Poincare Journal of Analysis & Applications
Vol. 11, No. 2 (2024), 181-188
©*Poincare Publishers*
DOI: 10.46753/pjaa.2024.v011i02.007
Online Published on 13. 11. 2024

SOME REMARKS ON RETRO BANACH FRAMES

DOLLY JAIN AND B. SEMTHANGA†

Date of Receiving	:	20. 09. 2023
Date of Revision	:	24. 10. 2024
Date of Acceptance	:	07. 11. 2024

Abstract. In this paper, we study exact retro Banach frames and give a sufficient condition for its existence. Also, we prove that if Y is a subspace of infinite co-dimensions of a separable Banach space $\mathcal{B}$ and if $(\mathcal{B}/Y)^*$ has a retro Banach frame, then $\mathcal{B}^*$ has a retro Banach frame. Finally, we define retro Banach frame of type P and proved that a separable Banach space has a retro Banach frame of type P.

1. Introduction

Duffin and Schaeffer [8] developed the notion of frame in the context of Hilbert spaces in 1952. In fact, Duffin and Schaeffer borrowed the Hilbert frame definition technique from Gabor's work [11]. Many years later, in 1986, Daubechies, Grossmann, and Meyer [9] recognized the potential of frames and discovered new uses for them in wavelets, Gabor transforms, and other pure and applied mathematics. For example, strong instruments from operator theory and Banach spaces are being used to investigate frames [12, 14, 18, 19, 20, 21, 23, 28, 31]. The seminal publications by O. Christensen [6, 7] go into further information about the use of frames. Following the publication by Daubechies, Grossmann, and Meyer [9], the idea of frames attracted a lot of attention.

A frame in the Hilbert space $\mathcal{H}$ is a sequence $\{h_i\}_{i=1}^{\infty} \subset \mathcal{H}$ of vectors that fulfills

$$M\|l\|^2 \leq \sum_{i=1}^{\infty} |\langle l, h_i\rangle|^2 \leq K\|l\|^2, \quad \forall\ l \in \mathcal{H},$$

where $0 < M \leq K < \infty$. M is called the lower frame bound and K is called the upper frame bound of the frame $\{h_i\}_{i=1}^{\infty}$ respectively. These frame bounds are not unique. If $M = K$, then the frame $\{h_i\}_{i=1}^{\infty}$ is said to be a tight frame and if $M = K = 1$, then $\{h_i\}_{i=1}^{\infty}$ is called Parseval frame. The above inequality is called the frame inequality.

2010 *Mathematics Subject Classification.* 46B15, 42C15.
Key words and phrases. Banach frame, Retro Banach frame, Exact retro Banach frame.
Communicated by: A. M. Jarrah
† *Corresponding author*

The operator $\mathcal{L} : \ell^2 \to \mathcal{H}$ given by

$$\mathcal{L}\left(\{\beta_i\}_{i\in\mathbb{N}}\right) = \sum_{i=1}^{\infty} \beta_i h_i, \quad \forall\ \{\beta_i\}_{i\in\mathbb{N}} \in \ell^2$$

is called the pre-frame operator or the synthesis operator. The adjoint operator $\mathcal{L}^* : \mathcal{H} \to \ell^2$ given by

$$\mathcal{L}^*(h) = \{\langle h, h_i\rangle\}_{i\in\mathbb{N}}, \quad \forall\ h \in \mathcal{H}$$

is called the analysis operator.

The composition of the pre-frame operator $\mathcal{L}$ and its adjoint operator $\mathcal{L}^*$ defined by $\mathcal{N} = \mathcal{L}\mathcal{L}^* : \mathcal{H} \to \mathcal{H}$ is called the frame operator. Therefore,

$$\mathcal{N}(h) = \sum_{i=1}^{\infty} \langle h, h_i\rangle\, h_i, \quad \forall\ h \in \mathcal{H}.$$

Then,

$$\langle \mathcal{N}h, h\rangle = \sum_{i=1}^{\infty} |\langle h, h_i\rangle|^2, \quad \forall\ h \in \mathcal{H}.$$

The frame operator $\mathcal{N}$ therefore is positive, self adjoint and invertible operator on $\mathcal{H}$. For all $h \in \mathcal{H}$, the reconstruction formula is given by

$$\begin{aligned} h &= \mathcal{N}\mathcal{N}^{-1}h \\ &= \sum_{i=1}^{\infty} \left\langle \mathcal{N}^{-1}h, h_i\right\rangle h_i \\ &= \sum_{i=1}^{\infty} \left\langle h, \mathcal{N}^{-1}h_i\right\rangle h_i. \end{aligned}$$

In several branches of engineering sciences, including pure and applied mathematics, frames are becoming indispensable. The ability of frames to represent each element of a space as a linear combination of its constituent elements is a significant and highly practical feature. However, linear independence between frame elements is not necessary, meaning that the corresponding coefficients need not always be unique.

Obtaining the reconstruction formula using the norm equivalency hypothesis is a noteworthy observation in Hilbert spaces. This does not hold in general in Banach spaces. Because of this, Feichtinger and Grochenig [10] created atomic decomposition in 1988 and studied the decomposition of a Banach space. Grochenig [14] proposed Banach frame in the context of Banach spaces later, in 1991. Numerous writers have expanded frames, creating fusion frames [4], G-frames [24, 30], p-frames [2], PG-frames [1], K-atomic decomposition [26], Λ-Banach frame [12], $\tilde{X}$ -frame [15], X_d-frames [5], Schauder frames [3], Frechet frame [27], and more.

Retro Banach frames were introduced in 2004 and Banach frames were extended to the conjugate Banach spaces by Jain et al. [18]. Raj Kumar and Sharma [25] examined the exactness of retro Banach frames in 2008 and presented near-exact versions of the frames. S. Jahan presented the concept of a $\mathcal{J}$-frame [17] and a strong retro Banach frame [16] in Banach spaces in 2017. Block sequences are an essential component of the retro Banach frame that have been the subject of recent work by N. N. Jha and Shalu Sharma [22]. Certain classes of retro Banach frames were studied by M.P Goswami[13].

1.1. Purpose and Significance of the Main Results. We will explore exact retro Banach frames in this research article and provide a sufficient condition for their existence. Furthermore, we demonstrate that $\mathcal{B}^*$ has a retro Banach frame if Y is a subspace of infinite co-dimensions of a separable Banach space $\mathcal{B}$ and if $(\mathcal{B}/Y)^*$ has a retro Banach frame. We finally define a retro Banach frame of type P and show that there exists a retro Banach frame of type P for a separable Banach space.

2. Preliminaries

In this paper a Banach space and its first dual over the scalar field $\mathbb{K}(\mathbb{R}$ or $\mathbb{C})$ shall be referred to as $\mathcal{B}$ and $\mathcal{B}^*$ respectively. $\mathcal{B}_d$ and $\mathcal{B}_d^*$ will denote the Banach spaces of scalar-valued sequences associated with $\mathcal{B}$ and $\mathcal{B}^*$ respectively. Also, $[b_n]_{n\in\mathbb{N}}$ denotes the closed linear subspace of $\mathcal{B}$ spanned by $\{b_n\}_{n\in\mathbb{N}}$. $\mathbb{N}$ denote the set of positive integers and RBF refers to the abbreviation of retro Banach frame.

In a Banach space $\mathcal{B}$ a sequence $\{b_n\}_{n\in\mathbb{N}}$ is complete if $[b_n]_{n\in\mathbb{N}} = \mathcal{B}$ and total if $\{h \in \mathcal{B}^* : h(b_n) = 0, \forall n \in \mathbb{N}\} = \{0\}$.

Grochenig [14] defined Banach frame as follows:

Definition 2.1. ([14]). Let $\mathcal{B}$ be a Banach space and $\mathcal{B}_d$ be an associated Banach space of scalar valued sequences, indexed by $\mathbb{N}$. Let $\{h_n\} \subset \mathcal{B}^*$ and $\mathcal{S} : \mathcal{B}_d \to \mathcal{B}$ be given. The pair $(\{h_n\}, \mathcal{S})$ is called a Banach frame for $\mathcal{B}$ with respect to $\mathcal{B}_d$ if

(i) $\{h_n(y)\} \in \mathcal{B}_d$, for each $y \in \mathcal{B}$,

(ii) there exist constants α and β with $0 < \alpha \le \beta < \infty$ such that
$$\alpha\|y\|_{\mathcal{B}} \le \|\{h_n(y)\}\|_{\mathcal{B}_d} \le \beta\|y\|_{\mathcal{B}}, \quad y \in \mathcal{B} \tag{2.1}$$

(iii) $\mathcal{S}$ is a bounded linear operator such that
$$\mathcal{S}(\{h_n(y)\}) = y, \quad y \in \mathcal{B}.$$

The positive constants α and β are called lower and upper frame bounds of the Banach frame $(\{h_n\}, \mathcal{S})$, respectively. The operator $S : \mathcal{B}_d \to \mathcal{B}$ is called the reconstruction operator. The inequality (2.1) is called the Banach frame inequality.

The Banach frame $(\{h_n\}, \mathcal{S})$ is said to be exact if there exists no reconstruction operator $\mathcal{S}_0$ such that $\left(\{h_n\}_{n\neq i}, \mathcal{S}_0\right)$ is a Banach frame for $\mathcal{B}$.

Remark 2.2. ([19]). Let $(\{h_n\}, \mathcal{S})$ be an exact Banach frame for $\mathcal{B}$. Then there exists a sequence $\{y_n\} \subset \mathcal{B}$, called an admissible sequence of vectors to $(\{h_n\}, \mathcal{S})$ such that
$$\begin{aligned} h_i(y_j) &= \delta_{i,j} \\ &= \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \quad \forall\ i, j \in \mathbb{N}\,. \end{aligned}$$

We now give the definition of retro Banach frame which is the extension of frames in conjugate Banach spaces.

Definition 2.3. ([18]). Let $\mathcal{B}$ be a Banach space and $\mathcal{B}^*$ be its conjugate space. Let $\mathcal{B}_d^*$ be a Banach space of scalar valued sequences indexed by $\mathbb{N}$ associated with $\mathcal{B}^*$. Let $\{y_n\}$ be a sequence in $\mathcal{B}$ and $\mathcal{T} : \mathcal{B}_d^* \to \mathcal{B}^*$ be given. The pair $(\{y_n\}, \mathcal{T})$ is called a retro Banach frame (RBF) for $\mathcal{B}^*$ with respect to $\mathcal{B}_d^*$ if

(i) $\{h(x_n)\} \in \mathcal{B}_d^*$, for each $h \in \mathcal{B}^*$,

(ii) there exist constants α and β with $0 < \alpha \leq \beta < \infty$ such that

$$\alpha \|h\|_{\mathcal{B}^*} \leq \|\{h(y_n)\}\|_{\mathcal{B}_d^*} \leq \beta \|h\|_{\mathcal{B}^*}, \quad \forall\, h \in \mathcal{B}^* \tag{2.2}$$

(iii) $\mathcal{T}$ is a bounded linear operator such that

$$\mathcal{T}(\{h(y_n)\}) = h, \quad \forall\, h \in \mathcal{B}^*.$$

The positive constants α and β are called lower and upper frame bounds of the retro Banach frame $(\{y_n\}, \mathcal{T})$, respectively. The operator $\mathcal{T} : \mathcal{B}_d^* \to \mathcal{B}^*$ is called the reconstruction operator for the retro Banach frame $(\{y_n\}, \mathcal{T})$. The inequality (2.2) is called the retro frame inequality.

A RBF $(\{y_n\}, \mathcal{T})$ is said to be tight if $\alpha = \beta$ and is called normalized tight if $\alpha = \beta = 1$. If removal of one y_n renders the collection $\{y_n\} \subset \mathcal{B}$ no longer a RBF for $\mathcal{B}^*$, then $(\{y_n\}, \mathcal{T})$ is called an exact RBF.

Lemma 2.4. *([29]). Let $\mathcal{B}$ be a Banach space and $\{b_n^*\}_{n\in\mathbb{N}}$ be a complete sequence in $\mathcal{B}^*$. Then $\mathcal{B}^*$ is linearly isometric to the BK-space $\mathcal{B}_d^* = \{\{b^*(b_n) : b^* \in \mathcal{B}^*\}\}$ with norm defined as $\left\|\{b^*(b_n)\}_{n\in\mathbb{N}}\right\|_{\mathcal{B}_d^*}, b^* \in \mathcal{B}^*$.*

Lemma 2.5. *([18]). Let $(\{y_n\}, \mathcal{T})\,(\{y_n\} \subset \mathcal{B}, \mathcal{T} : \mathcal{B}_d^* \to \mathcal{B}^*)$ be a retro Banach frame for $\mathcal{B}^*$ with respect to E_d^*. Then $(\{y_n\}, \mathcal{T})$ is exact if and only if $y_n \notin [y_i]_{i\neq n}$.*

3. Main Results

In this section, we discuss various results concerning retro Banach frames (RBF). We begin with the following result which provide sufficient condition for the existence of an exact RBF.

Theorem 3.1. *Let $\{b_n\}_{n\in\mathbb{N}}$ be a sequence in a separable Banach space $\mathcal{B}$ such that $b_n \notin [b_i]_{i\neq n}, n \in \mathbb{N}$. Then, $\mathcal{B}^*$ has an exact retro Banach frame associated with $\{b_n\}_{n\in\mathbb{N}}$.*

Proof. Without any loss of generality, we may assume that $\dim[b_n]_{n\in\mathbb{N}} < \infty$. Consider the quotient space $\mathcal{B}/[b_n]_{n\in\mathbb{N}}$. Since $b_n \notin [b_i]_{i\neq n}, n \in \mathbb{N}$, there exists a sequence $\{b_n^*\}_{n\in\mathbb{N}}$ in $\mathcal{B}^*$ such that $b_i^*(b_j) = \delta_{ij}, \forall i, j \in \mathbb{N}$.

Also, since $\mathcal{B}$ is separable, one can find a sequence $\{B_n\}_{n\in\mathbb{N}}$ in $\mathcal{B}/[b_n]_{n\in\mathbb{N}}$ such that $\dim B_n = \infty, n \in \mathbb{N}$ and $B_n \neq 0, n \in \mathbb{N}$.

Let $\beta_n \in B_n$ be any arbitrary element. Write

$$\phi_n = \beta_n - \sum_{j=1}^{n} b_j^*(\beta_n) b_j, n \in \mathbb{N}.$$

Then $\phi_n \in \beta_n + [b_n]_{n\in\mathbb{N}} = B_n, n \in \mathbb{N}$.

Also, if $\phi_n = 0$, then $\beta_n = \sum_{j=1}^{n} b_j^*(\beta_n) b_j \in [b_n]_{n\in\mathbb{N}}$ and so $B_n = 0$ which is a contradiction. So, $\phi_n \neq 0, n \in \mathbb{N}$. Further, note that

$$\begin{aligned} b_k^*(\phi_n) &= b_k^*(\beta_n) - \sum_{j=1}^{n} b_j^*(\beta_n) b_k^*(b_j) \\ &= b_k^*(\beta_n) - b_k^*(\beta_n) \\ &= 0, \quad \forall k = 1, 2, \ldots, n. \end{aligned}$$

Now we prove that $[\psi_n]_{n\in\mathbb{N}} = \mathcal{B}$, where $\{\psi_n\}_{n\in\mathbb{N}} = \{\phi_n\}_{n\in\mathbb{N}} \cup \{b_n\}$.
Let u be the canonical mapping of $\mathcal{B}$ onto $\mathcal{B}/[b_n]_{n\in\mathbb{N}}$. Then, for $b \in \mathcal{B}$ and $\epsilon > 0$, one can find constants $a_1, a_2, \ldots, a_k$ such that

$$\left\| u\left(b - \sum_{j=1}^{k} a_j \phi_j\right) \right\| = \left\| u(b) - \sum_{j=1}^{k} a_j B_j \right\| < \frac{\epsilon}{4}.$$

Moreover, there is an element $\beta \in [b_n]_{n\in\mathbb{N}}$ such that

$$\left\| b - \sum_{j=1}^{n} a_j \phi_j - \beta \right\| < \frac{\epsilon}{2}$$

and constants $p_1, p_2, \ldots, p_l$ satisfying

$$\left\| \beta - \sum_{j=1}^{l} p_j b_j \right\| < \frac{\epsilon}{2}.$$

Then,

$$\left\| b - \sum_{j=1}^{m} a_j \phi_j - \sum_{j=1}^{l} p_j b_j \right\| < \epsilon.$$

Thus, $[\psi_n]_{n\in\mathbb{N}} = \mathcal{B}$. In view of Lemma 2.4, one can find a sequence space $\mathcal{B}_d^* = \{\{b^*(\psi_n)\}_{n\in\mathbb{N}} : b^* \in \mathcal{B}^*\}$ with norm given by $\|\{b^*(\psi_n)\}_{n\in\mathbb{N}}\|_{\mathcal{B}_d^*} = \|b^*\|_{\mathcal{B}^*}, b^* \in \mathcal{B}^*$ and an operator $\mathcal{T} : \mathcal{B}_d^* \to \mathcal{B}^*$ such that $(\{\psi_n\}_{n\in\mathbb{N}}, \mathcal{T})$ is a retro Banach frame for $\mathcal{B}^*$. Next, we prove that $(\{\psi_n\}_{n\in\mathbb{N}}, \mathcal{T})$ is exact.

Let $\{y_n\}_{n\in\mathbb{N}} \in \mathcal{B}/[b_n]_{n\in\mathbb{N}}$ be such that $y_i(B_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$. Define

$$h_n(b) = y_n\left(b + [b_n]_{n\in\mathbb{N}}\right)$$

and

$$f_n = b_n^* - \sum_{i=1}^{n-1} b_n^*(\phi_i) h_i, n \in \mathbb{N}.$$

Then

$$h_n(b_j) = y_n\left(b_j + [b_n]_{n\in\mathbb{N}}\right) = y_n\left([b_n]_{n\in\mathbb{N}}\right) = 0j, n \in \mathbb{N}$$

$$h_n(\phi_j) = y_n\left(\phi_j + [b_n]_{n\in\mathbb{N}}\right) = y_n\left(B_j\right) = \delta_{nj}, \forall n, j \in \mathbb{N}.$$

$$f_n(b_j) = b_n^*(b_j) - \sum_{i=1}^{n-1} b_n^*(\phi_i)h_i(b_j) = \delta_{nj}, \forall n, j \in \mathbb{N}.$$

and

$$\begin{aligned} f_n(\phi_j) &= b_n^*(\phi_j) - \sum_{i=1}^{n} b_n^*(\phi_i)h_i(\phi_j) \\ &= \begin{cases} b_n^*(\phi_j) - b_n^*(\phi_j) = 0, j = 1, 2, \dots, n-1, n \in \mathbb{N} \\ b_n^*(\phi_j) = 0, j = n, n+1, \dots, n \in \mathbb{N}. \end{cases} \end{aligned}$$

Write $\{\psi_n^*\}_{n\in\mathbb{N}} = \{h_n\}_{n\in\mathbb{N}} \cup \{f_n\}_{n\in\mathbb{N}}$. Then $\psi_n^*(\psi_m) = \delta_{nm}, \forall n, m \in \mathbb{N}$. Hence $(\{\psi_n\}_{n\in\mathbb{N}}, \mathcal{T})$ is a retro Banach frame for $\mathcal{B}^*$. □

In the following result, we consider a subspace of a separable Banach space $\mathcal{B}$ which is of infinite co-dimension such that $(\mathcal{B}/Y)^*$ has a retro Banach frame and prove that in this case $\mathcal{B}^*$ has a retro Banach frame.

Theorem 3.2. *Let Y be a subspace of infinite co-dimension of a separable Banach space $\mathcal{B}$. If $(\mathcal{B}/Y)^*$ has a retro Banach frame, then $\mathcal{B}^*$ has a retro Banach frame.*

Proof. Suppose $(\{G_n\}_{n\in\mathbb{N}}, \mathcal{T})$ $\left(\{G_n\}_{n\in\mathbb{N}} \subseteq \mathcal{B}/Y, \mathcal{T} : \left(\frac{\mathcal{B}}{Y}\right)_d^* \to \left(\frac{\mathcal{B}}{Y}\right)^*\right)$ be a retro Banach frame for $\frac{\mathcal{B}}{Y}$ satisfying $G_n \neq 0, \forall n \in \mathbb{N}$. Then

$$\left\{\frac{1}{n\|G_n\|}G_n \to 0\right\}.$$

So, there exists $g_n \in \frac{G_n}{n\|G_n\|}, n \in \mathbb{N}$ such that $[g_n]_{n\in\mathbb{N}} = \mathcal{B}$. Write $b_n = n\|G_n\|g_n, n \in \mathbb{N}$. Then, there exists an associated BK-space given by

$$\mathcal{B}_d^* = \{\{\phi(b_n)\} : \phi \in \mathcal{B}^*\}$$

with norm given by $\|\{\phi(b_n)\}\|_{\mathcal{B}_d^*} = \|\phi\|_{\mathcal{B}^*}, \phi \in \mathcal{B}^*$ and an operator $\mathcal{S} : \mathcal{B}_d^* \to \mathcal{B}^*$ such that $(\{b_n\}, \mathcal{S})$ is a retro Banach frame for $\mathcal{B}^*$. □

Next, we define a retro Banach frame of type P.

Definition 3.3. A retro Banach frame $(\{b_n\}_{n\in\mathbb{N}}, \mathcal{T})$ $(\{b_n\}_{n\in\mathbb{N}} \subset \mathcal{B}, \mathcal{T} : \mathcal{B}_d^* \to \mathcal{B}^*)$ is said to be of type P if it is strictly bounded and there exists a constant $\rho > 0$ such that for all finite sequences $c_1, c_2, \dots, c_n \geq 0$ satisfying

$$\left\|\sum_{j=1}^{n} c_j b_j\right\| \geq \rho \sum_{j=1}^{n} c_j.$$

In conclusion, we establish the subsequent finding concerning the presence of a retro Banach frame of type P.

Theorem 3.4. *A separable Banach frame has a retro Banach frame of type P.*

Proof. Let $\mathcal{B}$ be a separable Banach space. Then there exists a sequence $\{b_n\}_{n\in\mathbb{N}}$ in $\mathcal{B}$ and a sequence $\{b_n^*\}_{n\in\mathbb{N}}$ in $\mathcal{B}^*$ such that $[b_n]_{n\in\mathbb{N}} = \mathcal{B}, b_i^*(b_j) = \delta_{ij}, i, j \in \mathbb{N}, \sup_{1\leq n<\infty} \|b_n\| < \infty$ and $\sup_{1\leq n<\infty} \|b_n^*\| < \infty$.

Define $\beta_n = b_n + b_{n+1}, n \in \mathbb{N}; \beta_n^* = b_{n+1}^*, n \in \mathbb{N}$. Then $\beta_i^*(\beta_j) = \delta_{ij}, i, j \in \mathbb{N}$ and $\sup_{1 \leq n < \infty} \|\beta_n\| < \infty$. Also, for every finite sequence $a_1, a_2, \ldots, a_n \geq 0$, we obtain

$$\left\|\sum_{k=1}^{n} a_k \beta_k\right\| \geq \rho \sum_{k=1}^{n} a_k, \text{where } \rho = \frac{1}{\|\beta_1^*\|}.$$

We claim that there exists a bounded linear operator $\mathcal{T} : \mathcal{B}^*{}_d \to \mathcal{B}^*$ such that $\left(\{\beta_n\}_{n\in\mathbb{N}}, \mathcal{T}\right)$ is a retro Banach frame for $\mathcal{B}^*$. On the contrary if $\left(\{\beta_n\}_{n\in\mathbb{N}}, \mathcal{T}\right)$ is not a retro Banach frame for $\mathcal{B}^*$, then $[\beta_n]_{n\in\mathbb{N}} \neq \mathcal{B}$. Therefore, one can find a $0 \neq \beta^*$ in $\mathcal{B}^*$ such that $\beta^*(\beta_n) = 0, \forall n \in \mathbb{N}$. This gives

$$\beta(b_1) = -\beta(b_2) = -\beta(b_3) = \ldots$$

Thus $\beta^*(b_1) \neq 0$ as $\beta^* \neq 0$. Write $\beta_0 = b_1$ and $\beta_0^* = \frac{1}{\beta^*(b_1)}\beta^*$. Then $[\beta_n]_{n\in\mathbb{N}\cup\{0\}} = \mathcal{B}$.

So there exists a bounded linear operator $\mathcal{T}_0 : \mathcal{B}^*{}_d \to \mathcal{B}^*$ such that $\left\{\{\beta_n\}_{n\in\mathbb{N}\cup\{0\}}, \mathcal{T}_0\right\}$ is a retro Banach frame for $\mathcal{B}^*$ which is of type P.

□

Acknowledgements

We would like to thank the editor and the referees for their comments and suggestions, which have strengthened this manuscript significantly.

References

[1] M. R. Abdollahpour, M. H. Faroughi and A. Rahimi, *pg*-frames in Banach spaces, Methods Funct. Anal. Topology, 13 (2007), no. 3, 201–210.

[2] A. Aldroubi, Q. Sun and W.-S. Tang, *p*-frames and shift invariant subspaces of L^p, J. Fourier Anal. Appl., 7 (2001), no. 1, 1–21.

[3] P. G. Casazza, D. G. Han and D. R. Larson, Frames for Banach spaces, in *The functional and harmonic analysis of wavelets and frames (San Antonio, TX, 1999)*, 149–182, Contemp. Math., 247, Amer. Math. Soc., Providence, RI, 1999.

[4] P. G. Casazza and G. Kutyniok, Frames of subspaces, in *Wavelets, frames and operator theory*, 87–113, Contemp. Math., 345, Amer. Math. Soc., Providence, RI, 2004.

[5] P. G. Casazza, O. Christensen and D. T. Stoeva, Frame expansions in separable Banach spaces, J. Math. Anal. Appl., 307 (2005), no. 2, 710–723.

[6] O. Christensen, *An introduction to frames and Riesz bases*, Applied and Numerical Harmonic Analysis, Birkhäuser Boston, Boston, MA, 2003.

[7] O. Christensen, *Frames and bases*, Applied and Numerical Harmonic Analysis, Birkhäuser Boston, Boston, MA, 2008.

[8] R. J. Duffin and A. C. Schaeffer, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc., 72 (1952), 341–366.

[9] I. C. Daubechies, A. Grossmann and Y. F. Meyer, Painless nonorthogonal expansions, J. Math. Phys., 27 (1986), no. 5, 1271–1283.

[10] H. G. Feichtinger and K. Gröchenig, A unified approach to atomic decompositions via integrable group representations, in *Function spaces and applications (Lund, 1986)*, 52–73, Lecture Notes in Math., 1302, Springer, Berlin, 1986.

[11] D. Gabor, Theory of communication. J. Inst. Electr. Eng. Part III: Radio Commun., 93 (1946), 429–457.

[12] M. P. Goswami and H. K. Pathak, Some results on Λ-Banach frames for operator spaces, Jordan J. Math. Stat., 11 (2018), no. 2, 169–194.

[13] M. P. Goswami, A note on certain classes of retro Banach frames, Sahand Commun. Math. Anal., 20 (2023), no. 1, 1–17.
[14] K. Gröchenig, Describing functions: atomic decompositions versus frames, Monatsh. Math., 112 (1991), no. 1, 1–42.
[15] M. I. Ismailov and A. Jabrailova, On $\tilde{X}$-frames and conjugate systems in Banach spaces, Sahand Commun. Math. Anal., 1 (2014), no. 2, 19–29.
[16] S. Jahan, V. Kumar and S. K. Kaushik, On the existence of non-linear frames, Arch. Math. (Brno), 53 (2017), no. 2, 101–109.
[17] S. Jahan and V. Kumar, Some results on the existence of frames in Banach spaces, Poincare J. Anal. Appl., 2018, no. 1, 25–33.
[18] P. K. Jain, S. K. Kaushik and L. K. Vashisht, Banach frames for conjugate Banach spaces, Z. Anal. Anwendungen, 23 (2004), no. 4, 713–720.
[19] P. K. Jain, S. K. Kaushik and L. K. Vashisht, On Banach frames, Indian J. Pure Appl. Math., 37 (2006), no. 5, 265–272.
[20] P. K. Jain, S. K. Kaushik and N. Gupta, On near exact Banach frames in Banach spaces, Bull. Aust. Math. Soc., 78 (2008), no. 2, 335–342.
[21] P. K. Jain, S. K. Kaushik and N. Gupta, On frame systems in Banach spaces, Int. J. Wavelets Multiresolut. Inf. Process., 7 (2009), no. 1, 1–7.
[22] N. N. Jha and S. Sharma, Block sequences of retro Banach frames, Poincare J. Anal. Appl., 7 (2020), no. 2, 267–274.
[23] S. K. Kaushik, Some results concerning frames in Banach spaces, Tamkang J. Math., 38 (2007), no. 3, 267–276.
[24] M. Rashidi Kouchi and A. Nazari, Some relationships between G-frames and frames, Sahand Commun. Math. Anal., 2 (2015), no. 1, 1–7.
[25] R. Kumar and S. K. Sharma, A note on retro Banach frames, Int. J. Pure Appl. Math., 48 (2008), no. 1, 111–115.
[26] K. T. Poumai and S. Jahan, On K-atomic decompositions in Banach spaces, Electron. J. Math. Anal. Appl., 6 (2018), no. 1, 183–197.
[27] A. Rahimi, Invariance of Frechet frames under perturbation, Sahand Commun. Math. Anal., 1 (2014), no. 1, 41–51.
[28] S. Sharma, On bi-Banach frames in Banach spaces, Int. J. Wavelets Multiresolut. Inf. Process., 12 (2014), no. 2, 1450015, 10 pp.
[29] I. Singer, *Bases in Banach spaces. II*, Ed. Acad. R.S. România, Bucharest, 1981 Springer, Berlin-New York, 1981.
[30] W.C. Sun, G-frames and g-Riesz Bases, J. Math. Anal. Appl., 322 (2006), no. 1, 437–452.
[31] R. M. Young, *An introduction to nonharmonic Fourier series*, Pure and Applied Mathematics, 93, Academic Press, New York-London, 1980.

(Dolly Jain) Department of Mathematics, Indraprastha college for women, University of Delhi, Delhi-110054, india
E-mail address: djain@ip.du.ac.in

(B. Semthanga) Department of Mathematics, Kirori Mal College, University of Delhi, Delhi-110007, india
E-mail address: bsemthanga@kmc.du.ac.in