\alpha+1}-p_\alpha)(E)$ are also in $\mathcal{P}^*$ for each $\alpha\in[\omega,\mu)$.

The definition given above is on the lines of the definition of a class $\mathcal{P}$ of Banach spaces in ([2], p.286). If the PRI in the definition of the class $\mathcal{P}$ is of type I, then we get the class $\mathcal{P}^*$. Every WCD Asplund space possesses a PRI of type I and its complemented subspaces also happens to be WCD Asplund. Thus, every WCD Asplund space belongs to the class $\mathcal{P}^*$.

Furthermore, if E is a quasi reflexive Banach space, then so is its dual E^* and both E and E^* are WCD spaces. So, all quasi reflexive Banach spaces are in the class $\mathcal{P}^*$.

We know that $\phi\in E^{**}$, considered as a functional on E^*, is weak* continuous if and only if $\phi\in\pi(E)$ Further, ϕ is said to be weak* sequentially continuous , if for any sequence $\{f_n\}$ in E^*, we have $\lim_{n\to\infty}\phi(f_n)=\phi(f)$. Clearly, every weak* continuous functional is weak* sequentially continuous while the converse may not hold.

A Banach space is said to have the Mazur property if every weak* sequentially continuous functional in the second dual E^{**} is weak* continuous, i.e., each $\phi\in E^{**}$ satisfying

$$\lim_{n\to\infty}f_n(x)=f(x)\quad(x\in E)\qquad\Longrightarrow\qquad\lim_{n\to\infty}\phi(f_n)=\phi(f)$$

for every weak* convergent sequence $\{f_n\}$ in the dual E^*, is in the canonical image of E in E^{**}.

Separable Banach spaces enjoy Mazur property as do reflexive spaces. Further, it is known that every WCG Banach space has the Mazur property. In fact, every Banach space with weak* angelic dual unit ball and, in particular, every WCD Banach space has the Mazur property while ℓ^∞ does not posess this property [3]. Also, it is known that Mazur property is inherited by closed subspaces. The following result is due to Kappeler [10].

Theorem 1.2. *If B_E is weak* sequentially dense in $B_{E^{**}}$ and B_{E^*} is weak* sequentially compact, then E has Mazur property.*

Let $(\Omega, \mathcal{F}, \sigma)$ be a probability space and E be a Banach space. A function $\psi\colon \Omega \to E$ is said to be scalarly measurable if the function $\tau \to f(\psi(\tau))$ is measurable for all $f \in E^*$. Further the function ψ is said to be Pettis integrable if, for every $A \in \mathcal{F}$, there is an $x_A \in E$ such that

$$f(x_A) = \int_A f(\psi(\tau))\, \mathrm{d}\sigma(\tau) \qquad (f \in E^*).$$

The Banach space E is said to have the Pettis Integration property if every bounded scalarly measurable function $\psi\colon \Omega \to E$ is Pettis integrable.

2. Main Result

It is shown in [7] that if a Banach space with dense $E = \text{card}\ \omega_1$ possesses a PRI of type I, then E has the Mazur property. Now, we shall prove the existence of Mazur property in every Banach space in class $\mathcal{P}^*$.

To begin with we prove a lemma. For simplicity of exposition, let π_p and π_α denote the canonical maps of $p_\alpha(E)$ and $(p_{\alpha+1} - p_\alpha)(E)$ to their respective second duals.

Lemma 2.1. *If E is a Banach space having PRI $\{p_\alpha\}_{\omega\leq\alpha\leq\mu}$ of type I such that $codim_{E^{**}}\pi(E) > 0$, then there is an ordinal $\omega < \lambda < \mu$ such that either*

$$codim_{((p_{\lambda+1}-p_\lambda)(E))^{**}}\pi_\lambda\left((p_{\lambda+1} - p_\lambda)(E)\right) > 0,$$

or

$$codim_{(p_\alpha(E)^{**})}\pi_{p_\alpha}(p_\alpha(E)) > 0.$$

Proof. Let $\phi \in E^{**} \setminus \pi(E)$. First, assume that μ is a non limit ordinal. Then, there is an ordinal $\omega < \gamma < \mu$ such that $\mu = \gamma + 1$. We claim that either

$$p_\gamma^{**} \cdot \phi \notin \pi_{p\gamma}(p_\gamma(E))$$

or

$$(p_{\gamma+1}^{**} - p_\gamma^{**}) \cdot \phi \notin \pi_\gamma((p_{\gamma+1} - p_\gamma)(E)).$$

Indeed, for otherwise, there are x, $y \in E$ such that

$$\phi(f) = f\left(p_\gamma(x) + (p_{\gamma+1} - p_\gamma)(y)\right) \quad (f \in E^*).$$

This is a contradiction.

Consider now the case when μ is a limit ordinal. We claim that there is an ordinal λ with $\omega < \lambda < \mu$ such that

$$p_\lambda^{**} \cdot \phi \notin \pi_{p_\lambda}(p_\lambda(E)).$$

Assume on the contrary that there is an $x \in E$ such that

$$(p_\alpha^* \cdot f)(x) = \phi(p_\alpha^* \cdot f) \qquad (f \in E^*, \omega < \alpha < \mu).$$

Taking limit as $\alpha < \mu$, and since the PRI is of type I, we have

$$\lim_{\alpha<\mu} p_\alpha^* \cdot f = f$$

whence

$$\phi(f) = f(x) \quad (f \in E^*).$$

This is again a contradiction. □

Now, we shall prove the main result.

Theorem 2.2. *Every Banach space in the class $\mathcal{P}^*$ has the Mazur property.*

Proof. Let E be a Banach space in the class $\mathcal{P}^*$ with dens $E =$ card μ. Then, E has a PRI $\{p_\alpha\}_{\omega\leq\alpha\leq\mu}$ of type I such that, for each $\omega \leq \alpha < \mu$, both $p_\alpha(E)$ and $(p_{\alpha+1}-p_\alpha)(E)$ are in $\mathcal{P}^*$.

If E is separable the dual unit ball is metrizable in the weak* topology, whence it is weak* angelic. It now follows that E has the Mazur property.

If E is non separable, i.e., dens $E > \aleph_0$, we set up an induction hypothesis on dens E. Let every Banach space F in the class $\mathcal{P}^*$ with dens $F <$ dens E has the Mazur property. Then, there is a weak* sequentially continuous functional $\phi \in E^{**}$ such that $\phi \notin \pi(E)$.

Let p be a bounded linear projection on E. Clearly, $p^{**}\cdot\phi$ considered as a functional on $(p^*(E))^*$ is $\sigma(p^*(E))^*, p(E))$-sequentially continuous. Now, since E has a PRI $\{p_\alpha\}_{\omega\leq\alpha\leq\mu}$, it follows that, for each $\omega \leq \alpha \leq \mu$, the functional $p_\alpha^{**}\cdot\phi$ and $(p_{\alpha+1}^{**}-p_\alpha^{**})\cdot\phi$ are sequentially continuous in the weak* topologies of $p_\alpha^*(E)$ and $(p_{\alpha+1}^*-p_\alpha^*)(E^*)$, respectively.

Now, since $\{p_\alpha\}_{\omega\leq\alpha\leq\mu}$ is a PRI of E of type I, by Lemma 2.1, there is an ordinal λ with $\omega < \lambda < \mu$ such that either

$$p_\lambda^{**}\cdot\phi \notin \pi_{p_\lambda}\left(p_\lambda(E)\right)$$

or

$$(p_{\lambda+1}^{**}-p_\lambda^{**})\cdot\phi \notin \pi_\lambda\left((p_{\lambda+1}-p_\lambda)(E)\right).$$

In other words, this means that either $p_\lambda(E)$ or $(p_{\lambda+1}-p_\lambda)(E)$ fails the Mazur property. However, since by the definition of a PRI,

$$\text{dens } p_\lambda(E) < \text{card } \mu$$

and

$$\text{dens } \left((p_{\lambda+1}-p_\lambda)(E)\right) < \text{card } \mu,$$

we have reached a contradiction. Hence, E has the Mazur property. □

Remark 2.3. It is not necessary that a Banach space having the Mazur property is in the class $\mathcal{P}^*$. By proposition 5.12 in [3], the Johnson-Lindenstrauss space (JL) has the Mazur property while it does not admit a PRI ([5], Remark 7).

Acknowledgement

We sincerely thank the referees for their valuable comments and useful suggestions, which remarkably improved this paper.

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