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A NOTE ON ITERATED $pg-$FRAMES IN BANACH SPACES

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Abstract. For a bounded operator Λ on a Banach space X, we propound the structure of the $pg-$frames generated by iterations of Λ and christen it as iterated $pg-$frame. We derive sufficient condition on the operator Λ for $\{\Lambda^i\}_{i\in\mathbb{N}}$ to be an iterated $pg-$frame. Further, we obtain some properties of the operator Λ in case $\{\Lambda^i\}_{i\in\mathbb{N}}$ is an iterated $pg-$frame for X. Towards the end, we analyze iterated $pg-$frames of the form $\{\psi\Lambda^i\}_{i\in\mathbb{N}}$ and $\{\Lambda^i\psi\}_{i\in\mathbb{N}}$, where $\psi \in B(X)$.

1. Introduction

Frames in Hilbert spaces is a permeance of a basis of a vector space to the sets which are not necessarily linearly independent. Frames can be visualized as the redundant bases which are generalization of orthonormal bases. The concept of Hilbert space frames were first propounded by Duffin and Schaeffer [10] to study some profound problems in non-harmonic Fourier series. Throughout this paper I denotes a subset of $\mathbb{N}$, H denotes a seperable Hilbert space and X denotes a Banach space.

Definition 1.1. [10] Let $I \subseteq \mathbb{N}$. A family of vectors $\{x_i\}_{i\in I}$ in a Hilbert space H is said to be a frame for H, if there exist two constants $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_{i\in I} |\langle x, x_i\rangle|^2 \leq B\|x\|^2, \text{ for all } x \in H.$$

The concept of p-frame for Banach spaces was introduced by Christensen and Stoeva [9] and p-frame of subspaces by Faroughi and Najati [15]. Frames have massive fruitful applications in quantum mechanics [4, 19] and Dynamical sampling [2, 3]. Elementary theory of frames and their applications in different directions are categorically explained in the books of Casazza and Kutyniok [4], Christensen [6, 7, 8], Han et.al.[11], Okoudjou [16], the tutorials of Casazza [5] and the memoir of Han and Larson [12]. Over past

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decades, plethora of frames have been introduced. One of them is the g-frame in Hilbert spaces , which was postulated by W. Sun [18].

Definition 1.2. [18] A sequence $\{\Lambda_i \in B(H, H_i) : i \in I\}$ is known as g-frame for H with respect to $\{H_i\}_{i\in I}$ if there exist constants $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_{i\in I} \|\Lambda_i x\|^2 \leq B\|x\|^2, \text{ for every } x \in H.$$

Now, we encapsulate some key concepts and definitions related to g-frames and pg-frames. The theory of g-frames in Banach spaces was coined by Amir Khosravi and Behrooz Khosravi [13].

Definition 1.3. [13] Let X_d denote Banach space of scalar-valued sequences. It is known as BK-space if the coordinate functionals are continuous on X_d. Additionally, X_d is called a solid BK-space if whenever $\{b_i\}$ and $\{c_i\}$ are sequences with $\{c_i\} \in X_d$ and $|b_i| \leq |c_i|$, then it implies that $\{b_i\} \in X_d$ and $\|\{b_i\}\| \leq \|\{c_i\}\|$.

Let X, Y are Banach spaces and $\{Y_i\}_{i\in I}$ is a sequence of closed subspaces of Y, $B(X, Y_i)$ denotes the space of all bounded linear operators from X to Y_i, and $B(X, X)$ is denoted by $B(X)$.

Definition 1.4. [13] A sequence $\{\Lambda_i \in B(X, Y_i) : i \in I\}$ is known as *g-frame* for X with respect to $\{Y_i\}_{i\in I}$ if there exists a solid BK-space X_d and constants $A, B > 0$ such that
(i) for all $x \in X, \{\|\Lambda_i x\|\} \in X_d$.
(ii) for all $x \in X, A\|x\|_X \leq \|\{\|\Lambda_i x\|\}\|_{X_d} \leq B\|x\|_X$.
Here A and B are called bounds of g-frame.

The concept of pg-frames in Banach spaces was introduced by M.R. Abdollahpour et.al [1], which deals with the rudiments of the pg-frames and embellished with some salient results of pg-frames.

Definition 1.5. [1] Let $p \in (1, \infty)$. A sequence $\{\Lambda_i \in B(X, Y_i) : i \in I\}$ is known as *pg−frame* for X with respect to $\{Y_i\}_{i\in I}$ if there exist constants $A, B > 0$ such that

$$A\|x\|^p \leq \sum_{i\in I} \|\Lambda_i x\|^p \leq B\|x\|^p, \text{ for every } x \in X.$$

If $A = B$, then $\{\Lambda_i\}_{i\in I}$ is called tight pg-frame. If $A = B = 1$, then it is called parseval pg-frame. If only the second half of the above inequality is satisfied, we say that $\{\Lambda_i\}_{i\in I}$ is a pg-Bessel sequence.

Example 1.6. Let $X = c_0$, $X_d = \ell_\infty$, $\{\Lambda_i\}_{i\in\mathbb{N}}$ = canonical unit vectors as basis for ℓ^1 and $Y_i = \mathbb{C}$ for all i. Then the family $\{\Lambda_i\}_{i\in\mathbb{N}}$ is a g-frame for c_0 but not a pg-frame.

The paper [17] entails the narrative of iteration of a bounded operator Λ on a Hilbert space H as g-frames and some useful results in this regard, whereas [14] embraces the properties of bounded representation of g-frames in Hilbert spaces. Taking impetus from the aforementioned papers [1, 17, 14], we generalize their results in the form of iterated pg-frames in Banach spaces. We expound some properties of operator Λ, whenever $\{\Lambda^i\}_{i\in\mathbb{N}}$ is a pg-frame for X. We ruminate for the iterated pg-frames of the form $\{\psi\Lambda^i\}_{i\in\mathbb{N}}$ and $\{\Lambda^i\psi\}_{i\in\mathbb{N}}$, where $\psi \in B(X)$. Furthermore, we demonstrate a family which can never be an iterated pg-frame. Lastly, we evince a result dealing with necessary and sufficient condition for the family $\{\Lambda^i\psi\}_{i\in\mathbb{N}}$ to be a pg-frame, where $\psi \in B(X)$.

2. Iterated pg-frames in Banach spaces

In this paper, the focal point is pg-frames for a Banach space, which are generated by iterations of a bounded operator on the given Banach space X. For $\Lambda \in B(X)$, we envisage on the following questions:
(i) Which conditions on operator Λ are necessary or sufficient for $\{\Lambda^i\}_{i\in\mathbb{N}}$ to constitute a pg-frame for X?
(ii) If $\{\Lambda^i\}_{i\in\mathbb{N}}$ is a pg-frame for X, then what we can conclude about the operator Λ?
Let X be a Banach space, $\Lambda \in B(X)$ and $\{\Lambda^i\}_{i\in\mathbb{N}}$ be an iterated sequence of Λ. We try to find conditions on Λ such that $\{\Lambda^i\}_{i\in\mathbb{N}}$ becomes a pg-frame. Our main aim is to characterize these kinds of frames. Regarding the existence of such operators, we have the following example:

Example 2.1. Let $\Lambda : \ell^p \mapsto \ell^p$ defined by $\Lambda(x_1, x_2, ...) = \frac{1}{2}(0, x_1, x_2, ...)$ for $x = (x_1, x_2, ...)$. Thus $\Lambda^i(x) = \frac{1}{2^i}(\underbrace{0, 0, 0, ..., 0}_{\text{i-times}}, x_1, x_2, ...)$. Hence $\sum_{i=1}^{\infty} \left\|\Lambda^i x\right\|^p = \left(\sum_{i=1}^{\infty} \frac{1}{2^{ip}}\right) \|x\|^p$ and so $\{\Lambda^i\}_{i\in\mathbb{N}}$ is a tight pg-frame.

Now, we provide an example of an operator Λ such that $\{\Lambda^i\}_{i\in\mathbb{N}}$ is not a pg-frame.

Example 2.2. Let $\Lambda : \ell^p \mapsto \ell^p$ defined by $\Lambda(x_1, x_2, ...) = (0, x_1, x_2, ...)$ for $x = (x_1, x_2, ...)$. Then $\|\Lambda^i x\| = \|x\|$, for all $i \in \mathbb{N}$. Hence $\sum_{i=1}^{\infty} \left\|\Lambda^i x\right\|^p = \sum_{i=1}^{\infty} \|x\|^p = \infty$. Thus $\{\Lambda^i\}_{i\in\mathbb{N}}$ is not a pg-frame.

In the following proposition, we provide a sufficient condition for Λ, such that $\{\Lambda^i\}_{i\in\mathbb{N}}$ is a pg-frame.

Proposition 2.3. Let $\Lambda : X \to X$ be an operator such that for every $x \in X$,

$$a\|x\| \le \|\Lambda x\| \le b\|x\| \ ,$$

where $a, b \in (0, 1)$. Then $\{\Lambda^i\}_{i\in\mathbb{N}}$ is a pg-frame for X.

Proof. For each $x \in X$ and $i \in \mathbb{N}$, we have

$$a^i\|x\| \le \left\|\Lambda^i x\right\| \le b^i\|x\|.$$

Thus

$$\left(\sum_{i=1}^{\infty} a^{pi}\right) \|x\|^p \le \sum_{i=1}^{\infty} \left\|\Lambda^i x\right\|^p \le \left(\sum_{i=1}^{\infty} b^{pi}\right) \|x\|^p.$$

Hence

$$\left(\frac{a^p}{1-a^p}\right) \|x\|^p \le \sum_{i=1}^{\infty} \left\|\Lambda^i x\right\|^p \le \left(\frac{b^p}{1-b^p}\right) \|x\|^p.$$

□

Now, we present the pivotal theorem of this paper, which has far-flung consequences. For $\Lambda \in B(X)$, the following theorem provides an interval for the range of $\|\Lambda\|$, bounds for the spectral radius $r(\Lambda)$ and characterizes the range and eigenvalues of a pg-frame $\{\Lambda^i\}_{i\in\mathbb{N}}$. Final part of the theorem, showcases the fact that for any real r, we can construct a corresponding operator Γ_r on X, such that it turns out to be an iterated pg-frame.

Theorem 2.4. *Let $\Lambda \in B(X)$ be such that $\{\Lambda^i\}_{i\in\mathbb{N}}$ is a pg-frame for X with bounds A and B. Then following conditions hold:*
(i)

$$\left(\frac{A}{1+B}\right)\|x\|^p \leq \|\Lambda x\|^p \leq \left(\frac{B}{1+A}\right)\|x\|^p, \quad x \in X.$$

(ii) For each $k \in \mathbb{N}$,

$$\left(\frac{A}{1+B}\right)^k \|x\|^p \leq \|\Lambda^k x\|^p \leq \left(\frac{B}{1+A}\right)^k \|x\|^p, \quad x \in X.$$

(iii) For each $k \in \mathbb{N}$, the operator Λ^k has closed range and

$$M = \inf_{\|x\|=1} \|\Lambda x\| < 1.$$

(iv)The spectral radius $r(\Lambda)$ of Λ satisfies

$$\left(\frac{A}{1+B}\right)^{1/p} \leq r(\Lambda) \leq \left(\frac{B}{1+A}\right)^{1/p}.$$

(v)If β is any eigenvalue of Λ, then $0 < |\beta| < 1$.

(vi) For each real number $r > 1$, there exists a corresponding operator Γ_r on X such that $\|\Gamma_r\| = \frac{1}{r}$ and $\{\Gamma_r^i\}_{i\in\mathbb{N}}$ is a pg-frame for X.

Proof. (i) Since $\{\Lambda^i\}_{i\in\mathbb{N}}$ is a pg-frame for X , we have

$$A\|x\|^p \leq \sum_{i=1}^{\infty} \left\|\Lambda^i x\right\|^p \leq B\|x\|^p, \quad x \in X. \tag{2.1}$$

Considering Λx instead of x in (2.1), we obtain

$$A\|\Lambda x\|^p \leq \sum_{i=1}^{\infty} \left\|\Lambda^{i+1} x\right\|^p \leq B\|\Lambda x\|^p, \quad x \in X. \tag{2.2}$$

From (2.1) and (2.2) , we obtain

$$\left(\frac{A}{1+B}\right)\|x\|^p \leq \|\Lambda x\|^p \leq \left(\frac{B}{1+A}\right)\|x\|^p, \quad x \in X.$$

(ii) The result follows by induction on k and using part (i).

(iii) Using part (ii), for each $k \in \mathbb{N}, \Lambda^k$ is bounded below and so it has closed range. Suppose on contrary that $M = \inf_{\|x\|=1} \|\Lambda x\| \geq 1$. For each $x \in X$, we have $(M^p)^i \|x\|^p \leq \left\|\Lambda^i x\right\|^p$. Since $\sum_{i=1}^{\infty} (M^p)^i$ is divergent, so $\{\Lambda^i\}_{i\in\mathbb{N}}$ can not be a pg-frame for X. Hence $M = \inf_{\|x\|=1} \|\Lambda x\| < 1$.

(iv) By part (ii), for each $i \in \mathbb{N}$, we get

$$\left(\frac{A}{1+B}\right)^{\frac{i}{p}} \|x\| \leq \left\|\Lambda^i x\right\| \leq \left(\frac{B}{1+A}\right)^{\frac{i}{p}} \|x\|, \quad x \in X.$$

Thus

$$\left(\frac{A}{1+B}\right)^{\frac{i}{p}} \leq \left\|\Lambda^i\right\| \leq \left(\frac{B}{1+A}\right)^{\frac{i}{p}}$$

which implies

$$\left(\frac{A}{1+B}\right)^{1/p} \leq r(\Lambda) \leq \left(\frac{B}{1+A}\right)^{1/p}.$$

(v) Let β be an eigenvalue of Λ. Then there exists a $x \neq 0$ such that $\Lambda x = \beta x$. Therefore

$$A\|x\|^p \leq \sum_{i=1}^{\infty} \left\|\Lambda^i x\right\|^p = \sum_{i=1}^{\infty} \left(|\beta^p|^i\right) \|x\|^p \leq B\|x\|^p.$$

Since $0 < \sum_{i=1}^{\infty} \left(|\beta^p|^i\right)$ is convergent, so $0 < |\beta| < 1$.

(vi) By part (i), there exists a constant $a > 0$ such that

$$a\|x\| \leq \|\Lambda x\| \leq \|\Lambda\|\|x\|, \quad x \in X.$$

For all $r > 1$, put

$$\Gamma_r x = \frac{\Lambda x}{r\|\Lambda\|}, \quad x \in X.$$

Thus, we get

$$\frac{a}{r\|\Lambda\|}\|x\| \leq \|\Gamma_r x\| \leq \frac{1}{r}\|x\|, \quad x \in X.$$

Therefore by proposition (2.3), the proof follows. □

In the following corollary, we characterize the norm of the tight iterated *pg*-frame.

Corollary 2.5. Let $\Lambda \in B(X)$ and $\left\{\Lambda^i\right\}_{i\in\mathbb{N}}$ be a tight *pg*-frame for X. Then $\|\Lambda\| < 1$ and $\lim_{i\to\infty} \Lambda^i = 0$.

Proof. Let $\left\{\Lambda^i\right\}_{i\in\mathbb{N}}$ be A-tight *pg*-frame for X, then by theorem (2.4), we have

$$\|\Lambda x\|^p = \left(\frac{A}{1+A}\right) \|x\|^p, \quad x \in X.$$

Thus $\|\Lambda\| < 1$ and $\lim_{i\to\infty} \left\|\Lambda^i\right\| \leq \lim_{i\to\infty} \|\Lambda\|^i = 0$. □

Corollary 2.6. Let $\Lambda, \Gamma \in B(X)$ and $\left\{\Lambda^i\right\}_{i\in\mathbb{N}}$ be a *pg*-frame for X with bounds A and B. If there exist $a, b > 0$ such that $b < 1 - \|\Lambda\|$ and

$$a\|x\| \leq \|(\Lambda - \Gamma)x\| \leq b\|x\|, \quad x \in X.$$

Then $\left\{\Gamma^i\right\}_{i\in\mathbb{N}}$ is a *pg*-frame for X.

Proof. By theorem (2.4), we have

$$\left(\frac{A}{1+B}\right)^{1/p}\|x\| \leq \|\Lambda x\| \leq \|\Lambda\|\|x\|, \quad x \in X.$$

Thus for each $x \in X$,

$$\left|a - \left(\frac{A}{1+B}\right)^{1/p}\right|\|x\| \leq \|\Gamma x\| \leq (b + \|\Lambda\|)\|x\|.$$

Without loss of generality assume that $a \neq \left(\frac{A}{1+B}\right)^{1/p}$. Hence the result follows from the proposition (2.3). $\square$

The next theorem contains some implications, when we consider the family $\Gamma = \left\{\psi\Lambda^i\right\}_{i\in\mathbb{N}}$ to be a *pg*-frame for X.

Theorem 2.7. *Let $\Lambda, \psi \in B(X)$ such that $\Gamma = \left\{\psi\Lambda^i\right\}_{i\in\mathbb{N}}$ be a pg-frame with bounds A_Γ and B_Γ. Then following conditions hold:*
(i) If $\bigcap_{i=1}^n \ker\psi\Lambda^i \neq \{0\}$, for each $n \in \mathbb{N}$, then $\|\Lambda\| \geq 1$.
(ii) $\Lambda^n x \to 0$ as $n \to \infty$, for every $x \in X$.
(iii) If $\|\psi\| < A_\Gamma^{1/p}$, then Λ is injective.

Proof. (i) Let $\epsilon > 0$, and suppose on contrary that $\|\Lambda\| < 1$. Then there exists $N > 0$ such that $\sum_{i=N+1}^\infty \left\|\psi\Lambda^i\right\|^p < \epsilon$. Let $0 \neq x \in \bigcap_{i=1}^N \ker\psi\Lambda^i$ with $\|x\| = 1$. Then

$$A_\Gamma \leq \sum_{i=1}^N \left\|\psi\Lambda^i x\right\|^p + \sum_{i=N+1}^\infty \left\|\psi\Lambda^i x\right\|^p < \epsilon.$$

Therefore $A_\Gamma = 0$, a contradiction. Hence, $\|\Lambda\| \geq 1$.

(ii) For each $n \in \mathbb{N}$ and $x \in X$, we obtain

$$A_\Gamma \left\|\Lambda^{n-1}x\right\|^p \leq \sum_{i=1}^\infty \left\|\psi\Lambda^{i-1+n}x\right\|^p = \sum_{i=n}^\infty \left\|\psi\Lambda^i x\right\|^p.$$

Since $\sum_{i=1}^\infty \left\|\psi\Lambda^i x\right\|^p$ is convergent by assumption, we get $\sum_{i=n}^\infty \left\|\psi\Lambda^i x\right\|^p \to 0$ as $n \to \infty$. Therefore, the above inequality implies that $\Lambda^n x \to 0$ as $n \to \infty$.

(iii) We have

$$\begin{aligned} A_\Gamma\|x\|^p \leq \sum_{i=1}^\infty \left\|\psi\Lambda^i x\right\|^p &\leq \|\psi\|^p \sum_{i=1}^\infty \left\|\Lambda^i x\right\|^p, \quad x \in X \\ &\leq A_\Gamma \sum_{i=1}^\infty \left\|\Lambda^i x\right\|^p, \quad x \in X. \end{aligned}$$

Thus, $A_\Gamma \left(\sum_{i=1}^\infty \left\|\Lambda^i x\right\|^p - \|x\|^p\right) \geq 0$. Hence, Λ is injective. $\square$

We exhibit a family, which can never be an iterated *pg*-frame.

Corollary 2.8. For any isometry operator $\Lambda : X \mapsto X$ and every $\psi \in B(X)$, the sequence $\Gamma = \left\{\psi\Lambda^i\right\}_{i\in\mathbb{N}}$ is not a *pg*-frame.

Proof. For each $x \in X$, $\|\Lambda^n x\| = \|x\|$, for all $n \in \mathbb{N}$. Let $\Gamma = \left\{\psi\Lambda^i\right\}_{i\in\mathbb{N}}$ be a *pg*-frame. By theorem (2.7), we get $\| \lim_{n\to\infty} \Lambda^n x\| = \lim_{n\to\infty} \|x\|$, which is tantamount to $x = 0$, for all $x \in X$, a contradiction. □

The following theorem gives a necessary and sufficient condition for the family $\left\{\Lambda^i\psi\right\}_{i\in\mathbb{N}}$ to be a *pg*-frame, where $\left\{\Lambda^i\right\}_{i\in\mathbb{N}}$ is a *pg*-frame and $\psi \in B(X)$.

Theorem 2.9. *Let $\left\{\Lambda^i\right\}_{i\in\mathbb{N}}$ be a pg-frame for X and $\psi \in B(X)$. Then $\left\{\Lambda^i\psi\right\}_{i\in\mathbb{N}}$ is a pg-frame for X if and only if ψ is bounded below.*

Proof. Let $\left\{\Lambda^i\psi\right\}_{i\in\mathbb{N}}$ be a *pg*-frame for X with bounds C, D. We have

$$C\|x\|^p \leq \sum_{i=1}^{\infty} \left\|\Lambda^i\psi x\right\|^p \leq D\|x\|^p, \quad x \in X. \tag{2.3}$$

Let A, B be *pg*-frame bounds of $\left\{\Lambda^i\right\}_{i\in\mathbb{N}}$. Thus we obtain

$$A\|\psi x\|^p \leq \sum_{i=1}^{\infty} \left\|\Lambda^i\psi x\right\|^p \leq B\|\psi x\|^p, \quad x \in X. \tag{2.4}$$

From (2.3) and (2.4), $C\|x\|^p \leq B\|\psi x\|^p$, and hence $\|\psi x\| \geq (\frac{C}{B})^{1/p}\|x\|$, for each $x \in X$.

Conversely, if there exists $k > 0$ such that for each $x \in X$, $\|\psi x\| \geq k\|x\|$. Then

$$Ak^p\|x\|^p \leq A\|\psi x\|^p \leq \sum_{i=1}^{\infty} \left\|\Lambda^i\psi x\right\|^p \leq B\|\psi x\|^p \leq B\|\psi\|^p\|x\|^p.$$

Hence, $\left\{\Lambda^i\psi\right\}_{i\in\mathbb{N}}$ is a *pg*-frame for X with bounds Ak^p and $B\|\psi\|^p$. □

The above theorem does not hold if ψ is not bounded below.

Example 2.10. Let $\Lambda : \ell^p \mapsto \ell^p$ defined by $\Lambda(x_1, x_2, ...) = \frac{1}{2}(0, x_1, x_2, ...)$. Then $\{\Lambda^i\}_{i\in\mathbb{N}}$ is a $pg-$frame for $X = \ell^p$, where $1 < p < \infty$. Define $\psi : \ell^p \mapsto \ell^p$ by $\psi(x_1, x_2, ...) = (x_2, x_3, ...)$. For $x = (1, 0, 0, 0, ...) \in \ell^p$, the first half of the inequality $A\|x\|^p \leq \sum_{i=1}^{\infty} \left\|\Lambda^i\psi x\right\|^p \leq B\|x\|^p$ is not satisfied.

Since every bounded invertible operator is bounded below, we have the following corollary.

Corollary 2.11. Let $\left\{\Lambda^i\right\}_{i\in\mathbb{N}}$ be a $pg-$frame for X . Let ψ be a bounded invertible operator on X. Then $\left\{\Lambda^i\psi\right\}_{i\in\mathbb{N}}$ is a $pg-$frame for X.

Theorem 2.12. *Let $\left\{\Lambda^i\right\}_{i\in\mathbb{N}}$ be a pg-frame for X and $(\beta_i)_{i\in\mathbb{N}}$ be a bounded sequence of complex numbers. If $\psi_i = \beta_i\psi$, where $\psi \in B(X)$ is invertible, then $\left\{\Lambda^i\psi_i\right\}_{i\in\mathbb{N}}$ is also a pg-frame for X.*

Proof. Let $\{\Lambda^i\}_{i\in\mathbb{N}}$ be a pg-frame for X with bounds A and B. Thus

$$A\|x\|^p \leq \sum_{i=1}^{\infty} \left\|\Lambda^i x\right\|^p \leq B\|x\|^p, \quad \text{for every } x \in X. \tag{2.5}$$

Since $(\beta_i)_{i\in\mathbb{N}}$ is a bounded sequence, there exist $k, K > 0$ such that $k \leq |\beta_i| \leq K$, for each $i \in \mathbb{N}$. Replacing x by $\psi(x)$ in (2.5), we get $A\|\psi x\|^p \leq \sum_{i=1}^{\infty} \left\|\Lambda^i \psi x\right\|^p \leq B\|\psi x\|^p$, for every $x \in X$. Thus, $\sum_{i=1}^{\infty} \left\|\beta_i \Lambda^i \psi x\right\|^p = \sum_{i=1}^{\infty} |\beta_i|^p \left\|\Lambda^i \psi x\right\|^p \leq K^p B\|\psi x\|^p \leq B\|\psi\|^p K^p \|x\|^p$.
Similarly, we can show that $\sum_{i=1}^{\infty} \left\|\beta_i \Lambda^i \psi x\right\|^p \geq \frac{Ak^p}{\|\psi^{-1}\|^p}\|x\|^p$, for every $x \in X$. Thus $\{\Lambda^i \psi_i\}_{i\in\mathbb{N}}$ is a pg-frame for X. □

Conclusion

While studying the literature on iteration operator frames, iterative action of normal operators and $pg-$frames in Banach spaces, the authors got the motivation to define iterated $pg-$frame in Banach spaces. We have established the existence of such frames with the help of examples and obtained necessary and sufficient conditions for the bounded operator Λ so that $\{\Lambda^i\}_{i\in\mathbb{N}}$ is an iterated pg-frame for X. The inquisitive readers can further study the properties and inequalities related to iterated $pg-$frames and might be able to obtain novel stability and duality results by using the concepts and results of this paper. Prudent researchers are also encouraged to think about the duals and generalized duals of iterated $pg-$frames.

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