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INTERPOLATION OF ENTIRE FUNCTIONS FROM LATTICE SAMPLING

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Abstract. We extend an interpolation formula for entire functions of order at most 2, minimal type, of J. M. Whittaker from the square lattice $\mathbb{Z}+\mathbb{Z}i$ in $\mathbb{C}$ to more general lattices $\mathbb{Z}\pi+\mathbb{Z}\tau\pi$ where $\Im m\,\tau > 0$. With it, we prove that an entire function of order at most 2, minimal type, which is bounded on a lattice in $\mathbb{C}$ is necessarily constant, thus generalizing a result of Pólya. We also derive identities about double series.

1. Introduction

Well-known for its relevance in signal processing, Shannon-Kotel'nikov interpolation theorem states that if a function f in one real variable is band-limited to $[-\pi;\pi]$, that is, if f is the Fourier transform of a function with support in $[-\pi;\pi]$, then it is completely determined by its values at the integers — in formula:

$$f(t) = \sin \pi t \sum_{n=-\infty}^{+\infty} \frac{f(n)}{(t-n)} \frac{1}{(-1)^n \pi} \tag{1.1}$$

(see [2] for the history of this result).

In [11, L_{501} p.72] J. M. Whittaker obtained the following two dimensional generalization of (1.1):

$$f(z) \ = \ \sigma(z) \sum_{(m,n)\in\mathbb{Z}^2} \frac{f(m+in)}{(z-m-in)} \frac{1}{\sigma'(m+in)} \tag{1.2}$$

for all entire functions f such that $\overline{\lim}_{r\to\infty} \log(\max_{|z|=r} |f(z)|)/r^2 < \pi/2$, where σ is the Weierstrass sigma function associated to the lattice $\mathbb{Z}+\mathbb{Z}i$ (a definition of σ is given

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below (2.1)). Surprisingly, it seems that a similar result for more general lattices has not yet been established.

This comes perhaps from the fact that a totally analogous formula for such a lattice is in general not true, since, for the Weierstrass sigma function associated to the lattice Γ, the limit $\lim_{\omega\in\Gamma,|\omega|\to\infty} |\sigma'(\omega)|$ needs not be $+\infty$ (for example if $\Gamma = \mathbb{Z} + \mathbb{Z}2i$, see remark 7 below) and so the analogue of the series in (1.2) needs not converge.

By finding an adequate function with which we can replace σ, we have been able to extend Whittaker's result from the square lattice $\mathbb{Z} + \mathbb{Z}i$ in $\mathbb{C}$ to more general lattices. The exact statement and its proof are in section 3, where we also derive from it that an entire function of order at most 2, minimal type, which is bounded on a lattice in $\mathbb{C}$ is necessarily constant. In section 4, we obtain as another consequence some double series identities. In section 5 at last, we make further remarks and, in particular, establish the above mentioned result on the behaviour of $|\sigma'(\omega)|$. But first we introduce in section 2 useful notations and auxiliary results.

2. Notations

We write $\mathbb{H} = \{\tau \in \mathbb{C} \ : \ \Im m\, \tau > 0\}$. A lattice in $\mathbb{C}$ is a discrete subgroup of $\mathbb{C}$ isomorphic to $\mathbb{Z} \times \mathbb{Z}$; any lattice Γ in $\mathbb{C}$ can be written $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\tau = \omega_2/\omega_1 \in \mathbb{H}$ [5, pp. 58 and 272]. The Weierstrass sigma function associated to such a lattice Γ is

$$\sigma(z) = z \prod_{\omega\in\Gamma,\, \omega\neq 0} \Big(1 - \frac{z}{\omega}\Big) \exp\Big(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\Big) \tag{2.1}$$

[1, (2.5) p.52] or [5, p.93]. We also need the Weiertrass zeta function associated to Γ:

$$\zeta(z) = \frac{1}{z} + \sum_{\omega\in\Gamma,\, \omega\neq 0} \frac{1}{z-\omega} + \frac{1}{z} + \frac{z}{\omega^2}$$

[1, (1.1) p.48] or [5, p.94]. With it we define two constants

$$\eta_1 = \zeta\left(\frac{\omega_1}{2}\right), \quad \eta_2 = \zeta\left(\frac{\omega_2}{2}\right) \tag{2.2}$$

(here we follow [1, Theorem 1 p.49]); they satisfy Legendre's relation:

$$\eta_1\omega_2 - \eta_2\omega_1 = \pi i. \tag{2.3}$$

[1, Theorem 2, p.50].

Closely related to η_1 is the conditionally convergent Eisenstein series

$$G_2(\tau) = \sum_{n\in\mathbb{Z},\, n\neq 0} \frac{1}{n^2} + \sum_{m\in\mathbb{Z},\, m\neq 0} \left(\sum_{n\in\mathbb{Z}} \frac{1}{(n+m\tau)^2}\right) \tag{2.4}$$

defined for all $\tau \in \mathbb{H}$ [6, III.6.1.(1) p.162] . It satisfies the relation

$$G_2(-1/\tau) = \tau^2 G_2(\tau) - 2\pi i\tau \tag{2.5}$$

[6, Satz III.6.1 p.163].

To replace σ we shall use (a well chosen modification of) the classical odd theta function, whose definition we recall now [7, (1.6.23) p.15]. If $\tau \in \mathbb{H}$, we put $q = \mathrm{e}^{\tau\pi i}$

and abbreviate $e^{\alpha\tau\pi i}$ to q^α when $\alpha \in \mathbb{R}$. Then, for $z \in \mathbb{C}$, $\tau \in \mathbb{H}$,

$$\vartheta(z,\tau) = 2q^{1/4} \sin z \prod_{m=1}^{+\infty} (1-q^{2m})(1-q^{2m} e^{2iz})(1-q^{2m} e^{-2iz}).$$

It is holomorphic on $\mathbb{C} \times \mathbb{H}$ and verifies, for all $(z,\tau) \in \mathbb{C} \times \mathbb{H}$, $m, n \in \mathbb{Z}$,

$$\vartheta(z + m\pi + n\tau\pi, \tau) = (-1)^{m+n} q^{-n^2} e^{-2inz} \vartheta(z,\tau). \tag{2.6}$$

(this follows from [7, (1.3.2) p.6]). Note also that $\vartheta(z,\tau)$ has simple zeros (in z) at the points of $\mathbb{Z}\pi + \mathbb{Z}\tau\pi$ (use [7, (1.3.10) p.6 and (1.5.7) p.12]) and that

$$\vartheta'(0,\tau) = \frac{\partial \vartheta}{\partial z}(0,\tau) = 2q^{1/4} \prod_{m=1}^{+\infty} (1-q^{2m})^3. \tag{2.7}$$

[7, Exercise 9 p.21] (this is also proved at [1, p.71] but with a definition of ϑ slightly different from the one in [7]).

Closely related to ϑ is the eta function of Dedekind, defined, for $\tau \in \mathbb{H}$ and $q = e^{\tau\pi i}$, by

$$\eta(\tau) = q^{1/12} \prod_{m=1}^{+\infty} (1-q^{2m}) \tag{2.8}$$

(not to be confused with η_1 and η_2 of (2.2)). It satisfies [6, Satz III.6.2A p.164 or Satz IV.4.9 p.219]

$$\eta(-1/\tau) = \sqrt{-i\tau}\, \eta(\tau). \tag{2.9}$$

Finally, we recall that an entire function f is of order at most 2, minimal type, if $f(z) e^{-\delta|z|^2}$ is bounded on $\mathbb{C}$ for every $\delta > 0$ [4, pp.59–61].

3. Interpolation.

Proposition 3.1. Let $\tau \in \mathbb{H}$ and f an entire function of order at most 2, minimal type. Then, for all $z \notin \mathbb{Z}\pi + \mathbb{Z}\tau\pi$,

$$f(z) = e^{iz^2/2\pi\tau} \frac{\vartheta(z,\tau)}{\vartheta'(0,\tau)} \sum_{(m,n)\in\mathbb{Z}^2} \frac{f(m\pi + n\tau\pi)}{(z - m\pi - n\tau\pi)} (-1)^{mn+m+n} q^{n^2/2} r^{m^2/2}$$

with $q = e^{\tau\pi i}$ and $r = e^{-\pi i/\tau}$.

Proof. For all $z \in \mathbb{C}$, $\tau \in \mathbb{H}$, we write

$$\Theta(z,\tau) = e^{iz^2/2\pi\tau} \vartheta(z,\tau). \tag{3.1}$$

It is a holomorphic function on $\mathbb{C} \times \mathbb{H}$, odd and with simple zeros (in z) at the points of $\mathbb{Z}\pi + \mathbb{Z}\tau\pi$ which satisfies, for all $(z,\tau) \in \mathbb{C} \times \mathbb{H}$, $m, n \in \mathbb{Z}$,

$$\Theta(z + m\pi + n\tau\pi, \tau) = (-1)^{mn+m+n} e^{-\pi i\tau n^2/2} e^{\pi i m^2/2\tau} e^{-inz} e^{imz/\tau} \Theta(z,\tau) \tag{3.2}$$

and, differentiating with respect to z,

$$\Theta'(m\pi + n\tau\pi, \tau) = (-1)^{mn+m+n} e^{-\pi i\tau n^2/2} e^{\pi i m^2/2\tau} \vartheta'(0,\tau). \tag{3.3}$$

Let $\Gamma = \mathbb{Z}\pi + \mathbb{Z}\tau\pi$ and fix $z \in \mathbb{C} \setminus \Gamma$. Take $N \in \mathbb{Z}$, $N > 0$, so big that z lies in $K_N = \{\lambda\pi + \mu\tau\pi \,|\, \lambda, \mu \in \mathbb{R},\ |\lambda|, |\mu| < N + \frac{1}{2}\}$. The residue theorem gives

$$\frac{1}{2\pi i}\int_{\partial K_N} \frac{f(\zeta)\, d\zeta}{(\zeta - z)\,\Theta(\zeta,\tau)} = \frac{f(z)}{\Theta(z,\tau)} + \sum_{\omega \in \Gamma \cap K_N} \frac{f(\omega)}{(\omega - z)\,\Theta'(\omega,\tau)}. \tag{3.4}$$

Since $\tau \in \mathbb{H}$, $-1/\tau \in \mathbb{H}$ and then $|q| = |\,\mathrm{e}^{\tau\pi i}\,| < 1$ as well as $|r| = |\,\mathrm{e}^{-\pi i/\tau}\,| < 1$. The expressions (3.2) and (3.3) and the hypothesis on f imply the convergence of the series and the vanishing of the integral in (3.4) when $N \to +\infty$. □

Remark 3.2. An adequate substitute for σ must have simple zeros at the point of the lattice, and grow as $\mathrm{e}^{an^2} \cdot \mathrm{e}^{bm^2}$ when $m, n \to +\infty$ for some $a, b > 0$. Choosing a modification of ϑ is suggested by J. M. Whittker himself, since he defines σ for the lattice $\mathbb{Z} + \mathbb{Z}i$ as $e^{\pi z^2/2}\theta_1(\pi z|i)/\theta_1'(0)\pi$ [11, (12.1) p.72]. The type of the modification is also given there: it must be $e^{c\cdot z^2}\vartheta(z,\tau)$. In (2.6) we have only q^{-n^2}: we must add r^{-bm^2}. We are thus led to (3.1); the only choice which is not totally determined is the one of the factor 1/2: it was made so as to render the right-hand sides of (3.2) and (3.3) as symmetric as possible in n and m.

Proposition 3.3. Let f an entire function of order at most 2, minimal type. If f is bounded on a lattice of $\mathbb{C}$, it is a constant.

Proof. We may suppose that f is bounded on a lattice of the form $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\tau = \omega_2/\omega_1 \in \mathbb{H}$. The function $g(z) = f(\omega_1 z/\pi)$ is then still of order at most 2, minimal type, and is moreover bounded on the lattice $\mathbb{Z}\pi + \mathbb{Z}\tau\pi$. Fix $z \in \mathbb{C}$, $z \neq 0$, and take the integer $k > 0$ so big that $z \notin k(\mathbb{Z}\pi + \mathbb{Z}\tau\pi)$. The function $h(z) = g(kz)$ is also of order at most 2, minimal type. Applied to h and to the point z/k, proposition 1 gives

$$\begin{aligned}
&g(z)\\
&= h(z/k)\\
&= \mathrm{e}^{iz^2/k^2 2\pi\tau}\,\frac{\vartheta(z/k,\tau)}{\vartheta'(0,\tau)}\sum_{(m,n)\in\mathbb{Z}^2}\frac{h(m\pi + n\tau\pi)}{((z/k) - m\pi - n\tau\pi)}(-1)^{mn+m+n}q^{n^2/2}\,r^{m^2/2}\\
&= \mathrm{e}^{iz^2/k^2 2\pi\tau}\,k\,\frac{\vartheta(z/k,\tau)}{\vartheta'(0,\tau)}\sum_{(m,n)\in\mathbb{Z}^2}\frac{g(km\pi + kn\tau\pi)}{(z - km\pi - kn\tau\pi)}(-1)^{mn+m+n}q^{n^2/2}\,r^{m^2/2}
\end{aligned}$$

Let us make k tend to $+\infty$. Since g is bounded on $\mathbb{Z}\pi + \mathbb{Z}\tau\pi$, each term of the series tends to 0 except the one with $(m,n) = (0,0)$. Moreover

$$\begin{aligned}
\lim_{k\to+\infty}\mathrm{e}^{iz^2/k^2 2\pi\tau}\,k\,\frac{\vartheta(z/k,\tau)}{\vartheta'(0,\tau)} &= z\cdot\lim_{k\to+\infty}\mathrm{e}^{iz^2/k^2 2\pi\tau}\,\frac{\vartheta(z/k,\tau)}{z/k}\,\frac{1}{\vartheta'(0,\tau)}\\
&= z\cdot 1\cdot\vartheta'(0,\tau)\cdot\frac{1}{\vartheta'(0,\tau)}\\
&= z.
\end{aligned}$$

Hence, when $k \to +\infty$ we arrive at $g(z) = z\cdot g(0)/z = g(0)$: g is constant, and so is f. □

Remark 3.4. This was proved by Pólya [9] for the lattice $\mathbb{Z} + \mathbb{Z}i$.

4. Double series.

Proposition 4.1. For all $\tau \in \mathbb{H}$ and $k \in \mathbb{Z}$ with $k \geq 0$, we have

$$\sum_{(m,n)\in\mathbb{Z}^2} (m\pi + n\tau\pi)^{2k} (-1)^{mn+m+n}\, q^{n^2/2}\, r^{m^2/2} = 0$$

with $q = \mathrm{e}^{\tau\pi i}$ and $r = \mathrm{e}^{-\pi i/\tau}$.

Proof. The function $f(z) = z$ satisfies the hypothesis of proposition 3.1. Hence, for all $z \notin \mathbb{Z}\pi + \mathbb{Z}\tau\pi$,

$$\begin{aligned} z &= \mathrm{e}^{iz^2/2\pi\tau}\, \frac{\vartheta(z,\tau)}{\vartheta'(0,\tau)} \sum_{(m,n)\in\mathbb{Z}^2} \frac{m\pi + n\tau\pi}{z - m\pi - n\tau\pi} (-1)^{mn+m+n}\, q^{n^2/2}\, r^{m^2/2} \\ &= \Theta(z,\tau)\, \frac{1}{\vartheta'(0,\tau)} \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{m\pi + n\tau\pi}{z - m\pi - n\tau\pi} (-1)^{mn+m+n}\, q^{n^2/2}\, r^{m^2/2} \end{aligned}$$

where Θ is given by (3.1). Let

$$\Xi_0(z) = \frac{1}{\vartheta'(0,\tau)} \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{m\pi + n\tau\pi}{(z - m\pi - n\tau\pi)} (-1)^{mn+m+n}\, q^{n^2/2}\, r^{m^2/2};$$

it is in particular a holomorphic function in z on a neighbourhood of 0 and we have $z = \Theta(z,\tau)\,\Xi_0(z)$. We differentiate both terms of this equation with respect to z and then evaluate at $z = 0$; we get $1 = \Theta'(0,\tau)\,\Xi_0(0) + \Theta(0,\tau)\,\Xi_0'(0)$. Since $\Theta'(0,\tau) = \vartheta'(0,\tau)$ (see (3.3)) and $\Theta(0,\tau) = 0$, we find $1 = \vartheta'(0,\tau)\,\Xi_0(0)$ or $\Xi_0(0) = 1/\vartheta'(0,\tau)$. But on the other hand

$$\begin{aligned} \Xi_0(0) &= \frac{1}{\vartheta'(0,\tau)} \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{m\pi + n\tau\pi}{(0 - m\pi - n\tau\pi)} (-1)^{mn+m+n}\, q^{n^2/2}\, r^{m^2/2} \\ &= \frac{1}{\vartheta'(0,\tau)} \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} (-1)\cdot(-1)^{mn+m+n}\, q^{n^2/2}\, r^{m^2/2}. \end{aligned}$$

Therefore

$$\frac{1}{\vartheta'(0,\tau)} = \frac{-1}{\vartheta'(0,\tau)} \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} (-1)^{mn+m+n}\, q^{n^2/2}\, r^{m^2/2}$$

or

$$1 = (-1)\cdot \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} (-1)^{mn+m+n}\, q^{n^2/2}\, r^{m^2/2}$$

and finally

$$\sum_{(m,n)\in\mathbb{Z}^2} (-1)^{mn+m+n}\, q^{n^2/2}\, r^{m^2/2} = 0.$$

The case $k = 0$ is proved.

Take now $k \in \mathbb{Z}$, $k > 0$. The function $f(z) = z^{2k+1}$ satisfies the hypothesis of proposition 3.1. Hence, for all $z \notin \mathbb{Z}\pi + \mathbb{Z}\tau\pi$,

$$\begin{aligned} & z^{2k+1} \\ &= \mathrm{e}^{iz^2/2\pi\tau}\, \frac{\vartheta(z,\tau)}{\vartheta'(0,\tau)} \sum_{(m,n)\in\mathbb{Z}^2} \frac{(m\pi + n\tau\pi)^{2k+1}}{z - m\pi - n\tau\pi} (-1)^{mn+m+n}\, q^{n^2/2}\, r^{m^2/2} \end{aligned}$$

$$= \Theta(z,\tau)\,\frac{1}{\vartheta'(0,\tau)}\sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)}\frac{(m\pi+n\tau\pi)^{2k+1}}{z-m\pi-n\tau\pi}(-1)^{mn+m+n}\,q^{n^2/2}\,r^{m^2/2}.$$

Let

$$\Xi_k(z)=\frac{1}{\vartheta'(0,\tau)}\sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)}\frac{(m\pi+n\tau\pi)^{2k+1}}{(z-m\pi-n\tau\pi)}(-1)^{mn+m+n}\,q^{n^2/2}\,r^{m^2/2};$$

it is in particular a holomorphic function in z on a neighbourhood of 0 and we have $z^{2k+1} = \Theta(z,\tau)\,\Xi_k(z)$. We differentiate both terms of this equation with respect to z and then evaluate at $z = 0$; we get $0 = \Theta'(0,\tau)\,\Xi_k(0) + \Theta(0,\tau)\,\Xi_k'(0)$. Since $\Theta'(0,\tau) = \vartheta'(0,\tau)$ and $\Theta(0,\tau) = 0$, we find $0 = \vartheta'(0,\tau)\,\Xi_k(0)$ or $\Xi_0(0) = 0$. But on the other hand

$$\begin{aligned}\Xi_k(0) &= \frac{1}{\vartheta'(0,\tau)}\sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)}\frac{(m\pi+n\tau\pi)^{2k+1}}{(0-m\pi-n\tau\pi)}(-1)^{mn+m+n}\,q^{n^2/2}\,r^{m^2/2}\\ &= \frac{1}{\vartheta'(0,\tau)}\sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)}(-1)(m\pi+n\tau\pi)^{2k}(-1)^{mn+m+n}\,q^{n^2/2}\,r^{m^2/2}.\end{aligned}$$

Therefore

$$\sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)}(m\pi+n\tau\pi)^{2k}(-1)^{mn+m+n}\,q^{n^2/2}\,r^{m^2/2}=0$$

and finally

$$\sum_{(m,n)\in\mathbb{Z}^2}(m\pi+n\tau\pi)^{2k}(-1)^{mn+m+n}\,q^{n^2/2}\,r^{m^2/2}=0.$$

□

5. Further remarks.

Remark 5.1. In proposition 3.1 take for f the constant function 1. This gives

$$\begin{aligned}&\mathrm{e}^{-iz^2/2\pi\tau}\,\frac{\vartheta'(0,\tau)}{\vartheta(z,\tau)}\\ &= \sum_{(m,n)\in\mathbb{Z}^2}\frac{(-1)^{mn+m+n}}{(z-m\pi-n\tau\pi)}\,\mathrm{e}^{\tau\pi i n^2/2}\,\mathrm{e}^{-\pi i m^2/2\tau}\\ &= \frac{1}{\tau}\sum_{(p,n)\in\mathbb{Z}^2}\frac{(-1)^{pn+p+n}}{((z/\tau)-n\pi-p(-1/\tau)\pi)}\,\mathrm{e}^{(-1/\tau)\pi i p^2/2}\,\mathrm{e}^{-\pi i n^2/2(-1/\tau)}\\ &= \frac{1}{\tau}\;\mathrm{e}^{-i(z/\tau)^2/2\pi(-1/\tau)}\;\frac{\vartheta'(0,-1/\tau)}{\vartheta(z/\tau,-1/\tau)}\end{aligned}$$

by letting $p = -m$ and using Proposition 3.1 again. So, if $z \notin \mathbb{Z}\pi + \mathbb{Z}\tau\pi$,

$$\mathrm{e}^{-iz^2/\pi\tau}\,\vartheta(z/\tau,-1/\tau) = \vartheta'(0,-1/\tau)\vartheta(z,\tau)/\tau\vartheta'(0,\tau). \tag{5.1}$$

Comparing (2.7) with (2.8), we see that $\eta^3(\tau) = \frac{1}{2}\vartheta'(0,\tau)$. From (2.9), we deduce that $\vartheta'(0,-1/\tau)/\vartheta'(0,\tau) = (-i\tau)\sqrt{-i\tau}$. Then (5.1) gives

$$i\;\mathrm{e}^{-iz^2/\pi\tau}\;\vartheta\left(\frac{z}{\tau},\frac{-1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\,\vartheta(z,\tau).$$

This is the famous Jacobi transformation formula for ϑ, true for every $(z, \tau) \in \mathbb{C} \times \mathbb{H}$ [7, p.17].

Remark 5.2. We prove here the assertion made in the introduction that, for a general lattice Γ and the Weierstrass sigma function associated to it, the limit $\lim_{\omega\in\Gamma, |\omega|\to\infty} |\sigma'(\omega)|$ needs not be $+\infty$.

According to [5, (3.12.6) p.103], we have $\sigma(z+\omega) = \varepsilon\sigma(z)\,\mathrm{e}^{2\eta(z+\omega/2)}$ where $\varepsilon = (-1)^{mn+m+n}$ and $\eta = m\eta_1 + n\eta_2$ if $\omega = m\omega_1 + n\omega_2$. Therefore,

$$\begin{aligned}\sigma'(\omega) &= \lim_{z\to 0} \frac{\sigma(\omega+z)-\sigma(\omega)}{z} = \lim_{z\to 0} \frac{\sigma(\omega+z)}{z} = \lim_{z\to 0} \frac{\sigma(\omega+z)}{\sigma(z)}\cdot\frac{\sigma(z)}{z} \\ &= \lim_{z\to 0} \varepsilon\,\mathrm{e}^{2\eta(z+\omega/2)}\cdot\frac{\sigma(z)}{z} = \varepsilon\,\mathrm{e}^{2\eta\cdot\omega/2}\cdot 1 = \varepsilon\,\mathrm{e}^{\eta\omega} \\ &= (-1)^{mn+m+n}\,\mathrm{e}^{m^2\eta_1\omega_1+mn(\eta_1\omega_2+\eta_2\omega_1)+n^2\eta_2\omega_2}\end{aligned}$$

if $\omega = m\omega_1 + n\omega_2$. From Legendre's relation (2.3) we deduce $\eta_2\omega_1 = \eta_1\omega_2 - \pi i$ and $\eta_2\omega_2 = \eta_1\frac{\omega_2^2}{\omega_1} - \pi i\frac{\omega_2}{\omega_1}$. Hence, writing $\tau = \omega_2/\omega_1$,

$$\begin{aligned}& m^2\eta_1\omega_1 + mn(\eta_1\omega_2+\eta_2\omega_1) + n^2\eta_2\omega_2 \\ &= m^2\eta_1\omega_1 + mn(2\eta_1\omega_1\tau - \pi i) + n^2(\eta_1\omega_1\tau^2 - \pi i\tau).\end{aligned}$$

We will now study the Eisenstein series $G_2(\tau)$ given by (2.4). Since

$$\frac{\pi^2}{\sin^2\pi u} = \sum_{n\in\mathbb{Z}} \frac{1}{(u-n)^2}$$

[5, Exercise 3J p.121] or [10, (I_3) p.141], we have

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n+m\tau)^2} = \frac{\pi^2}{\sin^2\pi m\tau}$$

and then

$$\sum_{m\in\mathbb{Z},\, m\neq 0}\left(\sum_{n\in\mathbb{Z}} \frac{1}{(n+m\tau)^2}\right) = \sum_{m\in\mathbb{Z},\, m\neq 0} \frac{\pi^2}{\sin^2\pi m\tau} = 2\sum_{m=1}^{+\infty} \frac{\pi^2}{\sin^2\pi m\tau}$$

([10, I.105 pp.182–183] prove that this last series does converge). Moreover

$$\sum_{n\in\mathbb{Z},\, n\neq 0} \frac{1}{n^2} = 2\sum_{n=1}^{+\infty}\frac{1}{n^2} = 2\,\frac{\pi^2}{6}$$

[10, I.69 pp.112–114]. Hence

$$G_2(\tau) = 2\pi^2\left(\frac{1}{6} + \sum_{n=1}^{+\infty}\frac{1}{\sin^2 n\pi\tau}\right).$$

Comparing with [10, (X_4) p.184] and taking into account the different notation there (see [10, (VI_4) p.163] where the generators of the lattice are $2\omega_1$ and $2\omega_3$), we find, for $\tau = \omega_2/\omega_1$,

$$\eta_1 = \frac{G_2(\tau)}{2\omega_1}.$$

So we obtain $\eta_1\omega_1 = \frac{1}{2}G_2(\tau)$, $2\eta_1\omega_1\tau - \pi i = G_2(\tau)\tau - \pi i$ and, using relation (2.5), $\eta_1\omega_1\tau^2 - \pi i\tau = \frac{1}{2}G_2(\tau)\tau^2 - \pi i\tau = \frac{1}{2}(G_2(\tau)\tau^2 - 2\pi i\tau) = \frac{1}{2}G_2(-1/\tau)$. We get finally:

$$\sigma'(\omega) = \varepsilon\, \mathrm{e}^{m^2\frac{1}{2}G_2(\tau)+mn(G_2(\tau)\tau-\pi i)+n^2\frac{1}{2}G_2(-1/\tau)}\,.$$

Now, suppose that $\tau = i\lambda$ with $\lambda \in \mathbb{R}$ and $\lambda > 0$. Then

$$G_2(i\lambda) = 2\pi^2\left(\frac{1}{6} + \sum_{n=1}^{+\infty}\frac{1}{\sin^2 n\pi i\lambda}\right) = 2\pi^2\left(\frac{1}{6} - \sum_{n=1}^{+\infty}\frac{1}{\sinh^2 n\pi\lambda}\right).$$

Clearly, the continuous function $\lambda \mapsto G_2(i\lambda)$ from $(0;+\infty)$ to $\mathbb{R}$ is strictly increasing, with $\lim_{\lambda\to 0+} G_2(i\lambda) = -\infty$ and $\lim_{\lambda\to+\infty} G_2(i\lambda) = \pi^2/3$. We find numerically $G_2(i\lambda_0) = 0$ for $\lambda_0 \cong 0.52352170$. Because $-1/\tau = i/\lambda$, we then have

$$\sigma'(\omega) = \varepsilon\, \mathrm{e}^{\frac{1}{2}(G_2(i\lambda)m^2+G_2(i/\lambda)n^2)}\, \mathrm{e}^{(G_2(i\lambda)\lambda-\pi)imn}$$

and

$$|\sigma'(\omega)| = \mathrm{e}^{\frac{1}{2}(G_2(i\lambda)m^2+G_2(i/\lambda)n^2)}\,.$$

We conclude that, for a lattice $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\omega_2/\omega_1 = i\lambda$, where $0 < \lambda < \lambda_0$ or $\lambda > \lambda_0^{-1}$, we do not have $\lim_{\omega\in\Gamma,|\omega|\to\infty} |\sigma'(\omega)| = +\infty$.

Remark 5.3. The result of J. M. Whittaker was extended by Higgins [3] to formulas where the sampled values are that of f and some of its derivatives, but always on the same lattice $\mathbb{Z} + \mathbb{Z}i$. And to understand better the convergence of the series in those formulas, Pogány [8] has studied the growth of the Weierstrass sigma function associated to $\mathbb{Z} + \mathbb{Z}i$.

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