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UNIQUENESS OF MEROMORPHIC FUNCTIONS WHEN TWO DIFFERENTIAL POLYNOMIALS SHARE A SET OF ROOTS OF UNITY

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Abstract. In this paper, we consider the uniqueness problem of meromorphic functions when two differential polynomials share a set of roots of unity ignoring multiplicities. Our results are inspired by a recent work of Khoai and An [Advanced Studies: Euro-Tbilisi Math. J., 15 (2022), 39–51].

1. Introduction, Definitions and Results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [4, 13]. Let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function f, we denote by $T(r,f)$ the Nevanlinna characteristic of f and by $S(r,f)$ any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $(r \to \infty, r \notin E)$. For $a \in \mathbb{C} \cup \{\infty\}$, the functions f and g are said to share the value a CM (counting multiplicities) if f and g have the same set of a-points with counting multiplicities. If f and g have the same set of a-points with ignoring multiplicities, then we say that f and g share the value a IM (ignoring multiplicities). For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_f(a)$ the set of a-points of f counted with its multiplicities and by $\overline{E}_f(a)$ the set of a-points of f where we do not count its multiplicity. For a nonempty subset $S \subset \mathbb{C} \cup \{\infty\}$, we define

$$E_f(S) = \cup_{a\in S} E_f(a) \quad and \quad \overline{E}_f(S) = \cup_{a\in S} \overline{E}_f(a).$$

Two functions f, g are said to share the set S CM, if $E_f(S) = E_g(S)$, and share the set S IM, if $\overline{E}_f(S) = \overline{E}_g(S)$.

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In 1997, Yang and Hua [12] proved the following result for nonconstant meromorphic functions when the first derivative of some power of those share a nonzero value.

Theorem A. Let f and g be two nonconstant meromorphic functions and $n(>11)$ be a positive integer. If $(f^n)'$ and $(g^n)'$ share 1 CM, then either $f(z) = c_1 exp(cz)$ and $g(z) = c_2 exp(-cz)$, where $c_1,\ c_2$ and c are three nonzero constants satisfying $(c_1c_2)^n(nc)^2 = -1$ or $f = tg$ for a constant t such that $t^n = 1$.

In 2007, Bhoosnurmath and Dyavanal [2] extended Theorem A by considering the general order derivatives and their result is as follows.

Theorem B. Let f and g be two nonconstant meromorphic functions and let $n,\ k$ be two positive integers with $n > 3k + 8$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then either $f(z) = c_1 exp(cz)$ and $g(z) = c_2 exp(-cz)$, where $c_1,\ c_2$ and c are three nonzero constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f = tg$ for a constant t such that $t^n = 1$.

In 2018, An and Khoai [1] improved Theorem B by considering set sharing instead of value sharing and proved the following result.

Theorem C. Let f and g be two nonconstant meromorphic functions. Suppose that $n,\ d,\ k$ be positive integers with $n > 2k + \frac{2k+8}{d}$, $d \geq 2$ and $S = \{z \in \mathbb{C} : z^d = 1\}$. If $E_{(f^n)^{(k)}}(S) = E_{(g^n)^{(k)}}(S)$, then one of the following holds:

(1) $f(z) = c_1 exp(cz)$ and $g(z) = c_2 exp(-cz)$, where $c_1,\ c_2$ and c are three nonzero constants such that $(-1)^{kd}(c_1c_2)^{nd}(nc)^{2kd} = 1$;

(2) $f = tg$ for a constant t such that $t^{nd} = 1$.

In 2020, Lahiri and Sinha [10] improved Theorem C by considering the case when two differential polynomials share the set of the roots of unity. The differential polynomials considered in [10] are as follows:

Definition 1.1. Let f be a nonconstant meromorphic function. Then $L(f)$ a differential polynomial of the following form: $L(f) = f^{(l)}$ for $l = 1,\ 2,\ 3$ and $L(f) = \sum_{j=1}^{l-3} a_j f^{(j)} + f^{(l)}$ for $l \geq 4$, where $a_1,\ a_2,\ \cdots, a_{l-3}$ are constants.

Moreover, they pointed out some gap in the proof of Theorem C and overcame this deficiency by the additional assumption that f and g share ∞ IM. Recently, Khoai and An [5] also studied the case when two differential polynomials share the set of the roots of unity. They proved following two theorems by avoiding the additional assumption of sharing ∞ posed in [10].

Let P be a polynomial of degree q,

$$P(z) = (z - a_1)^{m_1}(z - a_2)^{m_2} \ldots (z - a_l)^{m_l}, \tag{1.1}$$

where $a_1,\ a_2,\ \ldots,\ a_l$ are distinct zeros of $P(z)$ and $m_1 + m_2 + \ldots + m_l = q$.

We recall that a polynomial $P(z)$ is called a strong uniqueness polynomial for meromorphic functions if for two arbitrary nonconstant meromorphic functions f and g, and a nonzero constant c, the condition $P(f) = cP(g)$ implies $f = g$ ([7], [8]). We now state the results of Khoai and An.

Theorem D. Let f and g be two nonconstant meromorphic functions, P be a polynomial in the form (1.1), and let $n,\ k$ be positive integers with $n > 3k + \frac{4l+4}{q}$. If $(P(f)^n)^{(k)}$ and $(P(g)^n)^{(k)}$ share 1 CM, then one of the following holds:

(1) $l=1$; $f(z)=a_1+c_1exp(cz)$ and $g(z)=a_1+c_2exp(-cz)$, where $c_1,\ c_2$ and c are three nonzero constants such that $(-1)^k(c_1c_2)^{nq}(nqc)^{2k}=1$;
(2) $l=1$; $f-a_1=t(g-a_1)$ with $t^{nq}=1,\ t\in\mathbb{C}$;
(3) $l\geq 2$; $P(f)=tP(g)$ for a constant t such that $t^n=1,\ t\in\mathbb{C}$. Moreover, if P is a strong uniqueness polynomial, then $f=g$.
In particular, when $P(z)\ =z^q+az^{q-p}+b$ with $a,b\neq 0$ and $q\geq 2p+9$, we have $g=hf$ for a constant h such that $h^q=1$ and $h^p=1$ if $p\geq 4$; $f=g$ if $p\geq 2$ and $(q,p)\ =1$.

Theorem E. Let f and g be two nonconstant meromorphic functions, P be a polynomial in the form (1.1), and let $n,\ k,\ d$ be positive integers with $n>\max\{2k+\frac{2kq+4l+4}{qd},\frac{4k+5l-2}{2q}\}$, $d\geq 2$, and $S=\{z\in\mathbb{C}:z^d=1\}$. If $E_{(P(f)^n)^{(k)}}(S)\ =E_{(P(g)^n)^{(k)}}(S)$, then one of the following holds:

(1) $l=1$; $f(z)=a_1+ac_1exp(cz)$ and $g(z)=a_1+ac_2exp(-cz)$, for nonzero constants $a,\ c_1,\ c_2$ and c such that $a^{2nqd}=1,\ (-1)^k(c_1c_2)^{nq}(nqc)^{2k}=1$;
(2) $l=1$; $f-a_1=t(g-a_1)$ with $t^{nqd}=1,\ t\in\mathbb{C}$;
(3) $l\geq 2$; $P(f)=tP(g)$ for a constant t such that $t^{nd}=1,\ t\in\mathbb{C}$. Moreover, if P is a strong uniqueness polynomial, then $f=g$.
In particular, when $P(z)\ =z^q+az^{q-p}+b$ with $a,b\neq 0$ and $q\geq 2p+9$, we have $g=hf$ for a constant h such that $h^q=1$ and $h^p=1$ if $p\geq 4$, and $f=g$ if $p\geq 2$, $(q,p)\ =1$.

Regarding Theorems D and E it is natural to ask the following question which motivate us to write this paper.

Question 1.2. What will happen if we replace CM sharing by IM sharing in Theorems D and E?

We concentrate our attention to the above question and provide a possible answer in this direction. We now state our main results of the paper.

Theorem 1.3. *Let f and g be two nonconstant meromorphic functions, P be a polynomial of degree $q(>0)$ in the form (1.1), and let $n,\ k$ be two positive integers with $n>6k+\frac{3k+7l+7}{q}$. If $(P(f)^n)^{(k)}$ and $(P(g)^n)^{(k)}$ share 1 IM, then one of the following holds:*

(1) $l=1$; $f(z)=a_1+c_1exp(cz)$ and $g(z)=a_1+c_2exp(-cz)$, where $c_1,\ c_2$ and c are three nonzero constants such that $(-1)^k(c_1c_2)^{nq}(nqc)^{2k}=1$;
(2) $l=1$; $f-a_1=t(g-a_1)$ with $t^{nq}=1,\ t\in\mathbb{C}$;
(3) $l\geq 2$; $P(f)=tP(g)$ for a constant t such that $t^n=1,\ t\in\mathbb{C}$. Moreover, if P is a strong uniqueness polynomial, then $f=g$.
In particular, when $P(z)\ =z^q+az^{q-p}+b$ with $a,b\neq 0$ and $q\geq 2p+9$, we have $g=hf$ for a constant h such that $h^q=1$ and $h^p=1$ if $p\geq 4$; $f=g$ if $p\geq 2$ and $(q,p)\ =1$.

Theorem 1.4. *Let f and g be two nonconstant meromorphic functions, P be a polynomial of degree $q(>0)$ in the form (1.1), and let $n,\ k,\ d$ be positive integers with $n>\max\{2k+\frac{5kq+3k+7l+7}{qd},\frac{2k+2l-1}{q}\}$, $d\geq 2$, and $S=\{z\in\mathbb{C}:z^d=1\}$. If $\overline{E}_{(P(f)^n)^{(k)}}(S)\ =\overline{E}_{(P(g)^n)^{(k)}}(S)$, then one of the following holds:*

(1) $l = 1$; $f(z) = a_1 + ac_1 exp(cz)$ and $g(z) = a_1 + ac_2 exp(-cz)$, for nonzero constants a, c_1, c_2 and c such that $a^{2nqd} = 1$, $(-1)^k(c_1c_2)^{nq}(nqc)^{2k} = 1$;
(2) $l = 1$; $f - a_1 = t(g - a_1)$ with $t^{nqd} = 1$, $t \in \mathbb{C}$;
(3) $l \geq 2$; $P(f) = tP(g)$ for a constant t such that $t^{nd} = 1$, $t \in \mathbb{C}$. Moreover, if P is a strong uniqueness polynomial, then $f = g$.
In particular, when $P(z) = z^q + az^{q-p} + b$ with $a, b \neq 0$ and $q \geq 2p + 9$, we have $g = hf$ for a constant h such that $h^q = 1$ and $h^p = 1$ if $p \geq 4$, and $f = g$ if $p \geq 2$, $(q, p) = 1$.

Note 1.5. We note that from Theorem 1.3 one can obtain Theorem 1 of [16] in the case of sharing the value with ignoring multiplicities.

Though the standard definitions and notations of the value distribution theory are available in [4, 13], we give the following definitions and notations used in this paper.

Definition 1.6. [10] Let f be a nonconstant meromorphic function and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N_2(r, a; f)$ the counting function of the a-points of f, where a simple a-point is counted only once and a multiple a-point is counted only twice.

Definition 1.7. [10] Let f and g be two nonconstant meromorphic functions. For $a, b \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid g = b)$ and $N(r, a; f \mid g \neq b)$ the counting functions of those a-points of f, counted with multiplicities, which are the b-points of g and are not the b-points of g, respectively. By $\overline{N}(r, a; f \mid g = b)$ and $\overline{N}(r, a; f \mid g \neq b)$ we denote the corresponding reduced counting functions.

Definition 1.8. [15] Let f and g be two nonconstant meromorphic functions sharing a value $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f of order p and an a-point of g of order q. We denote by $\overline{N}_L(r, a; f)$ ($\overline{N}_L(r, a; g)$) the counting function of those common a-points of f and g where $p > q$ ($q > p$), each a-point being counted once only.

Definition 1.9. [4, 9] Let f and g share a value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those common a-points of f and g whose multiplicities are not the same. Clearly $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

2. Lemmas

Now we give some necessary lemmas which will be needed to prove the results of this paper.

Lemma 2.1. [4, p. 43] *Let f be a nonconstant meromorphic function and let $a_1, a_2, \cdots, a_q$ be q distinct points in $\mathbb{C} \cup \{\infty\}$. Then*

$$(q-2)T(r, f) \leq \sum_{j=1}^{q} \overline{N}(r, a_j; f) + S(r, f).$$

Lemma 2.2. [3] *Let f and g be two nonconstant meromorphic functions sharing* 1 *IM. Then one of the following cases holds.*

(1) $T(r, f) + T(r, g) \leq 2\{N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g)\}$
$$+3\overline{N}_*(r, 1; f, g) + S(r, f) + S(r, g),$$
(2) $f = \frac{(B+1)g+(A-B-1)}{Bg+(A-B)}$, *for some constants A, B.*

Lemma 2.3. *Let f and g be two nonconstant meromorphic functions sharing 1 IM. Then*

$$\overline{N}_*(r,1;f,g) \leq \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + S(r,f) + S(r,g).$$

Proof. The results follows from Lemma 8 of [14] and Definition 1.9. □

Lemma 2.4. *Let f be a nonconstant meromorphic function and k be a positive integer, and let $P(z)$ be a polynomial of degree $q(>0)$. Then*

$$T\left(r,(P(f))^{(k)}\right) \leq q(k+1)T(r,f) + S(r,f).$$

Proof. From Nevanlinna's first fundamental theorem we have

$$\begin{aligned} T\left(r,(P(f))^{(k)}\right) &= m\left(r,(P(f))^{(k)}\right) + N\left(r,(P(f))^{(k)}\right) \\ &\leq m(r,P(f)) + N(r,P(f)) + kN_1(r,P(f)) + S(r,f) \\ &\leq T(r,P(f)) + kT(r,P(f)) + S(r,f) \\ &\leq q(k+1)T(r,f) + S(r,f), \end{aligned}$$

where $N_1(r,P(f))$ denotes the counting function of poles of $P(f)$ counted once. Hence the lemma follows. □

Lemma 2.5. *Let f be a nonconstant meromorphic function and $n,\ k$ be positive integers with $n \geq k+1$, and let $P(z)$ be a polynomial of degree $q(>0)$. Then*

$$T(r,f) \leq T\left(r,(P(f)^n)^{(k)}\right) + S(r,f).$$

Proof. By a simple calculation we get

$$(P(f)^n)^{(k)} = P(f)^{n-k}L(f), \tag{2.1}$$

where $L(f)$ is a differential polynomial. Therefore,

$$N\left(r,\frac{1}{P(f)}\right) \leq N\left(r,\frac{1}{(P(f)^n)^{(k)}}\right) \quad \textit{and} \quad m\left(r,\frac{L(f)}{P(f)^k}\right) = m\left(r,\frac{(P(f)^n)^{(k)}}{P(f)^n}\right) = S(r,f).$$

Hence

$$\begin{aligned} m\left(r,\frac{1}{P(f)}\right) &\leq nm\left(r,\frac{1}{P(f)}\right) = m\left(r,\frac{1}{P(f)^n}\right) \\ &\leq m\left(r,\frac{1}{(P(f)^n)^{(k)}}\right) + m\left(r,\frac{L(f)}{P(f)^k}\right) \\ &= m\left(r,\frac{1}{(P(f)^n)^{(k)}}\right) + S(r,f). \end{aligned}$$

So by Nevanlinna's first fundamental theorem we get

$$\begin{aligned} T(r,P(f)) = T\left(r,\frac{1}{P(f)}\right) + O(1) &\leq T\left(r,\frac{1}{(P(f)^n)^{(k)}}\right) + S(r,f) \\ &\leq T\left(r,(P(f)^n)^{(k)}\right) + S(r,f). \end{aligned}$$

Hence the lemma follows. □

Lemma 2.6. [5] *Let f be a nonconstant meromorphic function and $n,\ k$ be two positive integers with $n > 2k$, and let $P(z)$ be a polynomial of degree $q(>0)$. Then*

(1) $(n-2k)qT(r,f)+kN(r,P(f))+N\left(r,\frac{P(f)^{n-k}}{(P(f)^n)^{(k)}}\right)\leq T(r,(P(f)^n)^{(k)})+S(r,f)$
$\leq (k+1)nqT(r,f)+S(r,f)$,

(2) $N\left(r,\frac{P(f)^{n-k}}{(P(f)^n)^{(k)}}\right)\leq kqT(r,f)+k\overline{N}(r,f)+S(r,f)$.

Lemma 2.7. *Let f be a nonconstant meromorphic function and $n,\ k$ be two positive integers with $n\geq k+2$, and let $P(z)$ be a polynomial of degree $q(>0)$. If $a\in\mathbb{C}\backslash\{0\}$, then*

$$\frac{n-k-2}{n+k}T(r,f)\leq \overline{N}\left(r,\frac{1}{(P(f)^n)^{(k)}-a}\right)+S(r,f).$$

Proof. To prove the lemma we follow the proof of Lemma 2.8 of [10]. Since by (2.1), $\frac{L(f)}{P(f)^k}=\frac{(P(f)^n)^{(k)}}{P(f)^n}$, we have

$$\begin{aligned}N(r,0;L(f)\mid P(f)\neq 0) &= N\left(r,0;\frac{L(f)}{P(f)^k}\mid P(f)\neq 0\right)\\ &\leq T\left(r,\frac{L(f)}{P(f)^k}\right)+O(1)\\ &= N\left(r,\frac{L(f)}{P(f)^k}\right)+m\left(r,\frac{L(f)}{P(f)^k}\right)+O(1)\\ &= N\left(r,\frac{(P(f)^n)^{(k)}}{P(f)^n}\right)+m\left(r,\frac{(P(f)^n)^{(k)}}{P(f)^n}\right)+O(1)\\ &= k\overline{N}(r,P(f))+k\overline{N}(r,0;P(f))+S(r,f). \qquad (2.2)\end{aligned}$$

Now by (2.1) and (2.2) we have

$$\begin{aligned}\overline{N}(r,0;(P(f)^n)^{(k)}) &= \overline{N}(r,0;P(f)^{n-k}L(f))\\ &\leq \overline{N}(r,0;P(f))+\overline{N}(r,0;L(f)\mid P(f)\neq 0)\\ &\leq (k+1)\overline{N}(r,0;P(f))+k\overline{N}(r,P(f))+S(r,f). \qquad (2.3)\end{aligned}$$

Since a zero of $P(f)$ with multiplicity p is a zero of $(P(f)^n)^{(k)}$ with multiplicity $np-k$, we obtain

$$\begin{aligned}&N(r,0;(P(f)^n)^{(k)})-\overline{N}(r,0;(P(f)^n)^{(k)})\\ &=N(r,0;(P(f)^n)^{(k)}\mid P(f)=0)-\overline{N}(r,0;(P(f)^n)^{(k)}\mid P(f)=0)\\ &\quad+N(r,0;(P(f)^n)^{(k)}\mid P(f)\neq 0)-\overline{N}(r,0;(P(f)^n)^{(k)}\mid P(f)\neq 0)\\ &\geq N(r,0;(P(f)^n)^{(k)}\mid P(f)=0)-\overline{N}(r,0;(P(f)^n)^{(k)}\mid P(f)=0)\\ &\geq (n-1)(k+1)\overline{N}_{(k+1}(r,0;P(f))+(n-k-1)\overline{N}_{k)}(r,0;P(f)), \qquad (2.4)\end{aligned}$$

where $\overline{N}_{(k}(r,0;P(f))$ and $\overline{N}_{k)}(r,0;P(f))$ are respectively the reduced counting functions of zeros of $P(f)$ with multiplicities not less than k and with multiplicities not greater than k.

Since $\overline{N}(r,0;P(f)) = \overline{N}_{k)}(r,0;P(f)) + \overline{N}_{(k+1}(r,0;P(f))$, we obtain from (2.3) and (2.4) that

$$\begin{aligned}\overline{N}(r,0;(P(f)^n)^{(k)}) &\leq (k+1)\overline{N}(r,0;P(f)) + k\overline{N}(r,P(f)) + S(r,f)\\ &\leq (k+1)\overline{N}_{k)}(r,0;P(f)) + (k+1)\overline{N}_{(k+1}(r,0;P(f))\\ &+ k\overline{N}(r,P(f)) + S(r,f)\\ &\leq \frac{k+1}{n-k-1}[N(r,0;(P(f)^n)^{(k)}) - \overline{N}(r,0;(P(f)^n)^{(k)})]\\ &+ k\overline{N}(r,P(f)) + (k+1)\overline{N}_{(k+1}(r,0;P(f))\\ &- \frac{(n-1)(k+1)^2}{n-k-1}\overline{N}_{(k+1}(r,0;P(f)) + S(r,f).\end{aligned}$$

This implies

$$\begin{aligned}\frac{n}{n-k-1}\overline{N}\left(r,0;(P(f)^n)^{(k)}\right) &\leq (k+1)\left[1-\frac{(n-1)(k+1)}{n-k-1}\right]\overline{N}_{(k+1}(r,0;P(f))\\ &+ \frac{k+1}{n-k-1}N\left(r,0;(P(f)^n)^{(k)}\right) + k\overline{N}(r,P(f))\\ &+ S(r,f).\end{aligned}$$

Since $\frac{(n-1)(k+1)}{n-k-1} > 1$, we obtain from above

$$\begin{aligned}\overline{N}\left(r,0;(P(f)^n)^{(k)}\right) &\leq \frac{k+1}{n}N\left(r,0;(P(f)^n)^{(k)}\right) + \frac{k(n-k-1)}{n}\overline{N}(r,P(f))\\ &+ S(r,f). \qquad (2.5)\end{aligned}$$

If α is a pole of $P(f)$ with multiplicity p_1, then α is a pole of $(P(f)^n)^{(k)}$ with multiplicity $np_1 + k \geq n + k$. Therefore we have

$$N(r,(P(f)^n)^{(k)}) \geq (n+k)\overline{N}(r,P(f)) \quad \textit{and} \quad \overline{N}(r,(P(f)^n)^{(k)}) = \overline{N}(r,P(f)).$$

Now applying Lemma 2.1 to $(P(f)^n)^{(k)}$ and using (2.5) we get

$$\begin{aligned}T(r,(P(f)^n)^{(k)}) &\leq \overline{N}(r,0;(P(f)^n)^{(k)}) + \overline{N}(r,a;(P(f)^n)^{(k)}) + \overline{N}(r,(P(f)^n)^{(k)})\\ &+ S(r,(P(f)^n)^{(k)})\\ &\leq \frac{k+1}{n}N(r,0;(P(f)^n)^{(k)}) + \frac{k(n-k-1)}{n}\overline{N}(r,P(f))\\ &+ \overline{N}(r,(P(f)^n)^{(k)}) + \overline{N}(r,a;(P(f)^n)^{(k)}) + S(r,f)\\ &\leq \left[\frac{k+1}{n} + \frac{n+k(n-k-1)}{n(n+k)}\right]T(r,(P(f)^n)^{(k)})\\ &+ \overline{N}(r,a;(P(f)^n)^{(k)}) + S(r,f).\end{aligned}$$

This implies by Lemma 2.5 that

$$\frac{n-k-2}{n+k}T(r,f) \leq \overline{N}\left(r,\frac{1}{(P(f)^n)^{(k)}-a}\right) + S(r,f).$$

This proves the lemma. □

Lemma 2.8. [11] *Let f and g be two nonconstant meromorphic functions and n, k be two positive integers with $n > 2k$. If $(f^n)^{(k)}(g^n)^{(k)} = 1$, then $f = c_1e^{cz}$ and $g = c_2e^{-cz}$ for three nonzero constants c_1, c_2 and c such that $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$.*

Lemma 2.9. [6] *Let* $p,\ q \in \mathbb{N}^*,\ c,\ d,\ e,\ u,\ v,\ t$ *be nonzero constants, and let* $q \geq 2p+4$, *and either* $p \geq 2,\ (p,q)=1,$ *or* $q \geq 4$. *Suppose that* (f,g) *is a nonconstant meromorphic solution of the equation*

$$cf^q + df^{q-p} + e \ = ug^q + vg^{q-p} + t.$$

Then $t = e$ *and there exist nonzero constant* h, *such that* $g = hf$, *with* $h^q = \frac{c}{u}$, $h^{q-p} = \frac{d}{v}$.

3. PROOF OF THE THEOREMS

Proof of Theorem 1.3. Let $F = (P(f)^n)^{(k)}$ and $G = (P(g)^n)^{(k)}$. Then

$$F = \ (P(f)^n)^{(k)} \ = P(f)^{n-k}L(f), \ \ and \ \ G = \ (P(g)^n)^{(k)} \ = P(g)^{n-k}L(g),$$

where L is a differential polynomial. By hypothesis, F and G share the value 1 IM. So by Lemma 2.2 we consider the following two cases:

Case 1. Let

$$\begin{aligned} T(r,F)+T(r,G) \ &\leq \ 2\{N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)\} \\ &+ \ 3\overline{N}_*(r,1;F,G)+S(r,F)+S(r,G). \end{aligned} \tag{3.1}$$

Again by Lemma 2.3 we have

$$\begin{aligned} \overline{N}_*(r,1;F,G) \ &\leq \ \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G) \\ &+ \ S(r,F)+S(r,G). \end{aligned} \tag{3.2}$$

Also by Lemmas 2.4 and 2.5 we see that $S(r,F) \ = S(r,f)$ *and* $S(r,G) \ = S(r,g)$. We note that

$$P(z) \ = (z-a_1)^{m_1}(z-a_2)^{m_2} \ \ldots \ (z-a_l)^{m_l},$$

where $a_1,\ a_2,\ \ldots,\ a_l$ are distinct zeros of $P(z)$ and $m_1+m_2+\ \ldots\ +m_l = q$. Therefore

$$N(r,P(f)) \ = qN(r,f) \ \ and \ \ \overline{N}(r,P(f)) \ = \overline{N}(r,f).$$

$$N_2(r,\infty;F) = 2\overline{N}(r,\infty;F) = 2\overline{N}(r,\infty;(P(f)^n)^{(k)}) = 2\overline{N}(r,\infty;f)$$

$$\text{and } N_2(r,\infty;G) = 2\overline{N}(r,\infty;G) = 2\overline{N}(r,\infty;(P(g)^n)^{(k)}) = 2\overline{N}(r,\infty;g). \tag{3.3}$$

$$\overline{N}(r,0;P(f)) \ = \sum_{i=1}^{l}\overline{N}(r,\frac{1}{f-a_i}) \leq lT(r,f)+S(r,f).$$

$$\begin{aligned} N_2(r,0;F) \ &\leq \ N_2\left(r,\frac{1}{P(f)^{n-k}}\right)+N\left(r,\frac{P(f)^{n-k}}{(P(f)^n)^{(k)}}\right) \\ &= \ 2\overline{N}\left(r,\frac{1}{P(f)}\right)+N\left(r,\frac{P(f)^{n-k}}{(P(f)^n)^{(k)}}\right) \\ &\leq \ 2lT(r,f)+N\left(r,\frac{P(f)^{n-k}}{(P(f)^n)^{(k)}}\right)+S(r,f). \end{aligned} \tag{3.4}$$

Similarly,

$$N_2(r,0;G) \leq 2lT(r,g)+N\left(r,\frac{P(g)^{n-k}}{(P(g)^n)^{(k)}}\right)+S(r,g). \tag{3.5}$$

$$\begin{aligned}\overline{N}(r,0;F) &= \overline{N}\left(r,\frac{1}{(P(f)^n)^{(k)}}\right)\\ &\leq \overline{N}\left(r,\frac{1}{P(f)^{n-k}}\right)+\overline{N}\left(r,\frac{P(f)^{n-k}}{(P(f)^n)^{(k)}}\right)\\ &= \overline{N}\left(r,\frac{1}{P(f)}\right)+\overline{N}\left(r,\frac{P(f)^{n-k}}{(P(f)^n)^{(k)}}\right)\\ &\leq lT(r,f)+\overline{N}\left(r,\frac{P(f)^{n-k}}{(P(f)^n)^{(k)}}\right)+S(r,f). \qquad (3.6)\end{aligned}$$

Similarly,

$$\overline{N}(r,0;G) = \overline{N}\left(r,\frac{1}{(P(g)^n)^{(k)}}\right) \leq lT(r,g)+\overline{N}\left(r,\frac{P(g)^{n-k}}{(P(g)^n)^{(k)}}\right)+S(r,g). \qquad (3.7)$$

Using (3.2)-(3.7) in (3.1) we get

$$\begin{aligned}T(r,F)+T(r,G) &\leq 7l(T(r,f)+T(r,g))+7(\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g))\\ &+ 2N\left(r,\frac{P(f)^{n-k}}{(P(f)^n)^{(k)}}\right)+2N\left(r,\frac{P(g)^{n-k}}{(P(g)^n)^{(k)}}\right)\\ &+ 3\overline{N}\left(r,\frac{P(f)^{n-k}}{(P(f)^n)^{(k)}}\right)+3\overline{N}\left(r,\frac{P(g)^{n-k}}{(P(g)^n)^{(k)}}\right)\\ &+ S(r,f)+S(r,g)\\ &\leq (7l+7)(T(r,f)+T(r,g))+5N\left(r,\frac{P(f)^{n-k}}{(P(f)^n)^{(k)}}\right)\\ &+ 5N\left(r,\frac{P(g)^{n-k}}{(P(g)^n)^{(k)}}\right)+S(r,f)+S(r,g). \qquad (3.8)\end{aligned}$$

For $n>2k$, using Lemma 2.6, we get from (3.8)

$$\begin{aligned}(n-2k)q\{T(r,f)+T(r,g)\} &\leq (7l+7)(T(r,f)+T(r,g))+4N\left(r,\frac{P(f)^{n-k}}{(P(f)^n)^{(k)}}\right)\\ &+ 4N\left(r,\frac{P(g)^{n-k}}{(P(g)^n)^{(k)}}\right)-k\left(N(r,P(f))+N(r,P(g))\right)\\ &+ S(r,f)+S(r,g)\\ &\leq (4kq+3k+7l+7)(T(r,f)+T(r,g))\\ &+ S(r,f)+S(r,g).\end{aligned}$$

Therefore $n\leq 6k+\frac{3k+7l+7}{q}$. This is a contradiction with the assumption that $n>6k+\frac{3k+7l+7}{q}$.

Case 2. Let

$$F=\frac{(B+1)G+(A-B-1)}{BG+(A-B)}. \qquad (3.9)$$

We now consider the following two subcases.

Subcase 2.1. Let $A = B$ ($\neq 0$). Then from (3.9) we get,

$$\frac{1}{F-1} = \frac{BG}{G-1}. \tag{3.10}$$

If $B = -1$, then from (3.10) we get $F.G = 1$, i.e., $(P(f)^n)^{(k)}(P(g)^n)^{(k)} = 1$. Suppose that $l = 1$. Then $P(z) = (z-a_1)^q$. Then we have

$$((f-a_1)^{nq})^{(k)}.((g-a_1)^{nq})^{(k)} = 1.$$

Using Lemma 2.8 and noting that $n > 6k + \frac{3k+7l+7}{q} > 2k$, we obtain $f = a_1 + c_1e^{cz}$ and $g = a_1 + c_2e^{-cz}$ for three nonzero constants c_1, c_2 and c such that $(-1)^k(c_1c_2)^{nq}(nqc)^{2k} = 1$, which is the case (1) of Theorem 1.3. Next we assume that $l \geq 2$. Then using Lemma 2.8 and noting that $n > 6k + \frac{3k+7l+7}{q} > 2k$, we obtain $P(f) = c_1e^{cz}$ and $P(g) = c_2e^{-cz}$ for three nonzero constants c_1, c_2 and c such that $(-1)^k(c_1c_2)^{nq}(nqc)^{2k} = 1$. From this it follows that f, g are entire functions. Since $P(z) = (z-a_1)^{m_1}(z-a_2)^{m_2}\ldots(z-a_l)^{m_l}$, $l \geq 2$, by Picard's theorem, $P(f)$ and $P(g)$ must have a zero, a contradiction.

If $B \neq -1$, then from (3.10) we get

$$\frac{1}{F} = \frac{BG}{(B+1)G-1} \text{ and } G = \frac{-1}{BF-(B+1)}.$$

Therefore,

$$\overline{N}(r,0;F) = \overline{N}\left(r, \frac{1}{B+1}; G\right) \text{ and } \overline{N}(r,\infty;G) = \overline{N}\left(r, \frac{B+1}{B}; F\right). \tag{3.11}$$

Using (3.11) and from Nevanlinna's second fundamental theorem we get

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,0;F) + \overline{N}\left(r, \frac{B+1}{B}; F\right) + \overline{N}(r,\infty;F) + S(r,F) \\ &= \overline{N}(r,0;F) + \overline{N}(r,\infty;G) + \overline{N}(r,\infty;F) + S(r,F) \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,0;G) + \overline{N}\left(r, \frac{1}{B+1}; G\right) + \overline{N}(r,\infty;G) + S(r,G) \\ &= \overline{N}(r,0;G) + \overline{N}(r,0;F) + \overline{N}(r,\infty;G) + S(r,G). \end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13), we obtain

$$\begin{aligned} T(r,F) + T(r,G) &\leq 2\overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) \\ &+ S(r,F) + S(r,G). \end{aligned} \tag{3.14}$$

Now since $\overline{N}(r,\infty;F) = \overline{N}(r,\infty;f)$ and $\overline{N}(r,\infty;G) = \overline{N}(r,\infty;g)$, using (3.6), (3.7) and (3.14) we get

$$\begin{aligned} T(r,F) + T(r,G) &\leq 2lT(r,f) + 2\overline{N}\left(r, \frac{P(f)^{n-k}}{(P(f)^n)^{(k)}}\right) + lT(r,g) + \overline{N}\left(r, \frac{P(g)^{n-k}}{(P(g)^n)^{(k)}}\right) \\ &+ \overline{N}(r,\infty;f) + 2\overline{N}(r,\infty;g) + S(r,f) + S(r,g). \end{aligned} \tag{3.15}$$

For $n > 2k$, using Lemma 2.6, we get from (3.15)

$$\begin{aligned}(n-2k)q\{T(r,f)+T(r,g)\} &\leq 2lT(r,f)+lT(r,g)+\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)\\ &+ 2\overline{N}\left(r,\frac{P(f)^{n-k}}{(P(f)^n)^{(k)}}\right)+\overline{N}\left(r,\frac{P(g)^{n-k}}{(P(g)^n)^{(k)}}\right)\\ &- k(N(r,P(f))+N(r,P(g)))-N\left(r,\frac{P(f)^{n-k}}{(P(f)^n)^{(k)}}\right)\\ &- N\left(r,\frac{P(g)^{n-k}}{(P(g)^n)^{(k)}}\right)+S(r,f)+S(r,g)\\ &\leq (2l+kq+1)T(r,f)+(l+2)T(r,g)+S(r,f)+S(r,g)\\ &\leq (2l+kq+1)(T(r,f)+T(r,g))+S(r,f)+S(r,g).\end{aligned}$$

Therefore $n \leq 3k+\frac{2l+1}{q}$, a contradiction with the assumption that $n > 6k+\frac{3k+7l+7}{q}$.

Subcase 2.2. Let $A \neq B$.

If $B \neq 0$, then from (3.9) we obtain $\overline{N}(r,\infty;F)=\overline{N}(r,\frac{B-A}{B};G)$. Also from (3.9) we can write

$$G=\frac{(B-A)F+(A-B-1)}{BF-(B+1)}.$$

From this it follows that $\overline{N}(r,\infty;G)=\overline{N}(r,\frac{B+1}{B};F)$.
Therefore from the second fundamental theorem of Nevanlinna we get

$$\begin{aligned}T(r,F) &\leq \overline{N}(r,0;F)+\overline{N}\left(r,\frac{B+1}{B};F\right)+\overline{N}(r,\infty;F)+S(r,F)\\ &= \overline{N}(r,0;F)+\overline{N}(r,\infty;G)+\overline{N}(r,\infty;F)+S(r,F)\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}T(r,G) &\leq \overline{N}(r,0;G)+\overline{N}\left(r,\frac{B-A}{B};G\right)+\overline{N}(r,\infty;G)+S(r,G)\\ &= \overline{N}(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+S(r,G).\end{aligned} \tag{3.17}$$

Combining (3.16) and (3.17) we get

$$\begin{aligned}T(r,F)+T(r,G) &\leq \{\overline{N}(r,0;F)+\overline{N}(r,0;G)\}+2\{\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\}\\ &+ S(r,F)+S(r,G).\end{aligned} \tag{3.18}$$

Since $\overline{N}(r,\infty;F)=\overline{N}(r,\infty;f)$ and $\overline{N}(r,\infty;G)=\overline{N}(r,\infty;g)$, using (3.6), (3.7) and (3.18) we get

$$\begin{aligned}T(r,F)+T(r,G) &\leq l(T(r,f)+T(r,g))+2(\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g))\\ &+ \overline{N}\left(r,\frac{P(f)^{n-k}}{(P(f)^n)^{(k)}}\right)+\overline{N}\left(r,\frac{P(g)^{n-k}}{(P(g)^n)^{(k)}}\right)\\ &+ S(r,f)+S(r,g).\end{aligned} \tag{3.19}$$

For $n > 2k$, using Lemma 2.6, we get from (3.19)

$$\begin{aligned}(n-2k)q(T(r,f)+T(r,g)) &\leq l(T(r,f)+T(r,g))+2(\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g))\\ &+ \overline{N}\left(r,\frac{P(f)^{n-k}}{(P(f)^n)^{(k)}}\right)+\overline{N}\left(r,\frac{P(g)^{n-k}}{(P(g)^n)^{(k)}}\right)\\ &- k(N(r,P(f))+N(r,P(g)))-N\left(r,\frac{P(f)^{n-k}}{(P(f)^n)^{(k)}}\right)\\ &- N\left(r,\frac{P(g)^{n-k}}{(P(g)^n)^{(k)}}\right)+S(r,f)+S(r,g)\\ &\leq (l+2)(T(r,f)+T(r,g))+S(r,f)+S(r,g).\end{aligned}$$

Therefore $n \leq 2k+\frac{l+2}{q}$, a contradiction with the assumption that $n > 6k+\frac{3k+7l+7}{q}$.

If $B=0$, then from (3.9) we obtain

$$AF = G+(A-1) \text{ and } G = AF-(A-1). \tag{3.20}$$

If $A\neq 1$, then $\overline{N}(r,0;F)=\overline{N}(r,1-A;G)$ and $\overline{N}(r,0;G)=\overline{N}\left(r,\frac{A-1}{A};F\right)$. Proceeding as above and using Nevanlinna's second fundamental theorem we get

$$\begin{aligned}T(r,F)+T(r,G) &\leq 2\{\overline{N}(r,0;F)+\overline{N}(r,0;G)\}+\{\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\}\\ &+ S(r,F)+S(r,G).\end{aligned}\tag{3.21}$$

Since $\overline{N}(r,\infty;F)=\overline{N}(r,\infty;f)$ and $\overline{N}(r,\infty;G)=\overline{N}(r,\infty;g)$, using (3.6), (3.7) and (3.21) we get

$$\begin{aligned}T(r,F)+T(r,G) &\leq 2l(T(r,f)+T(r,g))+(\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g))\\ &+ 2\overline{N}\left(r,\frac{P(f)^{n-k}}{(P(f)^n)^{(k)}}\right)+2\overline{N}\left(r,\frac{P(g)^{n-k}}{(P(g)^n)^{(k)}}\right)\\ &+ S(r,f)+S(r,g).\end{aligned}\tag{3.22}$$

For $n > 2k$, using Lemma 2.6, we get from (3.22)

$$\begin{aligned}(n-2k)q(T(r,f)+T(r,g)) &\leq 2l(T(r,f)+T(r,g))+(\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g))\\ &+ 2\overline{N}\left(r,\frac{P(f)^{n-k}}{(P(f)^n)^{(k)}}\right)+2\overline{N}\left(r,\frac{P(g)^{n-k}}{(P(g)^n)^{(k)}}\right)\\ &- k(N(r,P(f))+N(r,P(g)))-N\left(r,\frac{P(f)^{n-k}}{(P(f)^n)^{(k)}}\right)\\ &- N\left(r,\frac{P(g)^{n-k}}{(P(g)^n)^{(k)}}\right)+S(r,f)+S(r,g)\\ &\leq (2l+kq+1)(T(r,f)+T(r,g))+S(r,f)+S(r,g).\end{aligned}$$

Therefore $n \leq 3k+\frac{2l+1}{q}$, a contradiction with the assumption that $n > 6k+\frac{3k+7l+7}{q}$.

If $A=1$, then from (3.20) we get $F=G$, i.e., $(P(f)^n)^{(k)} = (P(g)^n)^{(k)}$. Since f, g are not constants, and $n > 6k+\frac{3k+7l+7}{q} > k$, by Lemma 2.5, we see that $(P(f)^n)^{(k)}$ and $(P(g)^n)^{(k)}$ are not constants. Integrating we obtain $P(f)^n = P(g)^n+s$, where s is a polynomial of degree at most $k-1$. If $s \not\equiv 0$, then we have

$$\frac{P(f)^n}{s} - 1 = \frac{P(g)^n}{s}.$$

Applying Lemma 2.1 to $\frac{P(f)^n}{s}$ with values $\infty,\ 0,\ 1$, we have

$$\begin{aligned} T\left(r, \frac{P(f)^n}{s}\right) &\le \overline{N}\left(r, \frac{P(f)^n}{s}\right) + \overline{N}\left(r, \frac{s}{P(f)^n}\right) + \overline{N}\left(r, \frac{s}{P(g)^n}\right) \\ &+ S\left(r, \frac{P(f)^n}{s}\right). \end{aligned} \tag{3.23}$$

We have

$$\begin{aligned} nqT(r,f) &= T(r, P(f)^n) + O(1) \\ &\le T\left(r, \frac{P(f)^n}{s}\right) + (k-1)\log\, r + O(1). \end{aligned}$$

Also

$$\begin{aligned} T\left(r, \frac{P(f)^n}{s}\right) &\le T(r, P(f)^n) + T(r,s) + O(1) \\ &\le nqT(r,f) + (k-1)\log\, r + O(1). \end{aligned}$$

From this we obtain

$$nqT(r,f) - (k-1)\log\, r + O(1) \le T\left(r, \frac{P(f)^n}{s}\right) \le nqT(r,f) + (k-1)\log\, r + O(1).$$

This shows that $S\left(r, \frac{P(f)^n}{s}\right) = S(r,f)$. Similarly, $S\left(r, \frac{P(g)^n}{s}\right) = S(r,g)$. Therefore, using (3.23) we obtain

$$\begin{aligned} T\left(r, P(f)^n\right) &\le T\left(r, \frac{P(f)^n}{s}\right) + (k-1)\log\, r + O(1) \\ &\le \overline{N}\left(r, \frac{1}{P(f)}\right) + \overline{N}\left(r, \frac{1}{P(g)}\right) + \overline{N}(r, P(f)) \\ &+ 2(k-1)\log\, r + S(r,f). \end{aligned} \tag{3.24}$$

Similarly,

$$\begin{aligned} T\left(r, P(g)^n\right) &\le \overline{N}\left(r, \frac{1}{P(f)}\right) + \overline{N}\left(r, \frac{1}{P(g)}\right) + \overline{N}(r, P(g)) \\ &+ 2(k-1)\log\, r + S(r,f). \end{aligned} \tag{3.25}$$

Since f and g are two nonconstant meromorphic functions, we have $T(r,f) \ge \log\, r + O(1)$ and $T(r,g) \ge \log\, r + O(1)$. Thus, from (3.24) and (3.25) we obtain

$$\begin{aligned} nq\{T(r,f) + T(r,g)\} &\le 2\left\{\overline{N}\left(r, \frac{1}{P(f)}\right) + \overline{N}\left(r, \frac{1}{P(g)}\right)\right\} + \overline{N}(r,f) + \overline{N}(r,g) \\ &+ 2(k-1)\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g) \\ &\le (2k+2l-1)\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g), \end{aligned}$$

a contradiction with the assumption that $n > 6k + \frac{3k+7l+7}{q}$. Hence $s \equiv 0$ and $P(f)^n = P(g)^n$. This gives $P(f) = tP(g)$ for a constant t such that $t^n = 1$. If P is a strong uniqueness polynomial, then $f = g$. If $l = 1$, then $(f - a_1) = t(g - a_1)$ with $t^{nq} = 1,\ t \in \mathbb{C}$.

We now consider the case where $P(z) = z^q + az^{q-p} + b$ with $a, b \neq 0$. By the similar argument as above, we see that there exist a constant t with $t^n = 1$ such that $P(f) = tP(g)$, i.e.

$$f^q + af^{q-p} + b = t(g^q + ag^{q-p} + b).$$

Applying Lemma 2.9 and using the same argument as in the proof of Theorem 1[5], we obtain $f = g$. This completes the proof of Theorem 1.3. $\square$

Proof of Theorem 1.4. Let $F_1 = \{(P(f)^n)^{(k)}\}^d$ and $G_1 = \{(P(g)^n)^{(k)}\}^d$. Also S be the set given by $\{z \in \mathbb{C} : z^d = 1\}$. Since $n > \max\{2k + \frac{5kq+3k+7l+7}{qd}, \frac{2k+2l-1}{q}\} > 2k$, by Lemma 2.7, we see that $(P(f)^n)^{(k)} = 1$ has solution. Similarly, $(P(g)^n)^{(k)} = 1$ has solution. Therefore, $E_{(P(f)^n)^{(k)}}(S) \neq \phi$ and $E_{(P(g)^n)^{(k)}}(S) \neq \phi$. Since $(P(f)^n)^{(k)}$ and $(P(g)^n)^{(k)}$ share the set S IM, we see that F_1 and G_1 share the value 1 IM. From Lemma 2.2, we now consider the following two cases:

Case 1. Let

$$\begin{aligned} T(r, F_1) + T(r, G_1) \quad &\leq \quad 2\{N_2(r, 0; F_1) + N_2(r, 0; G_1) + N_2(r, \infty; F_1) + N_2(r, \infty; G_1)\} \\ &+ \quad 3\overline{N}_*(r, 1; F_1, G_1) + S(r, F_1) + S(r, G_1). \end{aligned} \tag{3.26}$$

Since $d \geq 2$, we get

$$\begin{aligned} &N_2(r, \infty; F_1) = 2\overline{N}(r, \infty; F_1) = 2\overline{N}(r, \infty; f) \\ \text{and } &N_2(r, \infty; G_1) = 2\overline{N}(r, \infty; G_1) = 2\overline{N}(r, \infty; g). \end{aligned} \tag{3.27}$$

$$N_2(r, 0; F_1) = 2\overline{N}(r, 0; F_1) \leq 2lT(r, f) + 2\overline{N}\left(r, \frac{P(f)^{n-k}}{(P(f)^n)^{(k)}}\right) + S(r, f). \tag{3.28}$$

$$N_2(r, 0; G_1) = 2\overline{N}(r, 0; G_1) \leq 2lT(r, g) + 2\overline{N}\left(r, \frac{P(g)^{n-k}}{(P(g)^n)^{(k)}}\right) + S(r, g). \tag{3.29}$$

Again from Lemma 2.3 we have

$$\begin{aligned} \overline{N}_*(r, 1; F_1, G_1) \quad &\leq \quad \overline{N}(r, 0; F_1) + \overline{N}(r, 0; G_1) + \overline{N}(r, \infty; F_1) + \overline{N}(r, \infty; G_1) \\ &+ \quad S(r, F_1) + S(r, G_1). \end{aligned} \tag{3.30}$$

From Lemmas 2.4 and 2.5 we get

$$S(r, F_1) = S(r, (P(f)^n)^{(k)}) = S(r, f) \text{ and } S(r, G_1) = S(r, (P(g)^n)^{(k)}) = S(r, g). \tag{3.31}$$

Using (3.27)-(3.31) in (3.26) we get

$$\begin{aligned} &T(r, F_1) + T(r, G_1) \\ &\leq 7(\overline{N}(r, 0; F_1) + \overline{N}(r, 0; G_1)) + 7(\overline{N}(r, \infty; F_1) + \overline{N}(r, \infty; G_1)) \\ &\quad + S(r, F_1) + S(r, G_1) \\ &\leq 7l(T(r, f) + T(r, g)) + 7\overline{N}\left(r, \frac{P(f)^{n-k}}{(P(f)^n)^{(k)}}\right) + 7\overline{N}\left(r, \frac{P(g)^{n-k}}{(P(g)^n)^{(k)}}\right) \\ &\quad + 7(\overline{N}(r, \infty; f) + \overline{N}(r, \infty; g)) + S(r, f) + S(r, g). \end{aligned}$$

For $n > 2k$ and $d \geq 2$, using Lemma 2.6, we get from above

$$\begin{aligned} qd(n-2k)\{T(r,f)+T(r,g)\} &\leq (7l+7)(T(r,f)+T(r,g)) + 5N\left(r, \frac{P(f)^{n-k}}{(P(f)^n)^{(k)}}\right) \\ &+ 5N\left(r, \frac{P(g)^{n-k}}{(P(g)^n)^{(k)}}\right) - kd\left(N(r,P(f))+N(r,P(g))\right) \\ &+ S(r,f)+S(r,g) \\ &\leq (5kq+3k+7l+7)(T(r,f)+T(r,g)) \\ &+ S(r,f)+S(r,g). \end{aligned}$$

Therefore $n \leq 2k+\frac{5kq+3k+7l+7}{qd}$, a contradiction with the assumption that $n > \max\{2k+\frac{5kq+3k+7l+7}{qd}, \frac{2k+2l-1}{q}\}$.

Case 2. Let

$$F_1 = \frac{(B+1)G_1+(A-B-1)}{BG_1+(A-B)}.$$

Then proceeding similarly as in the proof of Case 2 of Theorem 1.3, we obtain either $F_1G_1 = 1$ or $F_1 = G_1$. Suppose that $F_1G_1 = 1$. Then $(P(f)^n)^{(k)}(P(g)^n)^{(k)} = h$, with $h^d = 1$. If $l \geq 2$, then proceeding similarly as in the proof of Theorem 1.3, we obtain a contradiction. Suppose that $l = 1$. Then $P(z) = (z-a_1)^q$. Then for some a with $a^{2nq} = h$, we have

$$\left(\left(\frac{f-a_1}{a}\right)^{nq}\right)^{(k)} \cdot \left(\left(\frac{g-a_1}{a}\right)^{nq}\right)^{(k)} = 1.$$

Using Lemma 2.8 and noting that $n > \max\{2k+\frac{5kq+3k+7l+7}{qd}, \frac{2k+2l-1}{q}\} > 2k$, we obtain $f = a_1 + ac_1e^{cz}$ and $g = a_1 + ac_2e^{-cz}$ for three nonzero constants c_1, c_2 and c such that $(-1)^k(c_1c_2)^{nq}(nqc)^{2k} = 1$, $a^{2nqd} = 1$, which is the case (1) of Theorem 1.4.

Suppose that $F_1 = G_1$. Then $(P(f)^n)^{(k)} = h(P(g)^n)^{(k)}$, with $h^d = 1$. Since f, g are not constants, and $n > \max\{2k+\frac{5kq+3k+7l+7}{qd}, \frac{2k+2l-1}{q}\} > k$, by Lemma 2.5, we see that $(P(f)^n)^{(k)}$ and $(P(g)^n)^{(k)}$ are not constants. Integrating we obtain

$$P^n(f) = hP^n(g) + s,$$

where s is a polynomial of degree $< k$. By similar argument as in the Case 2 of Theorem 1.3, it can be shown that $s = 0$. Therefore, $P(f) = tP(g)$ with $t^{nd} = 1$. If P is a strong uniqueness polynomial, then $f = g$. If $l = 1$, then $(f-a_1) = t(g-a_1)$ with $t^{nqd} = 1$, $t \in \mathbb{C}$.

Now we consider the case where $P(z) = z^q + az^{q-p} + b$ with $a, b \neq 0$. This part can be proved by similar argument as in Theorem 1.3. This completes the proof of Theorem 1.4. □

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