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ON PERFECTLY PRECONTINUOUS FUNCTIONS

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Abstract. In this paper, we introduce a new class of functions called perfectly precontinuous functions and study their basic properties.

1. Introduction

It is no doubt that generalized open sets play an important and useful role not only in topology but also with respect to its applications such as separation axioms, covering properties, multifunctions, digital topology, topological groups etc. One of the notions which has been the subject of research is preopen and preclosed sets and their applications. In 1964, Corson and Michael [2] introduced the notion of locally dense sets and Mashhour et al. [10] introduced quite independently the same notion but they called it preopen. Earlier in 1958, Ptak [13] used the word nearly open. A subset S of a topological space (X, τ) is said to be *preopen* if $S \subset Int(Cl(S))$. S is preclosed [10] if $X \setminus S$ is preopen or equivalently if $Cl(Int(S)) \subset S$. The intersection of all preclosed sets containing S is called the *preclosure* of S [3], and is denoted by $pCl(S)$. The family of all preopen sets of (X, τ) is denoted by $PO(X)$.

The aim of this paper is to introduce a new type of functions called perfectly precontinuous. This type of functions is weaker than strong continuity and is stronger than perfect continuity. Throughout this paper, X and Y denote topological spaces.

2. Perfectly precontinuous functions

Definition 2.1. A function $f : X \to Y$ is said to be

(1) *perfectly precontinuous* if $f^{-1}(V)$ is clopen in X for every subset $V \in PO(Y)$,

(2) *strongly M-precontinuous* [5] if $f^{-1}(V)$ is open in X for every preopen subset V in Y,

(3) *perfectly continuous* [12] if $f^{-1}(V)$ is clopen in X for every open subset V in Y,

(4) *strongly continuous* [9] if $f^{-1}(B)$ is clopen in X for every subset B in Y,

(5) *contra-strongly M-precontinuous* [5] if $f^{-1}(V)$ is closed in X for every preopen subset V in Y,

(6) *contra-(α, p)-continuous* if $f^{-1}(V)$ is α-open in X for every preopen subset V in Y,

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(7) *preirresolute* [14] if $f^{-1}(V)$ is preopen in X for every subset $V \in PO(Y)$,

(8) *precontinuous* [10] if $f^{-1}(V)$ is preopen in X for every open subset V in Y.

Remark 2.2. It is obvious that strong continuity implies perfect precontinuity and perfect precontinuity implies perfect continuity. But the converses are not necessarily true as shown by the following examples.

Example 2.3. Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b, c\}\}$, and $f : (X, \tau) \to (X, \tau)$ be the identity function. Then f is perfectly continuous but not perfectly precontinuous. Let $A = \{a, b\}$, then $Cl(A) = X$ and $A \subset Int(Cl(A))$. Hence A is preopen in X but $f^{-1}(A) = A$ is not clopen in (X, τ). Therefore, f is not perfectly precontinuous. It is obvious that f is perfectly continuous..

Example 2.4. Let X consists of the set of positive integers, i.e., $X = \mathbb{N}$ with odd-even topology, that is, generated by the base $B = \{\{2k-1, 2k\} \mid k \in \mathbb{N}\}$. The identity function defined on X is perfectly precontinuous but not strongly continuous since the inverse of singletons are not clopen.

Definition 2.5. A filterbase Ψ is said to be *pre-converges* (resp. *clopen-converges* (briefly *cl-converges* [8])) to a point x if every preopen set (resp. clopen set) containing x contains a member of Ψ.

Theorem 2.6. *For a function $f : X \to Y$ from a topological space X into a topological space Y, the following properties are equivalent:*
(1) f is prefectly precontinuous.
(2) The inverse image of every preclosed set D in Y is clopen in X.
(3) $f(\Psi)$ pre-converges to $f(x)$ for every filter Ψ cl-converging to x.
(4) $f(\nu_i)$ pre-converges to $f(x)$ for every net ν_i cl-converging to x.
(5) For every subset S of X, the image of the clopen-adherent points of S is a subset of the pre-adherent points of $f(S)$.

Proof. Obvious.

Definition 2.7. A subspace Y of a topological space W is said to be *pre-embedded* in W if every preopen set of Y is the restriction of a preopen set in W with Y. Equivalently this means that every preclosed set in Y is the restriction of a preclosed set in W with Y.

Proposition 2.8. Let $f : X \to Y$ be a perfectly precontinuous function. If $f(X)$ is pre-embedded in Y, then the surjection $f : X \to f(X)$ is perfectly precontinuous.

Proof. Assume that U is a preopen set of $f(X)$. There exists a set $V \in PO(Y)$ such that $U = V \cap f(X)$. But this implies that $f^{-1}(U) = f^{-1}(V) \cap X = f^{-1}(V)$. Hence $f^{-1}(U)$ is clopen in X.

Proposition 2.9. If $f : X \to Y$ is a perfectly precontinuous function and $g : Y \to Z$ is a precontinuous function, then $g \circ f$ is a perfectly continuous function.

Proof. Let W be any open set of Z. Since g is precontinuous, $g^{-1}(W)$ is preopen in Y. Since f is perfectly precontinuous, $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ is clopen in X and hence $g \circ f$ is perfectly continuous.

Proposition 2.10. If $f : X \to Y$ and $g : Y \to Z$ are perfectly precontinuous functions, then $g \circ f$ is a perfectly precontinuous function.

Proof. This is obvious from Proposition 2.9.

Theorem 2.11. *Let $f : X \to Y$ be a function and $\Lambda = \{X_\alpha \mid \alpha \in A\}$ a locally finite clopen cover of X. For each $\alpha \in A$, suppose $f_\alpha = f \mid X_\alpha$ denote the restriction map. Then f is perfectly precontinuous if and only if each f_α is perfectly precontinuous function.*

Proof. The necessity condition follows from the fact that perfectly precontinuity is invariant under the restriction of domain. The sufficiency condition: Suppose that V is a preopen set of Y. We have $f^{-1}(V) = \bigcup_{\alpha\in A}(f_\alpha)^{-1}(V) = \bigcup_{\alpha\in A}(f^{-1}(V) \cap X_\alpha)$. Since each $f^{-1}(V) \cap X_\alpha$ is clopen in X_α, then it is also in X. Since $f^{-1}(V)$ is the union of clopen sets, then it is open. Besides, since Λ is locally finite, then $\{f^{-1}(V) \cap X_\alpha \mid \alpha \in A\}$ is the union of a locally finite collection of closed sets which is closed. Therefore $f^{-1}(V)$ is also closed and hence clopen.

Definition 2.12. A topological space X is *equipped with pre-partition topology* if every preopen set in X is closed, or every preclosed set in X is open.

Theorem 2.13. *If $f : X \to Y$ is a surjective perfectly precontinuous function which maps clopen sets to closed sets (open sets), then Y is endowed with a pre-partition topology. Moreover, if f is also a bijection which maps preopen (preclosed) sets to preopen (preclosed) sets, then X is also endowed with a pre-partition topology.*

Proof. Straightforward.

Theorem 2.14. *Suppose that $f : X \to Y$ is a function and $g : X \to X\times Y$ is the graph function which is defined by $g(x) = (x, f(x))$ for each $x \in X$. If g is perfectly precontinuous, then f is perfectly precontinuous and X possesses a pre-partition topology. Moreover, if f is perfectly precontinuous and X is endowed with a pre-partition topology, then g is perfectly precontinuous.*

Proof. Let the graph function g be perfectly precontinuous. Since the projection map $p_y : X \times Y \to Y$ is open and continuous, for each preopen set V of Y, $p_y^{-1}(V)$ is preopen in $X \times Y$. Since g is perfectly precontinuous, $f^{-1}(V) = g^{-1}(p_y^{-1}(V))$ is clopen in X and thus the function $f = p_y \circ g$ is perfectly precontinuous. Now let $U \in PO(X)$. Then $U \times Y$ is a preopen set of $X \times Y$. By the fact that g is perfectly precontinuous, then $g^{-1}(U \times Y) = U$ is clopen in X which means that X is endowed with pre-partition topology.

Theorem 2.15. *Let $f_\alpha : X \to X_\alpha$ be a function for each $\alpha \in A$. Let also $f : X \to \prod_{\alpha\in A} X_\alpha$ be defined by $f(x) = (f_\alpha(x))$ for each $x \in X$. If f is perfectly precontinuous, then each f_α is perfectly precontinuous.*

Proof. For each $\alpha \in A$, the projection $p_\alpha : \prod_{\alpha\in A} X_\alpha \to X_\alpha$ is open and continuous and the inverse image of any preopen set in X_α is preopen. For any preopen set V_α in X_α, $p_\alpha^{-1}(V_\alpha)$ is preopen in $\Pi_{\alpha\in A}X_\alpha$. Since f is perfectly precontinuous, $f_\alpha^{-1}(V_\alpha) = (p_\alpha \circ f)^{-1}(V_\alpha)$ is clopen in X. Therefore, f_α is perfectly precontinuous.

Recall that a subset S of a topological space X is *mildly compact* [15] (resp. *strongly compact* [1], [11]) if every cover of S by clopen (resp. preopen) sets of X has a finite subcover. A topological space X is said to be *strongly compact* [1], [11] if every preopen cover of X admits a finite subcover.

Theorem 2.16. *If $f : X \to Y$ is perfectly precontinuous and $A \subset X$ mildly compact, then $f(A)$ is strongly compact. In addition, the perfectly precontinuous image of a mildly compact space is strongly compact.*

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be any cover of $f(A)$ by preopen sets of Y. Then $f(A) \subset \cup\{V_\alpha : \alpha \in \Delta\}$ and $A \subset f^{-1}(f(A)) \subset \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$. For each $\alpha \in \Delta$, $f^{-1}(V_\alpha)$ is clopen in X and A is mildly compact. Hence there exists a finite subset Δ_0 of Δ such that $A \subset \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\}$ and $f(A) \subset \cup\{V_\alpha : \alpha \in \Delta_0\}$. Therefore, $f(A)$ is strongly compact.

Definition 2.17. A topological space X is said to be

(1) *pre-T_0* [6] if for each pair of distinct points x and y in X, there exists a preopen set

containing one of the points x and y but not the other,

(2) *ultra Hausdorff* [15] if for each pair of distinct points x and y in X, there exist disjoint clopen sets U and V containing x and y, respectively.

Lemma 2.18. ([6], Theorem 3). *Every topological space is pre-T_0.*

Remark 2.19. The following proof is given by Professor M. Ganster in a private conversation in 2003. The proof is clear and we introduce it. Let X be a topological space and x be any point of X. Then either $Int\{x\} \neq \emptyset$ or $Int\{x\} = \emptyset$.

(i) If $Int\{x\} \neq \emptyset$, then we have $x \in Int\{x\} \subset \{x\}$ and $\{x\} = Int\{x\}$. Hence $\{x\}$ is open and hence preopen.

(ii) If $Int\{x\} = \emptyset$, then $X = X - Int\{x\} = Cl(X - \{x\})$. Therefore, $(X - \{x\}) \subset X = Int(Cl(X - \{x\}))$ and hence $X - \{x\}$ is preopen.

Theorem 2.20. *If $f : X \to Y$ is a perfectly precontinuous injection, then X is an ultra Hausdorff space.*

Proof. Let x and y be any two distinct points of X. Then $f(x) \neq f(y)$ since f is injective. By Lemma 2.18, there exists a preopen set V containing one of $f(x)$ or $f(y)$ but not the other. Now, assume that $f(x) \in V$. Since f is perfectly precontinuous, $f^{-1}(V)$ is a clopen set X such that $x \in f^{-1}(V)$ and $y \notin f^{-1}(V)$. Hence $f^{-1}(V)$ and $X \setminus f^{-1}(V)$ are disjoint clopen sets containing x and y, respectively. Therefore, X is ultra Hausdorff.

Theorem 2.21. *If $f : X \to Y$ is a perfectly precontinuous and S is a connected set in X, then $f(S)$ is a singleton. Moreover, every perfectly precontinuous function from a connected space to a space is constant and thus strongly continuous.*

Proof. Suppose that S is a connected set of X such that $f(S)$ is not a singleton. Then, there exist distinct points $x, y \in S$ such that $f(x) \neq f(y)$. If not so, then the proof finishes. Let $f(x), f(y)$ be two distinct points of $f(S)$. By Lemma 2.18, there exists a preopen set V containing one of $f(x)$ or $f(y)$ but not the other. Now, assume that $f(x) \in V$. Since f is perfectly precontinuous, $f^{-1}(V) \cap S$ is a nonempty clopen subset of S. On the other hand, since $y \notin f^{-1}(V)$ and $y \in S$, $y \in S \setminus f^{-1}(V)$ and $S \setminus f^{-1}(V)$ clopen in S and $(S \setminus f^{-1}(V)) \cap (S \cap f^{-1}(V)) = \emptyset$. This is contrary to the fact that S is connected.

Corollary 2.22. If $f : X \to Y$ is a perfectly continuous, where Y is a T_0 space, and S is a connected set in X, then $f(S)$ is a singleton. Moreover, every perfectly continuous function from a connected space to a T_0 space is constant and thus strongly continuous.

Proof. The proof is similar with those of Theorem 2.21.

A topological space X is said to be *sum connected* [7] if each point $x \in X$ has an open connected neighbourhood of x or equivalently each component of X is open in X.

Theorem 2.23. *If $f : X \to Y$ is a perfectly continuous surjection from a sum connected space X onto a T_0 space Y, then f is strongly continuous.*

Proof. Let X be a sum space. Then every component of X is clopen in X. Let B be any subset of Y. By Corollary 2.22, since any connected component containing a point of $f^{-1}(B)$ is contained in $f^{-1}(B)$. Then $f^{-1}(B)$ is the union of connected components and $f^{-1}(B)$ is clopen in X. Therefore, f is strongly continuous.

Corollary 2.24. Let X be a sum connected and Y be T_0. Then for a surjective function $f : X \to Y$, the following properties are equivalent:

(1) f is perfectly continuous;

(2) f is perfectly precontinuous;
(3) f is strongly continuous.

Proof. This follows immediately from Theorem 2.23 and Remark 2.2.

Lemma 2.25 ([4], Theorem 2.8). . *For a subset $A \subset (X, \tau)$, the following conditions are equivalent:*
(1) A is clopen.
(2) A is α-open and closed.
(3) A is nearly open (= preopen) and closed.

Now, the following result is an immediate consequence of Lemma 2.25.

Theorem 2.26. *For a $f : X \to Y$, the following are equivalent:*
(1) f is perfectly precontinuous.
(2) f is strongly M-precontinuous and contra-strongly M-precontinuous.
(3) f is contra-(α, p)-continuous and contra-strongly M-precontinuous.
(4) f is preirresolute and contra-strongly M-precontinuous.

Conclusion

In this paper, we introduced a new type of continuity called perfect precontinuity which is weaker than strong continuity and stronger than perfect continuity. We also presented some of its fundamental properties. This paper not only fills the gap in these types of continuities but also motivates for further research in different aspects.

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