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θ-SOMEWHAT NEARLY-OPEN SETS AND θ-SOMEWHAT NEARLY-CONTINUITY

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Abstract. In this paper, a new class of open set called θ-somewhat nearly-open set is introduced and studied. We then investigate its relation to other well-known types of open sets such as the classical open, θ-open, somewhat-open, and somewhat nearly-open sets in a topological space. Moreover, some characterizations of the notions of θ-somewhat nearly-continuous and strongly θ-somewhat nearly-continuous functions from an arbitrary topological space into the product space are obtained.

1. Introduction and Preliminaries

Numerous mathematicians have been developing different variations of open sets including its weaker and stronger versions. Levine [22] made the first attempt in 1963, when he introduced the concepts of semi-open set, semi-closed set, and semi-continuity of a function.

In 1968, Veličko [31] introduced the concept of θ-continuity between topological spaces as well as the concepts of θ-closure and θ-interior of a set. Several authors then investigated and discovered intriguing results concerning θ-open sets, see [1, 10, 12, 13, 18, 19, 20, 21, 23, 27, 29].

Let $(X, \mathfrak{T})$ be a topological space and $A \subseteq X$. The θ-closure and θ-interior of A are, respectively, denoted and defined by

$$Cl_\theta(A) = \{x \in X : Cl(U) \cap A \neq \varnothing \text{ for every open set U containing } x.\}$$

and

$$Int_\theta(A) = \{x \in X : Cl(U) \subseteq A \text{ for some open set U containing } x.\}$$

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where $Cl(U)$ is the closure of U in X. A subset A of X is θ-closed if $Cl_\theta(A) = A$ and θ-open if $Int_\theta(A) = A$. Equivalently, A is θ-open if and only if $X \setminus A$ is θ-closed. It is known that the collection $\mathcal{T}_\theta$ of all θ-open sets forms a topology on X, which is strictly coarser than $\mathcal{T}$.

In 1971, Gentry and Hoyle III [16] introduced the class of somewhat-continuous functions and somewhat-open functions. Somewhat-continuous function is a generalization of continuity requiring nonempty inverse images of open sets to have nonempty interiors instead of being open. The concepts of somewhat-interior and somewhat-closure of a subset of a topological space have been subsequently defined and the concepts of somewhat-open and somewhat-closed sets have been used to characterize somewhat-continuity, see [6, 7, 8]. Different modifications of somewhat-open sets were studied in some papers, see [2, 3, 30].

A subset U of a topological space X is said to be somewhat-open (briefly sw-open) if $U = \varnothing$ or if there exists $x \in U$ and an open set V such that $x \in V \subseteq U$. A set is called somewhat-closed if its complement is somewhat open. The somewhat-closure and somewhat-interior of $A \subseteq X$ are, respectively, denoted and defined by $swCl(A) = \bigcap\{F : F \text{ is somewhat-closed and } A \subseteq F\}$ and $swInt(A) = \bigcup\{U \subseteq A : U \text{ is somewhat-open}\}$.

In 1987, Piotrowski [28] introduced a new class of continuity called somewhat nearly-continuous functions. Due to the importance of this newly defined versions of continuity, Ameen [2, 3] continued the work of Piotrowski and proved some notable properties and characterizations.

A subset U of a topological space X is said to be somewhat nearly-open (briefly swn-open) if $U = \varnothing$ or $Int(Cl(U)) \neq \varnothing$. The complement of somewhat nearly-open is called somewhat nearly-closed (briefly swn-closed) that is, F is is called swn-closed if $Cl(Int(F)) \neq X$ or $F = X$. The somewhat nearly closure and somewhat nearly interior of $A \subseteq X$ are, respectively, denoted and defined by $swnCl(A) = \bigcap\{F : F \text{ is somewhat nearly-closed and } A \subseteq F\}$ and $swnInt(A) = \bigcup\{U \subseteq A : U \text{ is somewhat nearly-open}\}$.

A topological space X is said to be a *regular* if for each $x \in X$ and open set U containing x, there exists an open set W such that $x \in W \subseteq Cl(W) \subseteq U$. A subset A of X is said to be *regular open* (resp., *regular closed*) if $A = Int(Cl(A))$ (resp., $A = Cl(Int(A))$). It is known that every regular open set is an open set.

Let $\mathcal{A}$ be an indexing set and $\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a family of topological spaces. For each $\alpha \in \mathcal{A}$, let $\mathcal{T}_\alpha$ be the topology on Y_α. The Tychonoff topology on $\Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ is the topology generated by a subbase consisting of all sets $p_\alpha^{-1}(U_\alpha)$, where the projection map $p_\alpha : \Pi\{Y_\alpha : \alpha \in \mathcal{A}\} \to Y_\alpha$ is defined by $p_\alpha(\langle y_\beta \rangle) = y_\alpha$, U_α ranges over all members of $\mathcal{T}_\alpha$, and α ranges over all elements of $\mathcal{A}$. Corresponding to $U_\alpha \subseteq Y_\alpha$, denote $p_\alpha^{-1}(U_\alpha)$ by $\langle U_\alpha \rangle$. Similarly, for finitely many indices $\alpha_1, \alpha_2, \ldots, \alpha_n$, and sets $U_{\alpha_1} \subseteq Y_{\alpha_1}$, $U_{\alpha_2} \subseteq Y_{\alpha_2}, \ldots, U_{\alpha_n} \subseteq Y_{\alpha_n}$, the subset

$$\langle U_{\alpha_1} \rangle \cap \langle U_{\alpha_2} \rangle \cap \cdots \cap \langle U_{\alpha_n} \rangle = p_{\alpha_1}^{-1}(U_{\alpha_1}) \cap p_{\alpha_2}^{-1}(U_{\alpha_2}) \cap \cdots \cap p_{\alpha_n}^{-1}(U_{\alpha_n})$$

is denoted by $\langle U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n} \rangle$. We note that for each open set U_α subset of Y_α, $\langle U_\alpha \rangle = p_\alpha^{-1}(U_\alpha) = U_\alpha \times \Pi_{\beta \neq \alpha} Y_\beta$. Hence, a basis for the Tychonoff topology consists of sets of the form $\langle B_{\alpha_1}, B_{\alpha_2}, \ldots, B_{\alpha_n} \rangle$, where B_{α_i} is open in Y_{α_i} for every $i \in \{1, 2, \ldots, n\}$.

Now, the projection map $p_\alpha : \Pi\{Y_\alpha : \alpha \in \mathcal{A}\} \to Y_\alpha$ is defined by $p_\alpha(\langle y_\beta \rangle) = y_\alpha$ for each $\alpha \in \mathcal{A}$. It is known that every projection map is a continuous open surjection. Also, it is well known that a function f from an arbitrary space X into the Cartesian product

Y of the family of spaces $\{Y_\alpha : \alpha \in \mathcal{A}\}$ with the Tychonoff topology is continuous if and only if each coordinate function $p_\alpha \circ f$ is continuous, where p_α is the α-th coordinate projection map.

In this paper, we will introduce a new class of open set called θ-somewhat nearly-open (briefly θ-*swn*-open) set and investigate its relation to the other well-known types of open sets. Furthermore, some topological concepts related to θ-*swn*-open sets will be defined and studied.

2. Main Results

2.1. θ-Somewhat Nearly-Open and θ-Somewhat Nearly-Closed Functions. In this section, we define and characterize the concepts of θ-somewhat nearly-open and θ-somewhat nearly-closed functions.

Definition 2.1. Let X be a topological space. A subset A of X is said to be θ-*somewhat nearly-open* (briefly θ-*swn*-open) if for every $x \in A$, there exists an *swn*-open set U containing x such that $swnCl(U) \subseteq A$. A subset F of X is said to be θ-*somewhat nearly-closed* (briefly θ-*swn*-closed) if its complement $X \setminus F$ is θ-somewhat nearly-open.

Remark 2.2. Let A be a subset of a topological space X. If A is both *swn*-open and *swn*-closed set, then A is θ-*swn*-open.

Theorem 2.3. *Let X be a topological space and $A \subseteq X$.*

(i) *If A is θ-open, then A is θ-swn-open.*
(ii) *If A is θ-swn-open, then A is swn-open.*

Proof. (i) If A is θ-open, then for each $x \in A$, there exists an open set U containing x such that $Cl(U) \subseteq A$. It follows that $x \in U = Int(U) \subseteq Int(Cl(U)) \neq \emptyset$ so that U is an *swn*-open set that contains x. Hence, $swnCl(U) \subseteq Cl(U) \subseteq A$, see ([3], Diagram I).

(ii) Assume that A is θ-*swn*-open. If $A = \varnothing$, then we are done. Suppose that $A \neq \varnothing$. Then for every $x \in A$ there exists an *swn*-open set U containing x such that $swnCl(U) \subseteq A$. Since U is *swn*-open, using [3, Remark 3.1], there exists an open set G such that $\varnothing \neq G \subseteq Cl(U) \subseteq Cl(A)$. Consequently, $Int(Cl(A)) \neq \varnothing$. Thus, A is *swn*-open. $\square$

In view of [3, Diagram I] and Theorem 2.3, the following remark holds.

Remark 2.4. The following diagram holds for any subset of a topological space.

$$\begin{tikzcd}
\theta\text{-open} \arrow[r] \arrow[ddd] & \text{open} \arrow[d] \arrow[dr] & \\
& \text{pre-open} \arrow[d] & \alpha\text{-open} \arrow[l] \arrow[d] \\
& \beta\text{-open} \arrow[d] & \text{semi-open} \arrow[l] \arrow[d] \\
\theta\text{-}swn\text{-open} \arrow[r] & swn\text{-open} & sw\text{-open} \arrow[l]
\end{tikzcd}$$

We note that the above diagram is also true for their respective closed sets.

The following examples show that implications in the above diagram related to θ-*swn*-open are not reversible. Counterexamples for the other reverse implications are found in [3, p. 362] and other literature.

Example 2.5. (i) Consider the space $X = \{a, b, c\}$ with topology

$$\mathcal{T} = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}.$$

The sets $\{c\}$, $\{a, b\}$ and $\{b, c\}$ are θ-*swn*-open but not θ-open.

(ii) Consider the space $X = \{a, b, c\}$ with topology $\mathcal{T} = \{\varnothing, X, \{a\}, \{a, b\}, \{a, c\}\}$. The sets $\{a\}$, $\{a, b\}$ and $\{a, c\}$ are *swn*-open but not θ-*swn*-open.

Remark 2.6. θ-*swn*-open is independent to the notions of open, preopen [24], semi-open [22], α-open [26] , β-open [15] and *sw*-open.

To see this, consider the space $X = \{a, , b, c, d\}$ with topology $\mathcal{T} = \{\varnothing, X, \{a\}, \{c, d\}, \{a, c, d\}\}$. The sets $\{c\}$, $\{d\}$, $\{b, c\}$ and $\{b, d\}$ are θ-*swn*-open but not *sw*-open. Also, consider the topological space X given in Example 2.5 (i). The set $\{a, b\}$ is θ-*swn*-open but not β-open. By ([3], Diagram I), it follows that $\{a, b\}$ is also not open, preopen, semi-open and α-open.

On the other hand, consider the topological space X in Example 2.5 (ii). The set $\{a\}$ is open which implies that $\{a\}$ is also preopen, semi-open, α-open, β-open and *sw*-open. However, $\{a\}$ is not θ-*swn*-open.

Remark 2.7. The arbitrary union of θ-*swn*-open sets is θ-*swn*-open.

Remark 2.8. The finite intersection of θ-*swn*-open sets may not be a θ-*swn*-open. This means that the family of θ-*swn*-open sets does not form a topology on a set X.

Consider the topological space X in Example 2.5 (i). The sets $\{a, b\}$ and $\{b, c\}$ are θ-*swn*-open sets but $\{a, b\} \cap \{b, c\} = \{b\}$ is not a θ-*swn*-open.

Definition 2.9. Let X be a topological space and $A \subseteq X$.

(i) The θ-*swn*-interior of A is denoted and defined by $\theta\text{-}swnInt(A)=\bigcup\{U : U$ is θ-*swn*-open and $U \subseteq A\}$. We note that by Remark 2.7, $\theta\text{-}swnInt(A)$ is the largest θ-*swn*-open set contained in A. Moreover, $x \in \theta\text{-}swnInt(A)$ if and only if there exists a θ-*swn*-open set U containing x such that $U \subseteq A$.

(ii) The θ-*swn*-closure of A is denoted and defined by $\theta\text{-}swnCl(A)=\bigcap\{F : F$ is θ-*swn*-closed and $A \subseteq F\}$. We note that by Remark 2.7, $\theta\text{-}swnCl(A)$ is the smallest θ-*swn*-closed set containing A. Moreover, $x \in \theta\text{-}swnCl(A)$ if and only if for every θ-*swn*-open set U containing x, $U \cap A \neq \varnothing$.

Remark 2.10. Let X be a topological space and $A, B \subseteq X$. Then

(i) $x \in \theta\text{-}swnCl(A)$ if and only if for every *swn*-open set U containing x, $swnCl(U) \cap A \neq \varnothing$.

(ii) If $A \subseteq B$, then $\theta\text{-}swnCl(A) \subseteq \theta\text{-}swnCl(B)$.

(iii) A is θ-*swn*-closed if and only if $A = \theta\text{-}swnCl(A)$.

(iv) $\theta\text{-}swnCl(\theta\text{-}swnCl(A)) = \theta\text{-}swnCl(A)$.

(v) $\theta\text{-}swnCl(A) \cup \theta\text{-}swnCl(B) \subseteq \theta\text{-}swnCl(A \cup B)$.

(vi) $x \in \theta\text{-}swnInt(A)$ if and only if there exists an *swn*-open set U containing x such that $swnCl(U) \subseteq A$.

(vii) If $A \subseteq B$, then $\theta\text{-}swnInt(A) \subseteq \theta\text{-}swnInt(B)$.
(viii) A is θ-*swn*-open if and only if $A = \theta\text{-}swnInt(A)$.
(ix) $\theta\text{-}swnInt(A) = \theta\text{-}swnInt(\theta\text{-}swnInt(A))$.
(x) $\theta\text{-}swnInt(A \cap B) \subseteq \theta\text{-}swnInt(A) \cap \theta\text{-}swnInt(B)$.
(xi) $\theta\text{-}swnCl(X \setminus A) = X \setminus \theta\text{-}swnInt(A)$.
(xii) $A \subseteq swnCl(A) \subseteq \theta\text{-}swnCl(A) \subseteq Cl_\theta(A)$.
(xiii) $Int_\theta(A) \subseteq \theta\text{-}swnInt(A) \subseteq swnInt(A) \subseteq A$.

Next, we introduce and characterize the concepts of θ-somewhat nearly-open, θ-somewhat nearly-closed, strongly θ-somewhat nearly-open, and strongly θ-somewhat nearly-closed functions.

Definition 2.11. Let X and Y be topological spaces. A function $f : X \to Y$ is said to be

(i) θ-somewhat nearly-open (resp., θ-somewhat nearly-closed) on X if $f(G)$ is θ-somewhat nearly-open (resp., θ-somewhat nearly-closed) in Y for every θ-open (resp., θ-closed) set G in X.
(ii) strongly θ-somewhat nearly-open (resp., strongly θ-somewhat nearly-closed) on X if $f(G)$ is θ-somewhat nearly-open (resp., θ-somewhat nearly-closed) in Y for every open (resp., closed) set G in X.

Remark 2.12. Every strongly θ-somewhat nearly-open (resp., strongly θ-somewhat nearly-closed) function is a θ-somewhat nearly-open (resp., θ-somewhat nearly-closed) function.

The converse of Remark 2.12 is not necessarily true as shown in the following example.

Example 2.13. Let $X = Y = \{a, b, c\}$ with topologies $\mathcal{T}_X = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$ and $\mathcal{T}_Y = \{\varnothing, Y, \{a\}, \{a, b\}, \{a, c\}\}$. Define $f : X \to Y$ by $f := \{(a, a), (b, b), (c, c)\}$. Observe that the only θ-open sets in X are $\varnothing$ and X itself. Hence, f is θ-somewhat nearly-open on X. However, $f(\{a\}) = \{a\}$, $f(\{b\}) = \{b\}$ and $f(\{a, b\}) = \{a, b\}$ are not θ-somewhat nearly-open in Y. Thus, f is not strongly θ-somewhat nearly-open on X. A similar argument can be use to construct a θ-somewhat nearly-closed function that is not strongly θ-somewhat nearly-closed.

Remark 2.14. Let X and Y be topological spaces and $f : X \to Y$ be a bijective function. Then

(i) f is θ-somewhat nearly-open if and only if f is θ-somewhat nearly-closed.
(ii) f is strongly θ-somewhat nearly-open if and only if f is strongly θ-somewhat nearly-closed.

Theorem 2.15. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*

(i) *f is θ-somewhat nearly-open on X.*
(ii) *$f(Int_\theta(A)) \subseteq \theta\text{-}swnInt(f(A))$ for every $A \subseteq X$.*
(iii) *For each $x \in X$ and for every open set U in X containing x, there exists a somewhat nearly-open set V in Y containing $f(x)$ such that $swnCl(V) \subseteq f(Cl(U))$.*

Proof. (i)⇒(ii): Suppose that f is θ-somewhat nearly-open on X. Let $A \subseteq X$. Then $f(Int_\theta(A)) \subseteq f(A)$. Since $Int_\theta(A)$ is θ-open, $f(Int_\theta(A))$ is θ-*swn*-open in Y contained in $f(A)$. Since θ-$swnInt(f(A))$ is the largest θ-*swn*-open set contained in $f(A)$, $f(Int_\theta(A)) \subseteq \theta$-$swnInt(f(A))$.

(ii)⇒(iii): Let $x \in X$ and U be open in X containing x. Then there exists an open set W containing x such that $W \subseteq U$. It follows that $Cl(W) \subseteq Cl(U)$ so that $x \in Int_\theta(Cl(U))$. By assumption, $f(x) \in f(Int_\theta(Cl(U))) \subseteq \theta$-$swnInt(f(Cl(U)))$. By Remark 2.10 (vi), there exists an *swn*-open set V containing $f(x)$ such that $swnCl(V) \subseteq f(Cl(U))$.

(iii)⇒(i): Let U be θ-open in X. Let $y \in f(U)$. Then there exists $x \in U$ such that $f(x) = y$. Since U is θ-open, there exists an open set W containing x such that $Cl(W) \subseteq U$. By assumption, there exists an *swn*-open set V containing y such that $swnCl(V) \subseteq f(Cl(W)) \subseteq f(U)$. Thus, $f(U)$ is θ-*swn*-open. □

Theorem 2.16. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*

(i) *f is θ-somewhat nearly-closed on X.*
(ii) *θ-$swnCl(f(B)) \subseteq f(Cl_\theta(B))$ for each $B \subseteq X$.*

Proof. (i)⇒(ii): Suppose that f is θ-somewhat nearly-closed on X. Let $B \subseteq X$. Then $f(B) \subseteq f(Cl_\theta(B))$. Since $Cl_\theta(B)$ is θ-closed, $f(Cl_\theta(B))$ is θ-*swn*-closed in Y containing $f(B)$. Since θ-$swnCl(f(B))$ is the smallest θ-*swn*-closed set containing $f(B)$, θ-$swnCl(f(B)) \subseteq f(Cl_\theta(B))$.

(ii)⇒(i): Let U be θ-closed in X. Then $U = Cl_\theta(U)$. By assumption,

$$f(U) \subseteq \theta\text{-}swnCl(f(U)) \subseteq f(Cl_\theta(U)) = f(U).$$

This implies that f is θ-somewhat nearly-closed. □

Following the same argument as in Theorems 2.15 and 2.16, respectively, the following two results hold.

Theorem 2.17. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*

(i) *f is strongly θ-somewhat nearly-open on X.*
(ii) *$f(Int(A)) \subseteq \theta$-$swnInt(f(A))$ for every $A \subseteq X$.*
(iii) *$f(B)$ is θ-swn-open for every basic open set B in X.*
(iv) *For each $x \in X$ and for every open set U in X containing x, there exists an open set V in Y containing $f(x)$ such that $swnCl(V) \subseteq f(U)$.*

Theorem 2.18. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*

(i) *f is strongly θ-somewhat nearly-closed on X.*
(ii) *θ-$swnCl(f(G)) \subseteq f(Cl(G))$ for each $G \subseteq X$.*

Theorem 2.19. *Let X and Y be topological spaces and $f : X \to Y$ be a function. If X is a regular space, then f is strongly θ-somewhat nearly-open on X if and only if f is θ-somewhat nearly-open on X.*

Proof. By Remark 2.12, strongly θ-somewhat nearly-open function implies θ-somewhat nearly-open function. Conversely, assume that f is θ-somewhat nearly-open on X. Let

$x \in X$ and U be open in X containing x. Since X is regular, there exists an open set W such that $x \in W \subseteq Cl(W) \subseteq U$. By assumption, there exists an *swn*-open set V containing $f(x)$ such that $swnCl(V) \subseteq f(Cl(W)) \subseteq f(U)$. By Theorem 2.17, f is strongly θ-somewhat nearly-open on X. □

Theorem 2.20. *Let X, Y, and Z be topological spaces. If $f : X \to Y$ is both open* [14, p.86] *and closed* [14, p.86] *on X and $g : Y \to Z$ is θ-somewhat nearly-open on Y, then the composition $g \circ f : X \to Z$ is θ-somewhat nearly-open on X.*

Proof. Let $x \in X$ and U be open in X containing x. By assumption, there exists an open set V_Y in Y containing $f(x)$ such that $V_Y \subseteq f(U)$. Also, there exists an *swn*-open set V_Z in Z containing $g(f(x)) = (g \circ f)(x)$ such that $swnCl(V_Z) \subseteq g(Cl(V_Y))$. Since f is closed on X, by [14, Theorem 11.4], $Cl(f(U)) \subseteq f(Cl(U))$. Hence, $swnCl(V_Z) \subseteq g(Cl(V_Y)) \subseteq g(Cl(f(U))) \subseteq (g \circ f)(Cl(U))$. Thus, $g \circ f$ is θ-somewhat nearly-open on X. □

Theorem 2.21. *Let X and Y be topological spaces and $\mathcal{T}_A$ be the subspace topology on $A \subseteq X$. If $f : X \to Y$ is θ-somewhat nearly-open on X and A is open in X, then $f\big|_A : A \to Y$ is θ-somewhat nearly-open on A.*

Proof. Let $x \in A$ and O be open in A containing x. Then $O = A \cap U$, where U is open in X. Since A is open in X, O is also open in X. By assumption, there exists an *swn*-open set V in Y containing $f(x) = f\big|_A(x)$ such that $swnCl(V) \subseteq f(Cl(O)) = f\big|_A(Cl(O))$. In view of Theorem 2.15, $f\big|_A : A \to Y$ is θ-somewhat nearly-open on A. □

Theorem 2.22. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and $\mathcal{T}_A$ be the subspace topology on $A \subseteq X$. If $X = B \cup C$ and $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is a function such that $f\big|_B : (B, \mathcal{T}_B) \to (Y, \mathcal{T}_Y)$ and $f\big|_C : (C, \mathcal{T}_C) \to (Y, \mathcal{T}_Y)$ are θ-somewhat nearly-open, then $f : X \to Y$ is θ-somewhat nearly-open on X.*

Proof. Let $x \in X$ and $U \in \mathcal{T}_X$ containing x. Then $x \in B$ or $x \in C$. If $x \in B$, then $B \cap U \in \mathcal{T}_B$ containing x. By assumption, there exists an *swn*-open set V in Y containing $f\big|_B(x) = f(x)$ such that $swnCl(V) \subseteq f\big|_B(Cl(B \cap U)) \subseteq f(Cl(U))$. This means that $f : X \to Y$ is θ-somewhat nearly-open on X. The same conclusion holds if $x \in C$. □

2.2. θ-Somewhat Nearly-Continuity and other Versions of Continuity. In this section, we define and characterize the concept of θ-somewhat-continuous function and determine its connection to the other versions of continuity.

Definition 2.23. Let X and Y be topological spaces. A function $f : X \to Y$ is said to be

(i) *somewhat nearly-continuous* [3, 28] if $f^{-1}(G)$ is somewhat nearly-open in X for every open set G in Y;

(i) *strongly θ-somewhat nearly-continuous* (briefly strongly θ-*swn*-continuous) if $f^{-1}(G)$ is θ-*swn*-open in X for every open set G in Y;

(ii) *θ-somewhat nearly-continuous* (briefly θ-*swn*-continuous) if $f^{-1}(G)$ is θ-*swn*-open in X for every θ-open set G in Y;

(iv) *almost somewhat nearly-continuous* [28] if for each $x \in X$ and each open set V in Y containing $f(x)$, there exists somewhat nearly-open set U containing x such that $f(U) \subseteq Int(Cl(V))$;

(iii) *weakly somewhat nearly-continuous* (briefly weakly *swn*-continuous) if for every $x \in X$ and every open set V of Y containing $f(x)$, there exists an *swn*-open set U containing x such that $f(U) \subseteq Cl(V)$.

Theorem 2.24. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*

(i) *f is θ-swn-continuous on X.*
(ii) *$f^{-1}(F)$ is θ-swn-closed in X for each θ-closed subset F of Y.*
(iii) *For every $x \in X$ and every open set V in Y containing $f(x)$, there exists an swn-open set U containing x such that $f(swnCl(U)) \subseteq Cl(V)$.*
(iv) *$f(\theta\text{-}swnCl(A)) \subseteq Cl_\theta(f(A))$ for each $A \subseteq X$.*
(v) *$\theta\text{-}swnCl(f^{-1}(B)) \subseteq f^{-1}(Cl_\theta(B))$ for each $B \subseteq Y$.*
(vi) *$f^{-1}(V) \subseteq \theta\text{-}swnInt(f^{-1}(Cl(V)))$ for each open set $V \subseteq Y$.*
(vii) *$\theta\text{-}swnCl(f^{-1}(V)) \subseteq f^{-1}(Cl(V))$ for each open set $V \subseteq Y$.*

Proof. The equivalence of (i), (ii), (iii), (iv), and (v) can be proved using a similar argument in Theorem 2.15 and [9, Theorem 3.1].

(iii)⇒(vi): Let V be open in Y and let $x \in f^{-1}(V)$. Then $f(x) \in V$ so that there exists an *swn*-open set U containing x such that $f(swnCl(U)) \subseteq Cl(V)$. This means that $x \in U \subseteq swnCl(U) \subseteq f^{-1}(Cl(V))$. Using Remark 2.10 (vi), $x \in \theta\text{-}swnInt(f^{-1}(Cl(V)))$. Thus, (vi) holds.

(vi)⇒(vii): Let V be open in Y and let $x \in \theta\text{-}swnCl(f^{-1}(V))$ Suppose that $x \notin f^{-1}(Cl(V))$. Then $f(x) \notin Cl(V)$. This means that there exists an open set W containing $f(x)$ such that $W \cap V = \varnothing$. Since W and V are both open, $Cl(W) \cap V = \varnothing$ so that $f^{-1}(Cl(W)) \cap f^{-1}(V) = \varnothing$. By assumption, $x \in f^{-1}(W) \subseteq \theta\text{-}swnInt(f^{-1}(Cl(W)))$. This follows that there exists an *swn*-open set U containing x such that $swnCl(U) \subseteq f^{-1}(Cl(W))$. Hence, $swnCl(U) \cap f^{-1}(V) = \varnothing$, a contradiction since $x \in \theta\text{-}swnCl(f^{-1}(V))$. Thus, $x \in f^{-1}(Cl(V))$. Accordingly, (vii) holds.

(vii)⇒(iii): Let $x \in X$ and V be open in Y containing $f(x)$. Then $V \cap (Y \setminus Cl(V)) = \varnothing$ so that $f(x) \notin Cl(Y \setminus Cl(V))$. This means that $x \notin f^{-1}(Cl(Y \setminus Cl(V)))$. By assumption, $x \notin \theta\text{-}swnCl(f^{-1}(Y \setminus Cl(V)))$. Using Remark 2.10 (i), there exists an *swn*-open set U containing x such that

$$swCl(U) \cap f^{-1}(Y \setminus Cl(V)) = swnCl(U) \cap (X \setminus f^{-1}(Cl(V))) = \varnothing.$$

This means that $f(swnCl(U)) \subseteq Cl(V)$. □

To prove Theorem 2.26, we shall consider first the following lemma.

Lemma 2.25. *Let X be a topological space and V be an open subset of X. Then*

(i) *$Int(Cl(V))$ is regular open and*
(ii) *$Cl(V)$ is regular closed.*

Theorem 2.26. *Let X and Y be topological spaces and $f : X \to Y$ be a function. If f is almost somewhat nearly-continuous on X, then f is θ-somewhat nearly-continuous on X.*

Proof. Suppose that f is almost swn-continuous on X. Let $x \in X$ and V be open in Y containing $f(x)$. By Lemma 2.25, $Cl(V)$ is regular closed. By [2, Theorem 20], $f^{-1}(Int(Cl(V)))$ is in X swn-open and $f^{-1}(Cl(V))$ is swn-closed in X. Let $U = f^{-1}(Int(Cl(V)))$, which contains x. Observe that,

$$swnCl(U) = swnCl(f^{-1}(Int(Cl(V)))) \subseteq swnCl(f^{-1}(Cl(V))) = f^{-1}(Cl(V)).$$

Consequently, $f(swnCl(U)) \subseteq Cl(V)$. Therefore, by Theorem 2.24, f is θ-swn-continuous on X. □

In view of Definition 2.23, [2, Theorem 27], Theorem 2.24, and Theorem 2.26, we have the following remark.

Remark 2.27. The following diagram holds for a function $f : X \to Y$.

$$\begin{tikzcd}
\text{strongly } \theta\text{-}swn\text{-continuous} \arrow[r] & swn\text{-continuous} \arrow[d] \\
\theta\text{-}swn\text{-continuous} \arrow[d] & \text{almost } swn\text{-continuous} \arrow[l] \\
\text{weakly-}swn\text{-continuous} &
\end{tikzcd}$$

We note that none of the implications above is reversible as shown in the succeeding examples.

Example 2.28. Let X and Y be two topological spaces given by $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$ with respective topologies $\mathcal{T}_X = \{\varnothing, X, \{3\}, \{1, 3\}\}$ and $\mathcal{T}_Y = \{\varnothing, Y, \{c\}\}$. Define $f : X \to Y$ by $f := \{(1, a), (2, b), (3, c)\}$. Note that, $f^{-1}(\varnothing) = \varnothing$, $f^{-1}(Y) = X$ and $f^{-1}(\{c\}) = \{3\}$ are swn-open sets. Thus, f is a swn-continuous function. However, $f^{-1}(\{c\}) = \{3\}$ is not θ-swn-open in X. This means that f is not strongly θ-swn-continuous function.

Example 2.29. Consider the topological space X in Example 2.28 and space Y where $Y = \{a, b, c\}$ with topology $\mathcal{T}_Y = \{\varnothing, Y, \{b\}\}$. Define $f : X \to Y$ by $f := \{(1, a), (2, b), (3, c)\}$. Note that $\varnothing$ and Y are the only regular open sets in the space Y. Now, $f^{-1}(\varnothing) = \varnothing$ and $f^{-1}(Y) = X$ are swn-open sets. Thus, by [2, Theorem 27], f is almost swn-continuous on X. However, f is not an swn-continuous function since $f^{-1}(\{b\}) = \{2\}$ is not swn-open.

Example 2.30. Let X and Y be two topological spaces given by $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d\}$ with respective topologies $\mathcal{T}_X = \{\varnothing, X, \{1\}, \{3, 4\}, \{1, 3, 4\}\}$ and $\mathcal{T}_Y = \{\varnothing, Y, \{a\}, \{c, d\}, \{a, c, d\}\}$. Define $f : X \to Y$ by $f := \{(1, b), (2, a), (3, c), (4, d)\}$. Note that $\varnothing$ and Y are the only θ-open sets in Y and $f^{-1}(\varnothing) = \varnothing$ and $f^{-1}(Y) = X$ are θ-swn-open sets . Thus, f is a θ-swn-continuous function. However, note that $\{a\}$ is a regular open set and $f^{-1}(\{a\}) = \{2\}$ is not swn-open. Therefore, f is not almost swn-continuous function.

Example 2.31. Consider the sets $X = \{a, b, c\}$ and $Y = \{q, r, s\}$ with respective topologies $\mathcal{T}_X = \{\varnothing, X, \{b\}, \{a, b\}, \{b, c\}\} and \mathcal{T}_Y = \{\varnothing, Y, \{q, s\}, \{r\}\}$. Define $f : X \to Y$ by $f := \{(a, q), (b, s), (c, q)\}$. It can be verify that f is weakly-swn-continuous function. However, $f^{-1}(\{q, s\}) = \{a, b\}$ is not a θ-swn-open. Thus, f is not θ-swn-continuous function.

Following the same argument as in Theorem 2.24, the following result holds.

Theorem 2.32. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*

(i) *f is strongly θ-somewhat nearly-continuous on X.*
(ii) *$f^{-1}(F)$ is θ-somewhat nearly-closed in X for each closed subset F of Y.*
(iii) *$f^{-1}(B)$ is θ-somewhat nearly-open for each (subbasic) basic open set B in Y.*
(iv) *For every $x \in X$ and every open set V of Y containing $f(x)$, there exists a somewhat nearly-open set U containing x such that $f(swnCl(U)) \subseteq V$.*
(v) *$f(\theta\text{-}swnCl(A)) \subseteq Cl(f(A))$ for each $A \subseteq X$.*
(vi) *$\theta\text{-}swnCl(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$ for each $B \subseteq Y$.*

Remark 2.33. Let X be a topological space and $A \subseteq X$. Then

(i) $swnInt(A) = X \setminus swnCl(X \setminus A)$ and
(ii) $x \in swnCl(A)$ if and only if for every somewhat nearly-open set U containing x, $U \cap A \neq \varnothing$.

Theorem 2.34. *Let X and Y be topological spaces and $f : X \to Y$ be a function. If Y is a regular space, then the following statements are equivalent:*

(i) *f is strongly θ-swn-continuous on X.*
(ii) *f is almost swn-continuous on X.*
(iii) *f is swn-continuous on X.*
(iv) *f is θ-swn-continuous on X.*

Proof. (i)⇒(ii) Obvious.

(ii)⇒(iii): Let $x \in X$ and V be an open set of Y containing $f(x)$. Since Y is regular, there exists an open set W containing $f(x)$ such that $W \subseteq Cl(W) \subseteq V$. By assumption, there exists an swn-open set U containing x such that $f(U) \subseteq Cl(W) \subseteq V$. In view of [3, Remark 4.1], f is somewhat nearly-continuous on X.

(iii)⇒(i): Let $x \in X$ and V be an open set of Y containing $f(x)$. Since Y is regular, there exists an open set W containing $f(x)$ such that $W \subseteq Cl(W) \subseteq V$. By assumption and using [3, Remark 4.1], there exists an swn-open set U containing x such that $f(U) \subseteq W$. Next, we show that $f(swnCl(U)) \subseteq Cl(W)$ Let $y \in f(swnCl(U))$. Then there exists $x \in swnCl(U)$ such that $f(x) = y$. Let G be an open set in Y containing y. Since f is swn-continuous, $f^{-1}(G)$ is swn-open in X containing x. By Remark 2.33, $f^{-1}(G) \cap U \neq \varnothing$. It follows that

$$\varnothing \neq f\left(f^{-1}(G) \cap U\right) = G \cap f(U) \subseteq G \cap W$$

so that $y \in Cl(W)$. Thus, $f(swnCl(U)) \subseteq Cl(W)$, which implies that $f(swCl(U)) \subseteq V$. In view of Theorem 2.32, f is strongly θ-swn-continuous on X. □

Theorem 2.35. *Let X and Y be topological spaces, $f : X \to Y$ be a function, and $g : X \to X \times Y$ be the graph function of f defined by $g(x) = (x, f(x))$ for every $x \in X$. If g is θ-swn-continuous on X, then f is θ-swn-continuous on X.*

Proof. Suppose that g is θ-swn-continuous on X. Let $x \in X$ and V be an open set of Y containing $f(x)$. Then $X \times V$ is open in $X \times Y$ containing $g(x)$. By Theorem 2.24, there exists an swn-open set U containing x such that $g(swnCl(U)) \subseteq Cl(X \times V) = X \times Cl(V)$.

Next, we show that $f(swnCl(U)) \subseteq Cl(V)$. Let $y \in f(swnCl(U))$. Then there exists $x \in swnCl(U)$ such that $f(x) = y$. Then

$$g(x) = (x, f(x)) \in g(swnCl(U)) \subseteq X \times Cl(V)$$

so that $y = f(x) \in Cl(V)$. Thus, $f(swnCl(U)) \subseteq Cl(V)$. Accordingly, by Theorem 2.24, f is θ-*swn*-continuous on X. □

Theorem 2.36. *Let X and Y be topological spaces and $f_A : X \to \mathcal{D}$ the characteristic function of subset A of X, where $\mathcal{D}$ is the set $\{0, 1\}$ with discrete topology. Then the following statements are equivalent:*

(i) *f_A is strongly θ-somewhat nearly-continuous on X.*
(ii) *A is both θ-somewhat nearly-open and θ-somewhat nearly-closed.*
(iii) *f_A is θ-somewhat nearly-continuous on X.*

Proof. (i)⇔(ii): Trivial.
(i)⇒(iii): Follows from Remark 2.27.
(iii)⇒(ii): This is immediate since every set in $\mathcal{D}$ is θ-open □

Corollary 2.37. Let X and Y be topological spaces and $f_A : X \to \mathcal{D}$ the characteristic function of subset A of X, where $\mathcal{D}$ is the set $\{0, 1\}$ with discrete topology. Then the following statements are equivalent:

(i) f_A is strongly θ-somewhat nearly-continuous on X.
(ii) A is both θ-somewhat nearly-open and θ-somewhat nearly-closed.
(iii) f_A is θ-somewhat nearly-continuous on X.
(iv) f_A is almost somewhat nearly-continuous on X.
(iii) f_A is somewhat nearly-continuous on X.

2.3. Strongly θ-somewhat nearly-continuous Functions in the Product Space. This section provides a characterization of a θ-somewhat nearly-continuous function and strongly θ-somewhat nearly-continuous function from an arbitrary topological space into the product space.

In the following results, if $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ is a product space and $A_\alpha \subseteq Y_\alpha$ for each $\alpha \in \mathcal{A}$, we denote $A_{\alpha_1} \times \cdots \times A_{\alpha_n} \times \Pi\{Y_\alpha : \alpha \notin K\}$ by $\langle A_{\alpha_1}, \ldots, A_{\alpha_n}\rangle$, where $K = \{\alpha_1, \ldots, \alpha_n\}$. If $Y = \Pi\{Y_i : 1 \leq i \leq n\}$ is a finite product, denote $A_1 \times \cdots \times A_n$ by $\langle A_1, \ldots, A_n\rangle$.

The following theorem is analogous to [4, Theorem 3.4].

Theorem 2.38. *Let X be a topological space and $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a product space. A function $f : X \to Y$ is strongly θ-swn-continuous on X if and only if $p_\alpha \circ f$ is strongly θ-swn-continuous on X for every $\alpha \in \mathcal{A}$.*

Corollary 2.39. Let X be a topological space, $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a product space, and $f_\alpha : X \to Y_\alpha$ be a function for each $\alpha \in \mathcal{A}$. Let $f : X \to Y$ be the function defined by $f(x) = \langle f_\alpha(x)\rangle$. Then f is strongly θ-*swn*-continuous on X if and only if each f_α is strongly θ-*swn*-continuous on X for each $\alpha \in \mathcal{A}$.

Lemma 2.40. *Let $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a product space, $K := \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq \mathcal{A}$ and $\varnothing \neq U_{\alpha_i} \subseteq Y_{\alpha_i}$ for each $\alpha_i \in K$. Then*

(i) *$U = \langle U_{\alpha_1}, \ldots, U_{\alpha_n}\rangle$ is swn-open in Y if and only if each U_{α_i} is swn-open in Y_{α_i}; and*

(ii) $swnCl(\Pi\{A_\alpha : \alpha \in \mathcal{A}\}) \subseteq \Pi\{swnCl(A_\alpha) : \alpha \in \mathcal{A}\}$.

Proof. (i) Suppose that $U = \langle U_{\alpha_1}, ..., U_{\alpha_n}\rangle$ is *swn*-open in Y. By [3, Remark 3.1], there exists an open set W such that $\varnothing \neq W \subseteq Cl(U)$. Let $x \in W$. Since W is open, there exists a basic open set $O = \langle O_{\alpha_1}, ..., O_{\alpha_n}\rangle$ containing x such that $O \subseteq W$. Hence, $\varnothing \neq O = \langle O_{\alpha_1}, ..., O_{\alpha_n}\rangle \subseteq W \subseteq Cl(U) = \langle Cl(U_{\alpha_1}), ..., Cl(U_{\alpha_n})\rangle$. This means that for each $i = 1, \ldots, n$, $\varnothing \neq O_{\alpha_i} \subseteq Cl(U_{\alpha_i})$. By [3, Remark 3.1], each U_{α_i} is *swn*-open in Y_{α_i}.

Conversely, suppose that each U_{α_i} is *swn*-open in Y_{α_i}. By [3, Remark 3.1], there exists an open set W_{α_i} such that $\varnothing \neq W_{\alpha_i} \subseteq Cl(U_{\alpha_i})$. Let $W = \langle W_{\alpha_1}, ..., W_{\alpha_n}\rangle$ so that $W \neq \varnothing$. Then W is open in Y such that $\varnothing \neq W \subseteq \langle Cl(U_{\alpha_i})\rangle = Cl\,\langle U_{\alpha_i}\rangle = Cl(U)$. Thus, by [3, Remark 3.1], U is *swn*-open in Y.

(ii) Let $x = \langle x_\alpha\rangle \in swnCl(\Pi\{A_\alpha : \alpha \in \mathcal{A}\})$. Suppose that $x \notin \Pi\{swnCl(A_\alpha) : \alpha \in \mathcal{A}\}$. Then $x_\beta \notin swnCl(A_\beta)$ for some $\beta \in \mathcal{A}$. This means that there exists an *swn*-open set U_β containing x_β such that $U_\beta \cap A_\beta = \varnothing$. Since U_β is *swn*-open, by (i), $\langle U_\beta\rangle$ is *swn*-open that contains x. Thus,

$$\langle U_\beta\rangle \cap \Pi\{A_\alpha : \alpha \in \mathcal{A}\} = \Pi\{A_\alpha : \alpha \neq \beta\} \times (U_\beta \cap A_\beta) = \varnothing.$$

By Remark 2.33 (ii), $x \notin swnCl(\Pi\{A_\alpha : \alpha \in \mathcal{A}\})$, a contradiction. Therefore, $x \in \Pi\{swnCl(A_\alpha) : \alpha \in \mathcal{A}\}$. □

Theorem 2.41. *Let $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a product space and $A_\alpha \subseteq Y_\alpha$ for each $\alpha \in \mathcal{A}$. Then θ-$swnCl(\Pi\{A_\alpha : \alpha \in \mathcal{A}\}) \subseteq \Pi\{\theta\text{-}swnCl(A_\alpha) : \alpha \in \mathcal{A}\}$.*

Proof. Let $x = \langle x_\alpha\rangle \in \theta\text{-}swnCl(\Pi\{A_\alpha : \alpha \in \mathcal{A}\})$. Suppose that $x \notin \Pi\{\theta\text{-}swnCl(A_\alpha) : \alpha \in \mathcal{A}\}$. Then $x_\beta \notin swnCl_\theta(A_\beta)$ for some $\beta \in \mathcal{A}$. This means that there exists an *swn*-open set $U_\beta \ni x_\beta$ such that $swnCl(U_\beta) \cap A_\beta = \varnothing$. In view of Lemma 2.40, $\langle U_\beta\rangle$ is *swn*-open, $x \in \langle U_\beta\rangle$, and

$$\begin{aligned} swnCl(\langle U_\beta\rangle) \cap \Pi\{A_\alpha : \alpha \in \mathcal{A}\} &\subseteq \langle swnCl(U_\beta)\rangle \cap \Pi\{A_\alpha : \alpha \in \mathcal{A}\} \\ &= \Pi\{A_\alpha : \alpha \neq \beta\} \times (swnCl(U_\beta) \cap A_\beta) \\ &= \varnothing. \end{aligned}$$

A contradiction. Thus, $x \in \Pi\{\theta\text{-}swnCl(A_\alpha) : \alpha \in \mathcal{A}\}$. □

Theorem 2.42. *Let $Y = \Pi\{Y_i : 1 \le i \le n\}$ be a finite product space and $A_i \subseteq Y_i$ for each $i = 1, \ldots, n$. Then $\langle \theta\text{-}swnInt(A_1), \ldots, \theta\text{-}swnInt(A_n)\rangle \subseteq \theta\text{-}swnInt(\langle A_1, \ldots, A_n\rangle)$.*

Proof. Let $x = \langle x_i\rangle \in \langle \theta\text{-}swnInt(A_1), \ldots, \theta\text{-}swnInt(A_n)\rangle$. Then $x_i \in \theta\text{-}swnInt(A_i)$ for all $i = 1, 2, \ldots, n$. This means that there exists an *swn*-open set $U_i \ni x_i$ such that $swnCl(U_i) \subseteq A_i$. In view of Lemma 2.40, $\langle U_1, U_2, \ldots, U_n\rangle$ is *swn*-open containing x and

$$swnCl(\langle U_1, \ldots, U_n\rangle) \subseteq \langle swnCl(U_1), \ldots, swnCl(U_n)\rangle \subseteq \langle A_1, \ldots, A_n\rangle.$$

Thus, $x \in \theta\text{-}swnInt(\langle A_1, \ldots, A_n\rangle)$. □

Theorem 2.43. *Let $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a product space and $\varnothing \neq O_\alpha \subseteq Y_\alpha$ for each $\alpha \in \mathcal{A}$. Then $O = \langle O_{\alpha_1}, \ldots, O_{\alpha_n}\rangle$ is θ-swn-open in Y if each O_{α_i} is θ-swn-open in Y_{α_i}.*

Proof. Suppose that each $O_{\alpha_i} \neq \varnothing$ is θ-*swn*-open in Y_{α_i}. Let $x = \langle x_\alpha\rangle \in O$. Then $x_{\alpha_i} \in O_{\alpha_i}$ for all $\alpha_i \in K$. Hence, there exists an *swn*-open set $U_{\alpha_i} \ni x_{\alpha_i}$ such that

$swCl(U_{\alpha_i}) \subseteq O_{\alpha_i}$. Let $U = \langle U_{\alpha_1}, \ldots, U_{\alpha_n} \rangle$. By Lemma 2.40, U is *swn*-open containing x and

$$\begin{aligned} swnCl(U) &= swnCl(\langle U_{\alpha_1}, \ldots, U_{\alpha_n} \rangle) \\ &\subseteq \langle swnCl(U_{\alpha_1}), \ldots, swnCl(U_{\alpha_n}) \rangle \\ &\subseteq \langle O_{\alpha_1}, \ldots, O_{\alpha_n} \rangle \\ &= O. \end{aligned}$$

Thus, O is a θ-*swn*-open set in Y. □

Theorem 2.44. *Let $X = \Pi\{X_\alpha : \alpha \in \mathcal{A}\}$ and $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ be product spaces and for each $\alpha \in \mathcal{A}$, let $f_\alpha : X_\alpha \to Y_\alpha$ be a function. If each f_α is θ-swn-continuous on X_α, then the function $f : X \to Y$ defined by $f(\langle x_\alpha \rangle) = \langle f_\alpha(x_\alpha) \rangle$ is θ-swn-continuous on X.*

Proof. Assume that each f_α is θ-*swn*-continuous on X_α. Suppose that $x = \langle x_\alpha \rangle \in \Pi\{X_\alpha : \alpha \in \mathcal{A}\}$ and G be an open set in $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ containing $f(x)$. Then there exists a basic open set $\langle V_{\alpha_1}, \ldots, V_{\alpha_n} \rangle$ containing $f(x) = \langle f_\alpha(x_\alpha) \rangle$ such that $\langle V_{\alpha_1}, \ldots, V_{\alpha_n} \rangle \subseteq G$. This means that $f_{\alpha_i}(x_{\alpha_i}) \in V_{\alpha_i}$ for all $\alpha_i \in K$. Note that each V_{α_i} open in Y_{α_i}. Thus, by assumption, for all $\alpha_i \in K$, there exists an *swn*-open set U_{α_i} containing x_{α_i} such that $f_{\alpha_i}(swnCl(U_{\alpha_i})) \subseteq Cl(V_{\alpha_i})$. Let $U = \langle U_{\alpha_1}, \ldots, U_{\alpha_n} \rangle$. By Lemma 2.40 (i), U is *swn*-open containing $x = \langle x_\alpha \rangle$. Now, by Lemma 2.40 (ii) and by definition of f,

$$\begin{aligned} f(swnCl(U)) &= f(swnCl(\langle U_{\alpha_1}, \ldots, U_{\alpha_n} \rangle)) \\ &\subseteq f(\langle swnCl(U_{\alpha_1}), \ldots, swnCl(U_{\alpha_n}) \rangle) \\ &= \langle f_{\alpha_1}(swnCl(U_{\alpha_1})), \ldots, f_{\alpha_n}(swnCl(U_{\alpha_n})) \rangle \\ &\subseteq \langle Cl(V_{\alpha_1}), \ldots, Cl(V_{\alpha_n}) \rangle \\ &\subseteq Cl(G). \end{aligned}$$

Therefore, by Theorem 2.24, f is θ-*swn*-continuous on X. □

Theorem 2.45. *Let $X = \Pi\{X_\alpha : \alpha \in \mathcal{A}\}$ and $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ be product spaces and for each $\alpha \in \mathcal{A}$, let $f_\alpha : X_\alpha \to Y_\alpha$ be a function. If each f_α is strongly θ-swn-continuous on X_α, then the function $f : X \to Y$ defined by $f(\langle x_\alpha \rangle) = \langle f_\alpha(x_\alpha) \rangle$ is strongly θ-swn-continuous on X.*

Proof. Let $\langle V_{\alpha_1}, \ldots, V_{\alpha_n} \rangle$ be a basic open set in Y. Then

$$f^{-1}(\langle V_{\alpha_1}, \ldots, V_{\alpha_n} \rangle) = \left\langle f_{\alpha_1}^{-1}(V_{\alpha_1}), \ldots, f_{\alpha_n}^{-1}(V_{\alpha_n}) \right\rangle.$$

Since each f_{α_i} is strongly θ-*swn*-continuous, by Theorem 2.32 (iii), $f_{\alpha_i}^{-1}(V_{\alpha_i})$ is θ-*swn*-open in X_{α_i}. Let $x = \langle x_\alpha \rangle \in f^{-1}(\langle V_{\alpha_1}, \ldots, V_{\alpha_n} \rangle)$. Then $x_{\alpha_i} \in f_{\alpha_i}^{-1}(V_{\alpha_i})$ for all $\alpha_i \in K$. This means that there exists an *swn*-open set $O_{\alpha_i} \ni x_{\alpha_i}$ such that $swnCl(O_{\alpha_i}) \subseteq f_{\alpha_i}^{-1}(V_{\alpha_i})$. By Lemma 2.40, $\langle O_{\alpha_1}, \ldots, O_{\alpha_n} \rangle$ is *swn*-open in X containing x and

$$\begin{aligned} swnCl(\langle O_{\alpha_1}, \ldots, O_{\alpha_n} \rangle) &\subseteq \langle swnCl(O_{\alpha_1}), \ldots, swnCl(O_{\alpha_n}) \rangle \\ &\subseteq \left\langle f_{\alpha_1}^{-1}(V_{\alpha_1}), \ldots, f_{\alpha_n}^{-1}(V_{\alpha_n}) \right\rangle \\ &= f^{-1}(\langle V_{\alpha_1}, \ldots, V_{\alpha_n} \rangle). \end{aligned}$$

This implies that $f^{-1}(\langle V_{\alpha_1}, \ldots, V_{\alpha_n}\rangle)$ is θ-*swn*-open in X. Thus, f is strongly θ-*swn*-continuous on X. □

2.4. θ-somewhat nearly-connected Space and Versions of Separation Axioms and Compactness. This section characterizes the concepts of θ-somewhat nearly-connected space and some versions of separation axioms.

Definition 2.46. A topological space X is said to be a θ-somewhat nearly-connected if it is not the union of two nonempty disjoint θ-somewhat nearly-open sets. Otherwise, X is θ-somewhat nearly-disconnected. A subset B of X is θ-somewhat nearly-connected if it is θ-somewhat nearly-connected as a subspace of X.

In view of Theorem 2.36, the following result holds.

Theorem 2.47. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is θ-somewhat nearly-connected.*
(ii) *The only subsets of X that are both θ-somewhat nearly-open and θ-somewhat nearly-closed are $\varnothing$ and X.*
(iii) *No θ-somewhat nearly-continuous function $f : X \to \mathcal{D}$ is surjective.*
(iii) *No strongly θ-somewhat nearly-continuous function $f : X \to \mathcal{D}$ is surjective.*

A topological space X is said to be θ-connected (resp., connected) if it is not the union of two nonempty disjoint θ-open (resp., open)sets. Otherwise, X is θ-disconnected (resp., disconnected).

Theorem 2.48. *If a topological space X is θ-somewhat nearly-connected, then X is θ-connected.*

Proof. Follows directly from the fact that every θ-open set is a θ-*swn*-open set. □

Since connected and θ-connected spaces [18] are equivalent, the following result holds.

Corollary 2.49. If a topological space X is θ-somewhat nearly-connected, then X is connected.

Remark 2.50. The following diagram holds for a subset of a topological space.

$$\theta\text{-somewhat nearly-connected} \longrightarrow \theta\text{-connected} \longleftrightarrow \text{connected}$$

The reverse implication for θ-connected and θ-somewhat nearly-connected spaces is not true as shown in the following example.

Example 2.51. Consider the subspace $[0, 2)$ in real line $\mathbb{R}$ with the standard topology. By [25, Corollary 24.2], $[0, 2)$ is a connected space. Thus, $[0, 2)$ is also θ-connected. To show that $[0, 2)$ is θ-*swn*-disconnected, consider the two disjoint sets $[0, 1]$ and $(1, 2)$ whose union is $[0, 2)$. We will show that these sets are θ-*swn*-open in the subspace $[0, 2)$ of $\mathbb{R}$.

For $(1, 2)$, suppose that $x \in (1, 2)$. Then there exists $\epsilon > 0$ such that $x \in (x-\epsilon, x+\epsilon)$ and $(x - \epsilon, x + \epsilon) \subseteq Cl(x - \epsilon, x + \epsilon) = [x - \epsilon, x + \epsilon] \subseteq (1, 2)$. Hence, $(1, 2)$ is a θ-open so that $(1, 2)$ is θ-*swn*-open, by Theorem 2.3 (i). On the other hand, it can be easily verify that $[0, 1]$ is both *swn*-open and *swn*-closed. Hence, by Remark 2.2, $[0, 1]$ is θ-*swn*-open.

Definition 2.52. A topological space X is said to be

(i) *θ-somewhat nearly-Hausdorff* if given any pair of distinct points p, q in X there exist disjoint θ-somewhat nearly-open sets U and V such that $p \in U$ and $q \in V$;

(ii) *θ-somewhat nearly-regular* if for each closed set F and each point $x \notin F$, there exist disjoint θ-somewhat nearly-open sets U and V such that $x \in U$ and $F \subseteq V$;

(iii) *θ-somewhat nearly-normal* if for every pair of disjoint closed sets E and F of X, there exist disjoint θ-somewhat nearly-open sets U and V such that $E \subseteq U$ and $F \subseteq V$.

The following three results can be proved using the same techniques employed in [17].

Theorem 2.53. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is θ-somewhat nearly-Hausdorff.*

(ii) *Let $x \in X$. For $y \neq x$, there exists a θ-somewhat nearly-open set U containing x such that $y \notin \theta$-$swnCl(U)$.*

(iii) *For each $x \in X$, $C = \cap\{\theta$-$swnCl(U) : U$ is θ-somewhat nearly-open containing $x\}=\{x\}$.*

Theorem 2.54. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is θ-somewhat nearly-regular.*

(ii) *For each $x \in X$ and an open set U containing x, there exists θ-somewhat nearly-open set V such that $x \in V \subseteq \theta$-$swnCl(V) \subseteq U$.*

(iii) *For each $x \in X$ and closed set F with $x \notin F$, there exists a θ-somewhat nearly-open set V containing x such that $F \cap \theta$-$swnCl(V) = \varnothing$.*

Theorem 2.55. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is θ-somewhat nearly-normal.*

(ii) *For each closed set A and an open set $U \supseteq A$, there exists a θ-somewhat nearly-open set V containing A such that θ-$swnCl(V) \subseteq U$.*

(iii) *For each pair of disjoint closed sets A and B, there exists a θ-somewhat nearly-open set V containing A such that θ-$swnCl(V) \cap B = \varnothing$.*

A topological space X is said to be a T_1-space if for each $p, q \in X$ with $p \neq q$, there exist open sets U and V such that $p \in U$, $q \notin U$ and $q \in V$, $p \notin V$.

Theorem 2.56. *Let X be a T_1-space. Then*

(i) *If X is θ-somewhat nearly-regular, then X is θ-somewhat nearly-Hausdorff.*

(ii) *If X is θ-somewhat nearly-normal, then X is θ-somewhat nearly-regular.*

Proof. (i): Suppose that X is θ-somewhat nearly-regular. Since X is a T_1-space, for each $x, y \in X$ with $x \neq y$, there exist disjoint open sets U and V such that $x \in U$ and $y \notin U$, and $y \in V$ and $x \notin V$. This implies that $x \notin X \setminus U$ and $y \notin X \setminus V$. Since X is θ-somewhat nearly-regular, there exist disjoint θ-somewhat nearly-open sets A and B such that $x \in A$ and $X \setminus U \subseteq B$. Note that $y \in X \setminus U$. Hence, $y \in B$. Thus, X is θ-somewhat nearly-Hausdorff.

(ii) Suppose that X is θ–somewhat nearly-normal. Let F be a closed set and let $x \notin F$. Since X is a T_1-space, $\{x\}$ is closed in X. Since X is θ-*swn*-normal, there exist disjoint θ-*swn*-open sets U and V such that $\{x\} \subseteq U$ and $F \subseteq V$, which implies that $x \in U$. Thus, X is θ-somewhat nearly-regular. □

Definition 2.57. A topological space X is said to be

(i) *swn-closed* (resp., *swn-Lindelöf*) if every cover of X by somewhat nearly-open sets has a finite (resp., countable) subcollection whose somewhat nearly-closures cover X;
(ii) *countably swn-closed* (resp., *lightly compact* [5]) if every countable cover of X by somewhat nearly-open (resp., open) sets has a finite subcollection whose somewhat nearly-closures (resp., closures) cover X;
(iii) *quasi H-closed* [29] (resp., *almost Lindelöf* [11]) if every cover of X by open sets has a finite (resp., countable) subcollection whose closures cover X.

Definition 2.58. A subset A of a topological space X is said to be

(i) *swn-closed relative to* X if every cover of A by somewhat nearly-open sets in X has a finite subcollection whose somewhat nearly-closures cover A;
(ii) *quasi H-closed relative to* X [29] if every cover of A by open sets in X has a finite subcollection whose closures cover A.

Theorem 2.59. *Let X and Y be topological spaces and $f : X \to Y$ be θ-somewhat nearly-continuous on X. If $A \subseteq X$ is swn-closed relative to X, then $f(A)$ is quasi H-closed relative to Y.*

Proof. Let $\{V_\alpha : \alpha \in \mathcal{A}\}$ be a cover of $f(A)$ by open sets in Y. Then $f(A) \subseteq \cup\{V_\alpha : \alpha \in \mathcal{A}\}$. Let $x \in A$. Then $f(x) \in V_{\beta(x)}$ for some $\beta(x) \in \mathcal{A}$. Since f is θ-somewhat nearly-continuous on X, there exists a somewhat nearly-open set $U_{\beta(x)}$ containing x such that $f(swnCl(U_{\beta(x)})) \subseteq Cl(V_{\beta(x)})$. Note that $\{U_{\beta(x)} : x \in A\}$ is a collection of somewhat nearly-open sets in X that cover A. Since A is *swn*-closed relative to X, there exists a finite set $F \subseteq A$ such that $A \subseteq \cup\{swnCl(U_{\beta(x)}) : x \in F\}$. This means that

$$f(A) \subseteq \cup\{f(swnCl(U_{\beta(x)})) : x \in F\} \subseteq \cup\{Cl(V_{\beta(x)}) : x \in F\}.$$

Thus, $f(A)$ is quasi H-closed relative to Y. □

Corollary 2.60. Let X and Y be topological spaces and $f : X \to Y$ be θ-somewhat nearly-continuous and surjective. Then the following properties hold:

(i) If X is *swn*-closed, then Y is quasi H-closed.
(ii) If X is *swn*-Lindelöf, then Y is almost Lindelöf.
(iii) If X is countably *swn*-closed, then Y is lightly compact.

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