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# THE SHARP BOUND OF THE GENERALIZED ZALCMAN FOR THE INITIAL COEFFICIENT AND CERTAIN SECOND HANKEL DETERMINANTS OF $k^{th}$-ROOT TRANSFORMATION FOR A SUBCLASS OF HOLOMORPHIC FUNCTIONS

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***Abstract***. The objective of this paper is to estimate the sharp bound of the second Hankel determinants $H_{2,1,k}(f)$, $H_{2,2,k}(f)$ and generalized Zalcman for the initial coefficient for $k^{th}$-root transformation to the subclass of Holomorphic functions.

## 1. Introduction

Let $\mathcal{A}$ be the family of mappings $f$ of the type

$$f(z) = z + \sum_{t=2}^{\infty} a_t z^t \tag{1.1}$$

satisfying the normalized conditions $f(0) = 0$ and $f'(0) = 1$ in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{S}$ is the subfamily of $\mathcal{A}$, possessing univalent (schlicht) mappings.

Let $k$ be a positive integer. A domain $\mathbb{U} \in \mathbb{C}$ is said to be $k$-fold symmetric if a rotation of $\mathbb{U}$ about the origin through an angle $\frac{2\pi}{k}$ carries $\mathbb{U}$ to itself. A function $f$ is said to be $k$-fold symmetric in $\mathbb{D}$ if $f\left(e^{\frac{2\pi i}{k}} z\right) = e^{\frac{2\pi i}{k}} f(z)$ for every $z \in \mathbb{D}$. If $f$ is regular and $k$-fold symmetric in $\mathbb{D}$, then

$$f(z) = b_1 z + b_{k+1} z^{k+1} + b_{2k+1} z^{2k+1} + \cdots. \tag{1.2}$$

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Conversely, if $f$ is given by (1.2), then $f$ is $k$-fold symmetric inside the circle of convergence of the series (see [4]). The $k^{th}$-root transform for the mapping $f$ in (1.1) is

$$G(z) := \left[f(z^k)\right]^{\frac{1}{k}} = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1}. \tag{1.3}$$

At this sequel, we have introduced and interpreted the concept of Hankel determinant for $G(z)$ for $f$ in (1.1), with $q, t, k \in \mathbb{N} = \{1, 2, 3, ...\}$, as it's Hankel determinants,

$$H_{q,t,k}(f) = \begin{vmatrix} b_{k(t-1)+1} & b_{kt+1} & \cdots & b_{k(t+q-2)+1} \\ b_{kt+1} & b_{k(t+1)+1} & \cdots & b_{k(t+q-1)+1} \\ \vdots & \vdots & \vdots & \vdots \\ b_{k(t+q-2)+1} & b_{k(t+q-1)+1} & \cdots & b_{k[t+2(q-1)-1]+1} \end{vmatrix} (b_1 = 1). \tag{1.4}$$

Especially, when we set $k$ to equal 1 in equation (1.4), it simplifies to the Hankel determinant $H_{q,t,1}(f)$ as defined by Pommerenke in [13] for the function $f$ described in equation (1.1).

Ali et al. [1] derived exact estimates for $|b_{2k+1} - \mu b_{k+1}^2|$, represents the generalized Fekete-Szegö functional related to the function $G(z)$. The sharp bounds to the functional $|H_{2,2}(f)|$, obtained for $k = 1,\ q = 2$ and $t = 2$ in (1.4), called as Hankel determinant of order two, given by

$$H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2,$$

In the recent years, the research on the estimation of an upper bound to $|H_{2,2}(f)|$ has focused by many authors. The exact estimates of $|H_{2,2}(f)|$ for the functions namely, bounded turning, starlike and convex functions, sub families of $S$, symbolized as $\Re$, $S^*$, and $\mathcal{K}$, respectively fulfilling the conditions $\mathrm{Re} f'(z) > 0$, $\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$, and $\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0$ in the unit disc $\mathbb{D}$, were proved by Janteng et al. [7, 8] and derived the bounds as 4/9, 1, and 1/8 respectively.

Krishna et al [21] introduced the class $S^*\mathcal{K}_s(\beta), (0 \le \beta \le 1)$ with anaytic conditions

$$\mathrm{Re}\left[\frac{2\left\{zf'(z) + \beta z^2 f''(z)\right\}}{(1-\beta)\{f(z) - f(-z)\} + \beta\{zf'(z) + zf'(-z)\}}\right] > 0,\ z \in \mathbb{D}. \tag{1.5}$$

Vani et al. [22] estimated sharp bounds associated with the Zalcman conjecture for the initial coefficients and second Hankel determinants for the class $S^*\mathcal{K}_s(\beta), (0 \le \beta \le 1)$. Panigrahi et al. [11] investigated Fekete-Szego problems for the kth root transform of subclasses of starlike and convex functions with respect to symmetric points.Recently, Srivastava et al.[20] investigated the sharp bound of the third Hankel determinant of the kth-root transformation for bounded turning functions. Motivated by the results obtained by the earlier authors, for our study in this paper, we make an attempt to estimate the sharp bound of the second Hankel determinants $H_{2,1,k}(f)$, $H_{2,2,k}(f)$ and generalized Zalcman for the initial coefficient for $k^{th}$-root transformation to the subclass $S^*\mathcal{K}_s(\beta), (0 \le \beta \le 1)$ of Holomorphic functions

The collection $\mathcal{P}$, of all functions $p$, each one called as Carathéodory function [3] of the form,

$$p(z) = 1 + \sum_{t=1}^{\infty} c_t z^t, \tag{1.6}$$

holomorphic in $\mathbb{D}$ and $\text{Re}p(z) > 0$, for $z \in \mathbb{D}$. The classes $S^*\mathcal{K}_s(\beta)$ and $\mathcal{P}$ are invariant under the rotations, by Carathéodory Theorem (see [5, Vol. I, p. 80, Theorem 3]).

## 2. Preliminaries

For the proof of our main result we need the following lemmas. It contains the well-known formulas: for $c_2$ (e.g., [12, p. 166]), for $c_3$ due to Libera and Zlotkiewicz [9, 10] and for $c_4$ ([14]).

**Lemma 2.1.** ([12]) *For $p \in \mathcal{P}$, then $|c_t| \leq 2$, for $t \in \mathbb{N}$, equality occurs for the function $g(z) = \frac{1+z}{1-z}$, $z \in \mathbb{D}$.*

**Lemma 2.2.** ([6]) *If $p \in \mathcal{P}$, then $|c_i - \mu c_j c_{i-j}| \leq 2$, satisfies for the values $i, j \in \mathbb{N}$, with $i > j$ and $\mu \in [0,1]$, which is same as $|c_{n+k} - \mu c_n c_k| \leq 2$, for $n, k \in \mathbb{N}$, with $\mu \in [0,1]$.*

**Lemma 2.3.** ([2]) *If $p \in \mathcal{P}$, then*

$$|Jc_1^3 - Kc_1c_2 + Lc_3| \leq 2\big(|J| + |K - 2J| + |J - K + L|\big).$$

**Lemma 2.4.** ([15]) *If $p \in \mathcal{P}$, then*

$$2c_2 = c_1^2 + t\zeta,$$
$$4c_3 = c_1^3 + 2c_1t\zeta - c_1t\zeta^2 + 2t\left(1 - |\zeta|^2\right)\eta,$$

*and*

$$8c_4 = c_1^4 + 3c_1^2t\zeta + \left(4 - 3c_1^2\right)t\zeta^2 + c_1^2t\zeta^3 + 4t\left(1 - |\zeta|^2\right)\left(1 - |\eta|^2\right)\xi + 4t\left(1 - |\zeta|^2\right)\left(c_1\eta - c_1\zeta\eta - \bar{\zeta}\eta^2\right),$$

*where $t := 4 - c_1^2$, for some $\zeta$, $\eta$ and $\xi$ such that $|\zeta| \leq 1$, $|\eta| \leq 1$ and $|\xi| \leq 1$.*

**Result 2.5.** Suppose $\Psi : [0,2] \to \mathbb{R}$, $\beta \geq 0$, $k \in \mathbb{N}$ and $\Psi(c)$ defined as

$$\Psi(c) = \frac{\left(24\beta^5 + 72\beta^4 + 84\beta^3 + 40\beta^2 + 7\beta + 1\right)k^2 - 12\beta^3 - 16\beta^2 - 7\beta - 1}{192(\beta+1)^4(2\beta+1)^2(3\beta+1)k^4}c^2 - \frac{1}{4(\beta+1)(3\beta+1)k^2}.$$

Then

$$\Psi(c) \leq 0.$$

*Proof.* We can easily see that

$$\begin{aligned}
&192(\beta+1)^4(2\beta+1)^2(3\beta+1)k^4\Psi(c)\\
=&\left(\left(24\beta^5+72\beta^4+84\beta^3+40\beta^2+7\beta+1\right)k^2-12\beta^3-16\beta^2-7\beta-1\right)c^2\\
&-48(\beta+1)^3(2\beta+1)^2k^2\\
=&-\left[\beta^5\left(192-24c^2\right)k^2+\beta^4\left(768-72c^2\right)k^2\right.\\
&+\beta^3\left(12c^2k^2+\left(1200-84c^2\right)\right)+\beta^2\left(\left(912-40c^2\right)k^2+16c^2\right)\\
&\left.+\beta\left(\left(336-7c^2\right)k^2+7c^2\right)+\left(48-c^2\right)k^2+c^2\right]\leq 0.
\end{aligned}$$

Therefore, $\Psi(c)\leq 0$. □

## 3. Main Results: Bound for $\mathcal{S}^*\mathcal{K}_s$

**Theorem 3.1.** *If $f\in\mathcal{S}^*\mathcal{K}_s(\beta), 0\leq\beta\leq 1$ then*

$$\left|H_{2,1,k}(f)\right|\leq\frac{1}{(2\beta+1)k},$$

*and the result is sharp for $p_o(z)=\frac{1+z^2}{1-z^2}$.*

*Proof.* For $f\in S^*\mathcal{K}_s(\beta)$, there exists a holomorphic function $p\in\mathcal{P}$ such that

$$\frac{2\left\{zf'(z)+\beta z^2f''(z)\right\}}{(1-\beta)\{f(z)-f(-z)\}+\beta\{zf'(z)+zf'(-z)\}}=p(z),\quad z\in\mathbb{D}. \tag{3.1}$$

Substitute the values for $f, f', f''$ and $p$ in (3.1), it simplifies to

$$\begin{aligned}
&2a_2(1+\beta)+3a_3(1+2\beta)z+4a_4(1+3\beta)z^2+5a_5(1+4\beta)z^3+\cdots\\
&=c_1+z\left\{a_3(1+2\beta)+c_2\right\}+z^2\left\{a_3(1+2\beta)c_1+c_3\right\}\\
&\quad+z^3\left\{a_5(1+4\beta)+a_3(1+2\beta)c_2+c_4\right\}+\cdots.
\end{aligned}\tag{3.2}$$

Comparing the coefficients of $z^0, z^1, z^2$ and $z^3$ respectively in (3.1) we obtain

$$a_2=\frac{c_1}{2(1+\beta)};a_3=\frac{c_2}{2(1+2\beta)};a_4=\frac{c_1c_2+2c_3}{8(1+3\beta)};a_5=\frac{c_2^2+2c_4}{8(1+4\beta)}. \tag{3.3}$$

For the mapping $f$ in (1.1), a calculation gives

$$\begin{aligned}
\left[f(z^k)\right]^{\frac{1}{k}}&=\left[z^k+\sum_{n=2}^{\infty}a_nz^{nk}\right]^{\frac{1}{k}}=\left[z+\frac{1}{k}a_2z^{k+1}+\left\{\frac{1}{k}a_3+\frac{(1-k)}{2k^2}a_2^2\right\}z^{2k+1}\right.\\
&+\left\{\frac{1}{k}a_4+\frac{(1-k)}{k^2}a_2a_3+\frac{(1-k)(1-2k)}{6k^3}a_2^3\right\}z^{3k+1}\\
&+\left\{\frac{1}{k}a_5+\frac{(1-k)}{2k^2}(a_3^2+2a_2a_4)+\frac{(1-k)(1-2k)}{2k^3}a_2^2a_3\right.\\
&\left.\left.+\frac{(1-k)(1-2k)(1-3k)}{24k^4}a_2^4\right\}z^{4k+1}+\cdots\right].
\end{aligned}\tag{3.4}$$

Comparing the coefficients of $z^{k+1}$, $z^{2k+1}$, $z^{3k+1}$ and $z^{4k+1}$ in the expressions (1.2) and (3.3), we obtain

$$b_{k+1} = \frac{1}{k}a_2;\ b_{2k+1} = \frac{1}{k}a_3 + \frac{(1-k)}{2k^2}a_2^2;$$

$$b_{3k+1} = \left[\frac{1}{k}a_4 + \frac{(1-k)}{k^2}a_2a_3 + \frac{(1-k)(1-2k)}{6k^3}a_2^3\right];$$

$$b_{4k+1} = \left[\frac{1}{k}a_5 + \frac{(1-k)}{2k^2}(a_3^2 + 2a_2a_4) + \frac{(1-k)(1-2k)}{2k^3}a_2^2a_3 + \frac{(1-k)(1-2k)(1-3k)}{24k^4}a_2^4\right]. \tag{3.5}$$

From (3.3) and (3.5), we obtain

$$\begin{aligned}
b_{k+1} &= \frac{c_1}{2(\beta+1)k},\\
b_{2k+1} &= \frac{c_1^2(1-k)}{8(\beta+1)^2k^2} + \frac{c_2}{2(2\beta+1)k},\\
b_{3k+1} &= \frac{c_1^3(1-k)(1-2k)}{48(\beta+1)^3k^3} + \frac{c_2c_1(1-k)}{4(\beta+1)(2\beta+1)k^2} + \frac{c_2c_1+2c_3}{8(3\beta+1)k}\\
&\text{and}\\
b_{4k+1} &= \frac{(1-k)(1-2k)(1-3k)c_1^4}{384(\beta+1)^4k^4} + \frac{(1-k)(1-2k)c_1^2c_2}{16(1+\beta)^2(1+2\beta)k^3} + \frac{(1-k)c_2^2}{8(2\beta+1)^2k^2}\\
&\quad + \frac{(1-k)c_1\,(c_1c_2+2c_3)}{16(\beta+1)(3\beta+1)k^2} + \frac{c_2^2+2c_4}{(4\beta+1)k}.
\end{aligned} \tag{3.6}$$

Now, for $q = 2$ , $t = 1$ in (1.4), we have

$$H_{2,1,k}(f) = \begin{vmatrix} 1 & b_{k+1} \\ b_{k+1} & b_{2k+1} \end{vmatrix}, \tag{3.7}$$

Using the values of $b_j,\ (j = k+1, 2k+1)$ from (3.6) in (3.7), it simplifies to give

$$H_{2,1,k}(f) = \frac{c_2}{2(2\beta+1)k} - \frac{c_1^2(k+1)}{8(\beta+1)^2k^2}. \tag{3.8}$$

Taking modulus on both side and applying Lemma 2.2, we have

$$\left|H_{2,1,k}(f)\right| = \frac{1}{2(2\beta+1)k}\left|c_2 - \frac{(k+1)(2\beta+1)c_1^2}{4k(\beta+1)^2}\right| \leq \frac{1}{(2\beta+1)k}. \tag{3.9}$$

Form $p_o(z)$, we obtain $c_1 = 0,\ c_2 = 2$, further, $b_{k+1} = 0,\ b_{2k+1} = 1/(1+2\beta)k$ and it follows the result. □

**Theorem 3.2.** *If $f \in \mathcal{S}^*\mathcal{K}_s(\beta), 0 \leq \beta \leq 1$ then*

$$\left|H_{2,2,k}(f)\right| \leq \frac{1}{(2\beta k+k)^2},$$

*and the result is sharp for $p_o(z)$ given in Theorem 2.1.*

*Proof.* Now, for $q = 2$ , $t = 2$ in (1.4), we have

$$H_{2,2,k}(f) = \begin{vmatrix} b_{k+1} & b_{2k+1} \\ b_{2k+1} & b_{3k+1} \end{vmatrix}, \tag{3.10}$$

Putting the calculated values of $b_{jk+1}$, for $j \in \{1,2,3\}$ from (3.6) in (3.10), it simplifies to

$$\begin{aligned} H_{2,2,k}(f) =& \frac{c_1^4(k^2-1)}{192(\beta+1)^4k^4} + \frac{c_2c_1^2}{16(\beta+1)(3\beta+1)k^2} \\ &+ \frac{c_3c_1}{8(\beta+1)(3\beta+1)k^2} - \frac{c_2^2}{4(2\beta+1)^2k^2}. \end{aligned} \tag{3.11}$$

Substituting the values of $c_2$ and $c_3$ from Lemma 2.4 on the right side of the expression (3.11), it simplifies to

$$\begin{aligned} H_{2,2,k}(f) =& \frac{c_1^4\left(k^2-1\right)}{192(\beta+1)^4k^4} + \frac{\beta^2c_1^4}{16\left(3\beta^2+4\beta+1\right)(2\beta k+k)^2} + \\ &\left(4-c_1^2\right)\left(-\frac{c_1^2\zeta^2}{32\left(3\beta^2+4\beta+1\right)k^2} + \frac{c_1\left(1-|\zeta|^2\right)\eta}{16\left(3\beta^2+4\beta+1\right)k^2}\right. \\ &\left.-\frac{\left(4-c_1^2\right)\zeta^2}{16(2\beta k+k)^2} + \frac{(4\beta+1)c_1^2\zeta}{32\left(3\beta^2+4\beta+1\right)(2\beta k+k)^2}\right). \end{aligned} \tag{3.12}$$

Taking modulus on both sides and then applying the triangle inequality in the above expression, using $c_1 := c \in [0,2]$, $|\zeta| := x \in [0,1]$, and $|\eta| \leq 1$, we have

$$\begin{aligned} |H_{2,2,k}(f)| \leq& \frac{c^4\left(k^2-1\right)}{192(\beta+1)^4k^4} + \frac{\beta^2c^4}{16\left(3\beta^2+4\beta+1\right)(2\beta k+k)^2} + \\ &\left(4-c^2\right)\left(\frac{c^2x^2}{32\left(3\beta^2+4\beta+1\right)k^2} + \frac{c\left(1-x^2\right)}{16\left(3\beta^2+4\beta+1\right)k^2}\right. \\ &\left.+\frac{\left(4-c^2\right)x^2}{16(2\beta k+k)^2} + \frac{(4\beta+1)c^2x}{32\left(3\beta^2+4\beta+1\right)(2\beta k+k)^2}\right). \end{aligned}$$

Which is equivalent to

$$\begin{aligned} |H_{2,2,k}(f)| \leq& \frac{c^4\left(k^2-1\right)}{192(\beta+1)^4k^4} + \frac{\beta^2c^4}{16(\beta+1)(3\beta+1)(2\beta k+k)^2} \\ &+\left(4-c^2\right)\left(\frac{c}{16(\beta+1)(3\beta+1)k^2} + \frac{(4\beta+1)c^2x}{32(\beta+1)(2\beta+1)^2(3\beta+1)k^2}\right. \\ &\left.+\frac{(2-c)\left(4\left(3\beta^2+4\beta+1\right)+\left(2\beta^2+4\beta+1\right)c\right)x^2}{32(\beta+1)(3\beta+1)(2\beta k+k)^2}\right). \end{aligned}$$

For $c \in [0,2]$ and $\beta \in [0,1]$ all terms in RHS are positive, therefore,

$$\begin{aligned}
|H_{2,2,k}(f)| \leq & \frac{c^4\left(k^2-1\right)}{192(\beta+1)^4k^4} + \frac{\beta^2c^4}{16(\beta+1)(3\beta+1)(2\beta k+k)^2} \\
& + \left(4-c^2\right)\left(\frac{c}{16(\beta+1)(3\beta+1)k^2} + \frac{(4\beta+1)c^2}{32(\beta+1)(2\beta+1)^2(3\beta+1)k^2}\right. \\
& \left. + \frac{(2-c)\left(4\left(3\beta^2+4\beta+1\right)+\left(2\beta^2+4\beta+1\right)c\right)}{32(\beta+1)(3\beta+1)(2\beta k+k)^2}\right) \\
= & \frac{\left(24\beta^5+72\beta^4+84\beta^3+40\beta^2+7\beta+1\right)k^2-12\beta^3-16\beta^2-7\beta-1}{192(\beta+1)^4(2\beta+1)^2(3\beta+1)k^4}c^4 \\
& - \frac{c^2}{4(\beta+1)(3\beta+1)k^2} + \frac{1}{(2\beta+1)^2k^2} \\
= & \Psi(c)c^2 + \frac{1}{(2\beta k+k)^2} \leq \frac{1}{(2\beta k+k)^2}.
\end{aligned}$$

Since, by Result 2.5 $\Psi(c) \leq 0$.
Form $p_o(z)$, we obtain $c_1 = 0 = c_3$, $c_2 = 2$, further, $b_{k+1} = b_{3k+1} = 0$, $b_{2k+1} = 1/(1+2\beta)k$ and it follows the result. □

**Theorem 3.3.** *If $f \in \mathcal{S}^*\mathcal{K}_s(\beta), 0 \leq \beta \leq 1$ then*

$$|b_{3k+1} - b_{k+1}b_{2k+1}| \leq \frac{1}{2(3\beta+1)k},$$

*and the result is sharp for $p_1(z) = \frac{1+z^3}{1-z^3}$.*

*Proof.* Putting the calculated values of $b_{jk+1}$, for $j \in \{1,2,3\}$ from (3.6) in the expression $b_{3k+1} - b_{k+1}b_{2k+1}$, it simplifies to

$$b_{3k+1} - b_{k+1}b_{2k+1} = \frac{c_1^3(k^2-1)}{24(\beta+1)^3k^3} - \frac{\left(1+3\beta-2\beta^2\right)c_2c_1}{8(\beta+1)(2\beta+1)(3\beta+1)k} + \frac{c_3}{4(3\beta+1)k}. \tag{3.13}$$

Taking modulus on both sides and then applying the Lemma 2.3, we have

$$\begin{aligned}
|b_{3k+1} - b_{k+1}b_{2k+1}| \leq & 2\left[\left|\frac{k^2-1}{24(\beta+1)^3k^3}\right| + \left|\frac{1+3\beta-2\beta^2}{8(\beta+1)(2\beta+1)(3\beta+1)k} - \frac{2\left(k^2-1\right)}{24(\beta+1)^3k^3}\right|\right. \\
& \left. + \left|\frac{(k^2-1)}{24(\beta+1)^3k^3} - \frac{\left(1+3\beta-2\beta^2\right)}{8(\beta+1)(2\beta+1)(3\beta+1)k} + \frac{1}{4(3\beta+1)k}\right|\right] \\
= & 2\left[\frac{k^2-1}{24(\beta+1)^3k^3} + \frac{12\beta^2+10\beta+2+(1-\beta)\left(6\beta^3+9\beta^2+6\beta+1\right)k^2}{24(\beta+1)^3(2\beta+1)(3\beta+1)k^3}\right. \\
& \left. + \frac{18\beta^4k^2+45\beta^3k^2+\left(45k^2-6\right)\beta^2+\left(20k^2-5\right)\beta+4k^2-1}{24(\beta+1)^3(2\beta+1)(3\beta+1)k^3}\right] \\
= & \frac{1}{2(3\beta+1)k}.
\end{aligned}$$

Form $p_o(z)$, we obtain $c_1 = 0 = c_2$, $c_3 = 2$, further, $b_{k+1} = b_{2k+1} = 0$, $b_{3k+1} = 1/2(1+3\beta)k$ and it follows the result. □

## 4. Concluding Remarks and Observations

The problem of finding the sharp estimates of the Second-order Hankel determinant, which is the case when $q = 2$ and $t = 1, 2$ in (1.4) defining $H_{q,t,k}(f)$, is technically tougher than that in the case when $k = 1$.Here, in our present investigation, we have successfully derived the sharp bound of the second-order Hankel determinant $H_{2,t,k}\big(f\big), t = 1, 2$ for functions $f$ in a subclass $\mathcal{S}^*\mathcal{K}_s$ of normalized Holomorphic functions in the open unit disk $\mathbb{D}$. We have also described several earlier developments on the subject, which are related to our main results.

In conclusion, with a view to motivating further researches along the lines of our present investigation, we have chosen to refer the interested reader to several related recent developments (see, for example, [16], [17], [18], [19] and [20]) on bounds for the Hankel determinants $H_{q,t,k}(f)$ for many different values of the positive integers $q$ , $t$ and $k = 1$ except the last one.

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